New transformation of Wigner operator in phase space quantum mechanics for the two-mode entangled case*

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As a natural and important extension of Fan's paper (Fan Hong-Yi 2010 *Chin. Phys.* B **19** 040305) by employing the formula of operators' Weyl ordering expansion and the bipartite entangled state representation this paper finds a new two-fold complex integration transformation about the Wigner operator $\Delta(\mu,\nu)$ (in its entangled form) in phase space quantum mechanics, and its inverse transformation. In this way, some operator ordering problems regarding to $\left(a_1^\dagger - a_2\right)$ and $\left(a_1 + a_2^\dagger\right)$ can be solved and the contents of phase space quantum mechanics can be enriched, where $a_i, a_i^\dagger$ are bosonic creation and annihilation operators, respectively.

Keywords: Wigner operator in entangled form, Weyl ordering, two-fold complex integration transformation

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1. Introduction

Integration transformations are very useful in mathematical physics. In the preceding paper[1] we have reported a new integration transformation in $q-p$ phase space

$$\int \frac{\mathrm{d}p\mathrm{d}q}{\pi}\,\mathrm{e}^{2\mathrm{i}(p-x)(q-y)}h(p,q) = f(x,y)\,, \qquad (1)$$

which is invertible

$$\int \frac{\mathrm{d}x\mathrm{d}y}{\pi}\,\mathrm{e}^{-2\mathrm{i}(p-x)(q-y)}f(x,y) = h(p,q), \qquad (2)$$

and proved that this transformation obeys Parseval theorem. By virtue of the formula of operators' Weyl ordering expansion,[2,3] we find the following new two-fold q-number integration transformation about the Wigner operator $\Delta(q',p')$[4–6] in phase space quantum mechanics,[7]

$$\int \frac{\mathrm{d}p'\mathrm{d}q'}{\pi}\Delta(q',p')\,\mathrm{e}^{-2\mathrm{i}\left(p-p'\right)\left(q-q'\right)} = \delta(p-P)\,\delta(q-Q)\,, \qquad (3)$$

and its inverse

$$\int \mathrm{d}q\mathrm{d}p\delta(p-P)\,\delta(q-Q)\,\mathrm{e}^{2\mathrm{i}\left(p-p'\right)\left(q-q'\right)} = \Delta(q',p')\,, \qquad (4)$$

where Q, P are the coordinate and momentum operators, respectively, and $[Q,P]=i,(h=1)$. These two equations can be applied to studying mutual converting formulas among Q–P ordering, P–Q ordering and Weyl ordering of operators.

In the present paper, we extend Eqs. (3) and (4) to the two-mode entangled case and construct the two-fold complex integration transformation about the Wigner operator $\Delta(\mu,\nu)$ (in its entangled form, μ,ν are complex) in phase space quantum mechanics. By mentioning the two-mode entangled case, we naturally think of two mutually conjugate entangled state representations $|\xi\rangle$ and $|\eta\rangle$ (see Section 2 below). In Section 3, after deriving the Weyl ordering of $|\eta\rangle\langle\xi|$ by virtue of the formula of operator's Weyl ordering expansion, we give the Weyl ordering form of $\delta\left(\nu - a_1 + a_2^\dagger\right)\ \delta\left(\nu^* - a_1^\dagger + a_2\right)\ \delta\left(\mu - a_1 - a_2^\dagger\right)\ \delta\left(\mu^* - a_1^\dagger - a_2\right)$, where $a_i, a_i^\dagger$ are the bosonic creation and annihilation operators satisfying $\left[a_i, a_j^\dagger\right] = \delta_{ij}$. Then in Section 4, we find a new two-fold complex integration transformation relating the Wigner operator $\Delta(\mu,\nu)$ and the operator $\delta\left(\nu - a_1 + a_2^\dagger\right)\ \delta\left(\nu^* - a_1^\dagger + a_2\right)\ \delta\left(\mu - a_1 - a_2^\dagger\right)\ \delta\left(\mu^* - a_1^\dagger - a_2\right)$. In Section 5 we tackle some operator ordering problems with regard to $\left(a_1^\dagger - a_2\right)$ and

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$\left(a_1 + a_2^\dagger\right)$. In this way, the contents of phase space quantum mechanics can be further enriched.

2. Brief review of the two mutually conjugate entangled states

The conception of quantum entanglement has been paid much attention because of their wide uses in quantum communication and quantum computation. The original concept of quantum entanglement pointed out by Einstein, Podolsky and Rosen (EPR) in their famous paper[8] arguing the incompleteness of quantum mechanics is for a bipartite system characteristic of continuous variables. According to the original idea of EPR that two particles' relative coordinate operator commutes with their total momentum operator, $[Q_1 - Q_2, P_1 + P_2] = 0$, we have set up the bipartite entangled state representation of continuous variable in two-mode Fock space,[9] which is the common eigenvector $|\eta\rangle$ of $Q_1 - Q_2$ and $P_1 + P_2$, i.e.

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right)|00\rangle\,, \quad (5)$$

where $\eta \equiv \eta_1 + \mathrm{i}\eta_2$ is a complex number, and $|00\rangle$ is the two-mode vacuum state. Using $\left[a_i, a_j^\dagger\right] = \delta_{ij}$, we can show that $|\eta\rangle$ obeys the eigenvector equations

$$\begin{aligned}\left(a_1 - a_2^\dagger\right)|\eta\rangle &= \eta\,|\eta\rangle\,,\\ \left(a_2 - a_1^\dagger\right)|\eta\rangle &= -\eta^*\,|\eta\rangle\,.\end{aligned} \quad (6)$$

The real and imaginary parts of η are respectively the eigenvalue of $Q_1 - Q_2$ and $P_1 + P_2$,

$$\begin{aligned}(Q_1 - Q_2)\,|\eta\rangle &= \sqrt{2}\eta_1\,|\eta\rangle\,,\\ (P_1 + P_2)\,|\eta\rangle &= \sqrt{2}\eta_2\,|\eta\rangle\,.\end{aligned} \quad (7)$$

$|\eta\rangle$ is proved to have complete relation and orthonormal property

$$\int \frac{\mathrm{d}^2\eta}{\pi}\,|\eta\rangle\,\langle\eta| = 1 \quad (8)$$

and

$$\langle\eta'|\,\eta\rangle = \pi\delta\left(\eta' - \eta\right)\delta\left(\eta'^* - \eta^*\right) \equiv \pi\delta^{(2)}\left(\eta' - \eta\right). \quad (9)$$

On the other hand, $|\eta\rangle'$s conjugate state, which is the common eigenvector of two particles' centre-of-mass coordinate and relative momentum, is

$$|\xi\rangle = \exp\left(-\frac{|\xi|^2}{2} + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger\right)|00\rangle \quad (10)$$

with $\xi = \xi_1 + \mathrm{i}\xi_2$. The $|\xi\rangle$ obeys

$$\left(a_1 + a_2^\dagger\right)|\xi\rangle = \xi\,|\xi\rangle\,, \quad \left(a_1^\dagger + a_2\right)|\xi\rangle = \xi^*\,|\xi\rangle\,, \quad (11)$$

and

$$\begin{aligned}(Q_1 + Q_2)\,|\xi\rangle &= \sqrt{2}\xi_1\,|\xi\rangle\,,\\ (P_1 - P_2)\,|\xi\rangle &= \sqrt{2}\xi_2\,|\xi\rangle\,.\end{aligned} \quad (12)$$

The $|\xi\rangle$ is also completen and orthonormal,

$$\int \frac{\mathrm{d}^2\xi}{\pi}\,|\xi\rangle\,\langle\xi| = 1 \quad (13)$$

and

$$\langle\xi\,|\xi'\rangle = \pi\delta\left(\xi - \xi'\right)\delta\left(\xi^* - \xi'^*\right). \quad (14)$$

It is found that $|\xi\rangle$ and $|\eta\rangle$ are of equal importance because

$$\langle\eta|\,\xi\rangle = \frac{1}{2}\exp\left[\left(\eta^*\xi - \xi^*\eta\right)/2\right], \quad (15)$$

which is just a complex form of the Fourier transformation, since $(\eta^*\xi - \xi^*\eta)$ is pure imaginary, $\langle\eta|\,\xi\rangle$ is a Fourier transform kernel.

3. The Weyl ordering form of $\boldsymbol{\delta\left(\nu - a_1 + a_2^\dagger\right)\delta\left(\nu^* - a_1^\dagger + a_2\right)\delta\left(\mu - a_1 - a_2^\dagger\right)\delta\left(\mu^* - a_1^\dagger - a_2\right)}$

Using the complete relations Eqs. (8) and (13), as well as Eqs. (6), (11) and (15), we have

$$\begin{aligned}&\delta\left(\nu - a_1 + a_2^\dagger\right)\delta\left(\nu^* - a_1^\dagger + a_2\right)\delta\left(\mu - a_1 - a_2^\dagger\right)\delta\left(\mu^* - a_1^\dagger - a_2\right)\\ &= \int \frac{\mathrm{d}^2\eta\mathrm{d}^2\xi}{\pi^2}\delta^{(2)}\left(\nu - a_1 + a_2^\dagger\right)|\eta\rangle\,\langle\eta|\,\xi\rangle\,\langle\xi|\,\delta^{(2)}\left(\mu - a_1 - a_2^\dagger\right)\\ &= \frac{1}{2}\int \frac{\mathrm{d}^2\eta\mathrm{d}^2\xi}{\pi^2}\,|\eta\rangle\,\langle\xi|\,\mathrm{e}^{(\eta^*\xi - \eta\xi^*)/2}\delta^{(2)}\left(\nu - \eta\right)\delta^{(2)}\left(\mu - \xi\right)\end{aligned}$$

$$= \frac{1}{2} |\eta\rangle_{\eta=\nu} \langle\xi|_{\xi=\mu} \, \mathrm{e}^{(\nu^*\mu - \nu\mu^*)/2}, \tag{16}$$

where we have simplified the notation of the product of two Delta functions

$$\begin{aligned} &\delta^{(2)}\left(\nu - a_1 + a_2^\dagger\right) \delta^{(2)}\left(\mu - a_1 - a_2^\dagger\right) \\ &\equiv \delta\left(\nu - a_1 + a_2^\dagger\right) \delta\left(\nu^* - a_1^\dagger + a_2\right) \delta\left(\mu - a_1 - a_2^\dagger\right) \delta\left(\mu^* - a_1^\dagger - a_2\right). \end{aligned} \tag{17}$$

From Eq. (16) we see that once the Weyl ordering of $|\eta\rangle \langle\xi|$ is known, the Weyl ordering of $\delta\left(\nu - a_1 + a_2^\dagger\right)$ $\delta\left(\nu^* - a_1^\dagger + a_2\right)$ $\delta\left(\mu - a_1 - a_2^\dagger\right)$ $\delta\left(\mu^* - a_1^\dagger - a_2\right)$ is obtained. For this purpose, we recall the Weyl ordered expansion formula of two-mode operators[2, 3]

$$\rho = 4 \int \frac{\mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2}{\pi^2} \vdots \langle -\beta_1, -\beta_2| \, \rho \, |\beta_1, \beta_2\rangle \exp\left[2 \sum_{k=1}^{2} \left(\beta_k^* a_k - a_k^\dagger \beta_k + a_k^\dagger a_k\right)\right] \vdots \, , \tag{18}$$

where $\vdots \, \vdots$ denotes the Weyl ordering, within which $a_i, a_i^\dagger$ can be permuted, $|\beta_1, \beta_2\rangle \equiv |\beta_1\rangle \, |\beta_2\rangle$ is a two-mode coherent state with $|\beta_i\rangle = \exp[-|\beta_i|^2/2 + \beta_i a_i^\dagger] \, |0\rangle_i$.[10,11] For the operator $|\eta\rangle \langle\xi|$, using Eq. (18) and the overlaps

$$\langle\xi| \, \beta_1, \beta_2\rangle = \exp\left(-\frac{1}{2}|\xi|^2 + \xi^*\beta_1 + \xi\beta_2 - \beta_1\beta_2 - \frac{1}{2}|\beta_1|^2 - \frac{1}{2}|\beta_2|^2\right) \tag{19}$$

and

$$\langle -\beta_1, -\beta_2| \, \eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 - \eta\beta_1^* + \eta^*\beta_2^* + \beta_1^*\beta_2^* - \frac{1}{2}|\beta_1|^2 - \frac{1}{2}|\beta_2|^2\right), \tag{20}$$

we can derive the Weyl ordering of $|\eta\rangle \langle\xi|$ in the following way:

$$\begin{aligned} |\eta\rangle \langle\xi| &= 4 \int \frac{\mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2}{\pi^2} \vdots \langle -\beta_1, -\beta_2| \, \eta\rangle \langle\xi| \, \beta_1, \beta_2\rangle \exp\left[2 \sum_{k=1}^{2} \left(\beta_k^* a_k - a_k^\dagger \beta_k + a_k^\dagger a_k\right)\right] \vdots \\ &= 4 \int \frac{\mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2}{\pi^2} \vdots \exp\left[-\frac{|\eta|^2}{2} - \eta\beta_1^* + \eta^*\beta_2^* + \beta_1^*\beta_2^* - |\beta_1|^2 - |\beta_2|^2\right. \\ &\quad \left. - \frac{|\xi|^2}{2} + \xi^*\beta_1 + \xi\beta_2 - \beta_1\beta_2 + 2 \sum_{k=1}^{2} \left(\beta_k^* a_k - a_k^\dagger \beta_k + a_k^\dagger a_k\right)\right] \vdots \\ &= 4 \, \mathrm{e}^{(-|\eta|^2 - |\xi|^2)/2} \int \frac{\mathrm{d}^2\beta_2}{\pi} \vdots \exp\left[-2|\beta_2|^2 + \beta_2\left(\xi - 2a_2^\dagger - 2a_1 + \eta\right)\right. \\ &\quad \left. + \beta_2^*\left(\eta^* + 2a_2 + \xi^* - 2a_1^\dagger\right) + \left(\xi^* - 2a_1^\dagger\right)(2a_1 - \eta) + 2 \sum_{k=1}^{2} a_k^\dagger a_k\right] \vdots \\ &= 2 \vdots \exp\left[\frac{1}{2}\left(\xi\eta^* - \eta\xi^*\right) + \xi\left(a_2 - a_1^\dagger\right) + \eta\left(a_2 + a_1^\dagger\right)\right. \\ &\quad \left. - \eta^*\left(a_1 + a_2^\dagger\right) + \xi^*\left(a_1 - a_2^\dagger\right) + 2a_2^\dagger a_1^\dagger - 2a_2 a_1\right] \vdots \, , \end{aligned} \tag{21}$$

where we have used the integral formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp\left(\zeta |z|^2 + \xi z + \eta z^*\right) = -\frac{1}{\zeta} \exp\left[-\frac{\xi\eta}{\zeta}\right], \quad \mathrm{Re}\,(\zeta) < 0. \tag{22}$$

It then follows from Eq. (21) that

$$\frac{1}{2} |\eta\rangle \langle\xi| \, \mathrm{e}^{(\eta^*\xi - \eta\xi^*)/2} = \vdots \exp\left[\left(\xi - a_1 - a_2^\dagger\right)\left(\eta^* - a_1^\dagger + a_2\right) - \left(\eta - a_1 + a_2^\dagger\right)\left(\xi^* - a_1^\dagger - a_2\right)\right] \vdots \, . \tag{23}$$

Substituting Eq. (23) into Eq. (16) we have the Weyl ordered form

$$\begin{aligned}&\delta\left(\nu-a_1+a_2^{\dagger}\right)\delta\left(\nu^*-a_1^{\dagger}+a_2\right)\delta\left(\mu-a_1-a_2^{\dagger}\right)\delta\left(\mu^*-a_1^{\dagger}-a_2\right)\\&=\vdots\ \exp\left[\left(\mu-a_1-a_2^{\dagger}\right)\left(\nu^*-a_1^{\dagger}+a_2\right)-\left(\nu-a_1+a_2^{\dagger}\right)\left(\mu^*-a_1^{\dagger}-a_2\right)\right]\ \vdots,\end{aligned}\tag{24}$$

which is in exponential form within $\begin{matrix}\vdots\\ \vdots\\ \vdots\end{matrix}$ and is easy for remembering. Noting

$$\left[a_1^{\dagger}-a_2,a_1+a_2^{\dagger}\right]=-2,\tag{25}$$

we should consider another ordering other than Eq. (24), i.e.,

$$\begin{aligned}&\delta\left(\mu-a_1-a_2^{\dagger}\right)\delta\left(\mu^*-a_1^{\dagger}-a_2\right)\delta\left(\nu-a_1+a_2^{\dagger}\right)\delta\left(\nu^*-a_1^{\dagger}+a_2\right)\\&=\frac{1}{2}\int\frac{\mathrm{d}^2\eta\mathrm{d}^2\xi}{\pi^2}\left|\xi\right\rangle\left\langle\eta\right|\mathrm{e}^{(\eta\xi^*-\eta^*\xi)/2}\delta^{(2)}\left(\nu-\eta\right)\delta^{(2)}\left(\mu-\xi\right).\end{aligned}\tag{26}$$

Similar to the way of deriving Eq. (23), we deduce

$$\frac{1}{2}\left|\xi\right\rangle\left\langle\eta\right|\mathrm{e}^{(\eta\xi^*-\eta^*\xi)/2}=\vdots\ \exp\left[\left(\eta-a_1+a_2^{\dagger}\right)\left(\xi^*-a_1^{\dagger}-a_2\right)-\left(\xi-a_1-a_2^{\dagger}\right)\left(\eta^*-a_1^{\dagger}+a_2\right)\right]\ \vdots.\tag{27}$$

It then follows

$$\begin{aligned}&\delta\left(\mu-a_1-a_2^{\dagger}\right)\delta\left(\mu^*-a_1^{\dagger}-a_2\right)\delta\left(\nu-a_1+a_2^{\dagger}\right)\delta\left(\nu^*-a_1^{\dagger}+a_2\right)\\&=\vdots\ \exp\left[-\left(\mu-a_1-a_2^{\dagger}\right)\left(\nu^*-a_1^{\dagger}+a_2\right)+\left(\nu-a_1+a_2^{\dagger}\right)\left(\mu^*-a_1^{\dagger}-a_2\right)\right]\ \vdots.\end{aligned}\tag{28}$$

Comparing Eq. (24) with Eq. (28), we should point out that due to the existence of Weyl ordering symbol $\begin{matrix}\vdots\\ \vdots\\ \vdots\end{matrix}$,

$$\left(\vdots\ \mathrm{e}^{x}\ \vdots\right)^{-1}\neq\vdots\ \mathrm{e}^{-x}\ \vdots.$$

4. The mutual integration transformation between $\delta\left(\nu-a_1+a_2^{\dagger}\right)\delta\left(\nu^*-a_1^{\dagger}+a_2\right)\delta\left(\mu-a_1-a_2^{\dagger}\right)\delta\left(\mu^*-a_1^{\dagger}-a_2\right)$ and the Wigner operator

In Ref. [12] for the correlated two-body systems, we have successfully derived the Wigner operator in entangled form, expressed in the entangled state $\left\langle\eta\right|$ representation as

$$\Delta\left(\mu,\nu\right)=\int\frac{\mathrm{d}^2\eta}{\pi^3}\left|\nu-\eta\right\rangle\left\langle\nu+\eta\right|\mathrm{e}^{\eta\mu^*-\eta^*\mu},\tag{29}$$

$\Delta\left(\mu,\nu\right)$ plays the role of establishing the relationship between η–ξ phase space function and its Weyl–Wigner quantum correspondence operator. Using the similar method to derive Eq. (21), we find that the Weyl ordering of operator $\left|\nu-\eta\right\rangle\left\langle\nu+\eta\right|$ is

$$\left|\nu-\eta\right\rangle\left\langle\nu+\eta\right|=\vdots\ \delta\left(\nu-a_1+a_2^{\dagger}\right)\delta\left(\nu^*-a_1^{\dagger}+a_2\right)$$

$$\times \exp\left[\eta\left(\nu^{*}-2a_{1}^{\dagger}\right)-\eta^{*}\left(\nu-2a_{1}\right)-2\left(a_{1}^{\dagger}-\nu^{*}\right)\left(a_{1}-\nu\right)+2a_{2}^{\dagger}a_{2}\right]\vdots. \tag{30}$$

Substituting Eq. (30) into Eq. (29) and performing the integration over $\mathrm{d}^2\eta$, we obtain

$$\begin{aligned}\Delta\left(\mu,\nu\right) &= \vdots\,\delta\left(\mu-a_{1}-a_{2}^{\dagger}\right)\delta\left(\mu^{*}-a_{1}^{\dagger}-a_{2}\right)\delta\left(\nu-a_{1}+a_{2}^{\dagger}\right)\delta\left(\nu^{*}-a_{1}^{\dagger}+a_{2}\right)\vdots\\ &= \vdots\,\delta^{(2)}\left(\mu-a_{1}-a_{2}^{\dagger}\right)\delta^{(2)}\left(\nu-a_{1}+a_{2}^{\dagger}\right)\vdots,\end{aligned} \tag{31}$$

which is also a neat Dirac's δ-operator function within the Weyl ordering symbol.

As a result of Eqs. (24) and (31), we have

$$\begin{aligned}&\delta\left(\eta-a_{1}+a_{2}^{\dagger}\right)\delta\left(\eta^{*}-a_{1}^{\dagger}+a_{2}\right)\delta\left(\xi-a_{1}-a_{2}^{\dagger}\right)\delta\left(\xi^{*}-a_{1}^{\dagger}-a_{2}\right)\\ &= \vdots\,\exp\left[\left(\xi-a_{1}-a_{2}^{\dagger}\right)\left(\eta^{*}-a_{1}^{\dagger}+a_{2}\right)-\left(\eta-a_{1}+a_{2}^{\dagger}\right)\left(\xi^{*}-a_{1}^{\dagger}-a_{2}\right)\right]\vdots\\ &= \int\frac{\mathrm{d}^2\mu\mathrm{d}^2\nu}{\pi^2}\vdots\,\mathrm{e}^{(\xi-\mu)(\eta^{*}-\nu^{*})-(\eta-\nu)(\xi^{*}-\mu^{*})}\delta^{(2)}\left(\mu-a_{1}-a_{2}^{\dagger}\right)\delta^{(2)}\left(\nu-a_{1}+a_{2}^{\dagger}\right)\vdots\\ &= \int\frac{\mathrm{d}^2\mu\mathrm{d}^2\nu}{\pi^2}\,\mathrm{e}^{(\xi-\mu)(\eta^{*}-\nu^{*})-(\eta-\nu)(\xi^{*}-\mu^{*})}\Delta\left(\mu,\nu\right).\end{aligned} \tag{32}$$

This is the mutual transformation between the entangled form of Wigner operator $\Delta\left(\mu,\nu\right)$ and $\delta\left(\mu-a_{1}-a_{2}^{\dagger}\right)\delta\left(\mu^{*}-a_{1}^{\dagger}-a_{2}\right)\delta\left(\nu-a_{1}+a_{2}^{\dagger}\right)\delta\left(\nu^{*}-a_{1}^{\dagger}+a_{2}\right)$. Due to $\int\mathrm{d}^2\mu\mathrm{d}^2\nu\Delta\left(\mu,\nu\right)=1$, we may also consider Eq. (32) as the expansion of the four Delta operators' product in terms of $\Delta\left(\mu,\nu\right)$. Taking $W\left(\mu,\nu\right)=\left\langle\psi\right|\Delta\left(\mu,\nu\right)\left|\psi\right\rangle$, we see the following integration relation of the Wigner function of $\left|\psi\right\rangle$ from Eqs. (27) and (32),

$$\int\frac{\mathrm{d}^2\mu\mathrm{d}^2\nu}{\pi^2}\,\mathrm{e}^{(\xi-\mu)(\eta^{*}-\nu^{*})-(\eta-\nu)(\xi^{*}-\mu^{*})}W\left(\mu,\nu\right)=\frac{1}{2}\left\langle\psi\right|\left.\eta\right\rangle\left\langle\xi\right|\left.\psi\right\rangle\,\mathrm{e}^{(\eta^{*}\xi-\eta\xi^{*})/2}.$$

Moreover, the reciprocal transformation of Eq. (32) is

$$\begin{aligned}&\int\frac{\mathrm{d}^2\xi\mathrm{d}^2\eta}{\pi^2}\delta\left(\eta-a_{1}+a_{2}^{\dagger}\right)\delta\left(\eta^{*}-a_{1}^{\dagger}+a_{2}\right)\delta\left(\xi-a_{1}-a_{2}^{\dagger}\right)\delta\left(\xi^{*}-a_{1}^{\dagger}-a_{2}\right)\,\mathrm{e}^{-(\xi-\mu)(\eta^{*}-\nu^{*})+(\eta-\nu)(\xi^{*}-\mu^{*})}\\ &=\int\frac{\mathrm{d}^2\xi\mathrm{d}^2\eta}{\pi^2}\int\frac{\mathrm{d}^2\mu'\mathrm{d}^2\nu'}{\pi^2}\Delta\left(\mu',\nu'\right)\,\mathrm{e}^{(\xi-\mu')(\eta^{*}-\nu'^{*})-(\eta-\nu')(\xi^{*}-\mu'^{*})}\,\mathrm{e}^{-(\xi-\mu)(\eta^{*}-\nu^{*})+(\eta-\nu)(\xi^{*}-\mu^{*})}\\ &=\int\frac{\mathrm{d}^2\mu'\mathrm{d}^2\nu'}{\pi^2}\Delta\left(\mu',\nu'\right)\,\mathrm{e}^{\left(-\nu'\mu'^{*}+\mu'\nu'^{*}-\mu\nu^{*}+\nu\mu^{*}\right)}\int\frac{\mathrm{d}^2\xi\mathrm{d}^2\eta}{\pi^2}\,\mathrm{e}^{\xi\left(\nu^{*}-\nu'^{*}\right)+\xi^{*}\left(\nu'-\nu\right)}\,\mathrm{e}^{\eta^{*}\left(\mu-\mu'\right)+\eta\left(\mu'^{*}-\mu^{*}\right)}\\ &=\int\frac{\mathrm{d}^2\mu'\mathrm{d}^2\nu'}{\pi^2}\Delta\left(\mu',\nu'\right)\,\mathrm{e}^{\left(-\nu'\mu'^{*}+\mu'\nu'^{*}-\mu\nu^{*}+\nu\mu^{*}\right)}\delta^{(2)}\left(\nu'-\nu\right)\delta^{(2)}\delta\left(\mu'-\mu\right)\\ &=\Delta\left(\mu,\nu\right).\end{aligned} \tag{33}$$

So Eqs. (32) and (33) are qualified to be new two-fold complex integration transformations about the Wigner operator $\Delta\left(\mu,\nu\right)$ (in its entangled form) in phase space quantum mechanics.

Further, multiplying both sides of Eq. (33) from the left by $\int\mathrm{d}^2\mu\mathrm{d}^2\nu/\pi^2 D\left(\nu,\mu\right)$ and considering

$$Q_{j}=\frac{a_{j}+a_{j}^{\dagger}}{\sqrt{2}},\quad P_{j}=\frac{a_{j}-a_{j}^{\dagger}}{\mathrm{i}\sqrt{2}}, \tag{34}$$

we obtain

$$\int\frac{\mathrm{d}^2\mu\mathrm{d}^2\nu}{\pi^2}D\left(\nu,\mu\right)\Delta\left(\mu,\nu\right)$$

$$
\begin{aligned}
&= \int \frac{\mathrm{d}^2\xi \mathrm{d}^2\eta}{\pi^2}\delta^{(2)}\left(\eta - a_1 + a_2^\dagger\right)\delta^{(2)}\left(\xi - a_1 - a_2^\dagger\right)\int \frac{\mathrm{d}^2\mu \mathrm{d}^2\nu}{\pi^2}D\left(\nu,\mu\right)\,\mathrm{e}^{-(\xi-\mu)(\eta^*-\nu^*)+(\eta-\nu)(\xi^*-\mu^*)} \\
&= \int \frac{\mathrm{d}^2\xi \mathrm{d}^2\eta}{\pi^2}\delta^{(2)}\left(\eta - a_1 + a_2^\dagger\right)\delta^{(2)}\left(\xi - a_1 - a_2^\dagger\right)\mathcal{F}(\eta,\xi) \\
&= \int \frac{\mathrm{d}^2\xi \mathrm{d}^2\eta}{\pi^2}\delta\left(\eta_1 - \frac{Q_1-Q_2}{\sqrt{2}}\right)\delta\left(\eta_2 - \frac{P_1+P_2}{\sqrt{2}}\right)\delta\left(\xi_1 - \frac{Q_1+Q_2}{\sqrt{2}}\right)\delta\left(\xi_2 - \frac{P_1-P_2}{\sqrt{2}}\right)\mathcal{F}(\eta,\xi) \\
&= \mathcal{F}\left(\frac{Q_1-Q_2}{\sqrt{2}},\frac{P_1+P_2}{\sqrt{2}},\frac{Q_1+Q_2}{\sqrt{2}},\frac{P_1-P_2}{\sqrt{2}}\right),
\end{aligned}
\tag{35}
$$

where we have introduced

$$
\mathcal{F}(\eta,\xi) \equiv \int \frac{\mathrm{d}^2\mu \mathrm{d}^2\nu}{\pi^2}D\left(\nu,\mu\right)\,\mathrm{e}^{(\xi^*-\mu^*)(\eta-\nu)-(\eta^*-\nu^*)(\xi-\mu)},
\tag{36}
$$

which seems to be a new interesting transformation. We can prove that the inverse transform of Eq. (36) is

$$
\int \frac{\mathrm{d}^2\xi \mathrm{d}^2\eta}{\pi^2}\,\mathrm{e}^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)}\mathcal{F}(\eta,\xi) \equiv D\left(\nu,\mu\right).
\tag{37}
$$

In fact, substituting Eq. (37) into Eq. (36) yields

$$
\begin{aligned}
&\int \frac{\mathrm{d}^2\xi' \mathrm{d}^2\eta'}{\pi^2}\mathcal{F}(\eta',\xi')\int \frac{\mathrm{d}^2\mu \mathrm{d}^2\nu}{\pi^2}\,\mathrm{e}^{\left(\xi'-\mu\right)\left(\eta'^*-\nu^*\right)-\left(\eta'-\nu\right)\left(\xi'^*-\mu^*\right)+(\xi^*-\mu^*)(\eta-\nu)-(\eta^*-\nu^*)(\xi-\mu)} \\
&= \int \frac{\mathrm{d}^2\xi' \mathrm{d}^2\eta'}{\pi^2}\mathcal{F}(\eta',\xi')\,\mathrm{e}^{\xi'\eta'^*-\eta'\xi'^*+\xi^*\eta-\eta^*\xi}\int \frac{\mathrm{d}^2\mu \mathrm{d}^2\nu}{\pi^2}\,\mathrm{e}^{\left(\eta^*-\eta'^*\right)\mu+\left(\eta'-\eta\right)\mu^*}\,\mathrm{e}^{\left(\xi'^*-\xi^*\right)\nu+\left(\xi-\xi'\right)\nu^*} \\
&= \int \mathrm{d}^2\xi' \mathrm{d}^2\eta'\mathcal{F}(\eta',\xi')\,\mathrm{e}^{\xi'\eta'^*-\eta'\xi'^*+\xi^*\eta-\eta^*\xi}\delta^{(2)}\left(\eta'-\eta\right)\delta^{(2)}\delta\left(\xi-\xi'\right) \\
&= \mathcal{F}(\eta,\xi).
\end{aligned}
\tag{38}
$$

Due to

$$
\begin{aligned}
&\int \frac{\mathrm{d}^2\xi \mathrm{d}^2\eta}{\pi^2}\exp\left[\left(\xi-\mu\right)\left(\eta^*-\nu^*\right)-\left(\eta-\nu\right)\left(\xi^*-\mu^*\right)\right] \\
&= \int \mathrm{d}^2\xi\delta\left(\xi-\mu\right)\delta\left(\xi^*-\mu^*\right)\,\mathrm{e}^{\nu(\xi^*-\mu^*)-\nu^*(\xi-\mu)} = 1,
\end{aligned}
\tag{39}
$$

$\mathrm{e}^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)}$ can be considered as a basis function in ξ–η phase space, or Eq. (37) can be looked as an expansion of $D\left(\nu,\mu\right)$ in terms of $\mathrm{e}^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)}$, with the expansion coefficient being $\mathcal{F}(\eta,\xi)$.

Additionally, we can prove

$$
\begin{aligned}
&\int \frac{\mathrm{d}^2\xi \mathrm{d}^2\eta}{\pi^2}|\mathcal{F}(\eta,\xi)|^2 \\
&= \int \frac{\mathrm{d}^2\mu \mathrm{d}^2\nu}{\pi^2}|D\left(\nu,\mu\right)|^2\int \frac{\mathrm{d}^2\mu' \mathrm{d}^2\nu'}{\pi^2}\exp\left[\left(\mu^*\nu-\nu^*\mu\right)+\left(\mu'\nu'^*-\nu'\mu'^*\right)\right] \\
&\quad\times\int \frac{\mathrm{d}^2\xi \mathrm{d}^2\eta}{\pi^2}\exp\left[\left(\mu'^*-\mu^*\right)\eta+\left(\mu-\mu'\right)\eta^*+\left(\nu^*-\nu'^*\right)\xi+\left(\nu'-\nu\right)\xi^*\right] \\
&= \int \frac{\mathrm{d}^2\mu \mathrm{d}^2\nu}{\pi^2}|D\left(\nu,\mu\right)|^2\int \mathrm{d}^2\mu' \mathrm{d}^2\nu'\exp\left[\left(\mu^*\nu-\nu^*\mu\right)+\left(\mu'\nu'^*-\nu'\mu'^*\right)\right]\delta^{(2)}\left(\mu-\mu'\right)\delta^{(2)}\delta(\nu'-\nu) \\
&= \int \frac{\mathrm{d}^2\mu \mathrm{d}^2\nu}{\pi^2}|D\left(\nu,\mu\right)|^2,
\end{aligned}
\tag{40}
$$

which shows that the transformations of Eqs. (36) and (37) obey the Parseval-like theorem as well.

5. Application

By using the above transformation we can solve some operator ordering problems arising from Eq. (25), for example, we want to put $\left(a_1^\dagger - a_2\right)^n\left(a_1 + a_2^\dagger\right)^m$ into its Weyl ordered form. According to Eqs. (8), (13) and

(15), we have

$$\frac{1}{2}\int \frac{\mathrm{d}^2\eta \mathrm{d}^2\xi}{\pi^2}\left|\eta\right\rangle\left\langle\xi\right| \mathrm{e}^{(\eta^*\xi-\eta\xi^*)/2}=1 \tag{41}$$

and

$$\frac{1}{2}\int \frac{\mathrm{d}^2\eta \mathrm{d}^2\xi}{\pi^2}\left|\xi\right\rangle\left\langle\eta\right| \mathrm{e}^{(\eta\xi^*-\eta^*\xi)/2}=1. \tag{42}$$

As a result of Eqs. (41) and (23), using the eigenvector equations (6) and (11), we obtain

$$\begin{aligned}
&\left(a_1^\dagger-a_2\right)^n\left(a_1+a_2^\dagger\right)^m\\
&=\frac{1}{2}\int \frac{\mathrm{d}^2\eta \mathrm{d}^2\xi}{\pi^2}\eta^{*n}\xi^m\left|\eta\right\rangle\left\langle\xi\right| \mathrm{e}^{(\eta^*\xi-\eta\xi^*)/2}\\
&=\int \frac{\mathrm{d}^2\eta \mathrm{d}^2\xi}{\pi^2}\eta^{*n}\xi^m \vdots \exp\left[\left(\xi-a_1-a_2^\dagger\right)\left(\eta^*-a_1^\dagger+a_2\right)-\left(\eta-a_1+a_2^\dagger\right)\left(\xi^*-a_1^\dagger-a_2\right)\right] \vdots\\
&=\int \frac{\mathrm{d}\eta_1 \mathrm{d}\eta_2 \mathrm{d}\xi_1 \mathrm{d}\xi_2}{\pi^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}\mathrm{i}^{k+l}\eta_1^{n-k}\left(-\eta_2\right)^k\xi_1^{m-l}\xi_2^l\\
&\quad\times \vdots \exp\left\{2\mathrm{i}\left[-\eta_2+\frac{1}{\sqrt{2}}\left(P_1+P_2\right)\right]\left[\xi_1-\frac{1}{\sqrt{2}}\left(Q_1+Q_2\right)\right]\right\}\\
&\quad\times\exp\left\{2\mathrm{i}\left[\eta_1-\frac{1}{\sqrt{2}}\left(Q_1-Q_2\right)\right]\left[\xi_2-\frac{1}{\sqrt{2}}\left(P_1-P_2\right)\right]\right\} \vdots\\
&=-2^{-(m+n)/2}\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}\\
&\quad\times \vdots H_{n-k,l}\left[\left(Q_1-Q_2\right),\mathrm{i}\left(P_1-P_2\right)\right]H_{m-l,k}\left[\left(Q_1+Q_2\right),-\mathrm{i}\left(P_1+P_2\right)\right] \vdots,
\end{aligned} \tag{43}$$

where we have used the following integration formula:[13]

$$\int \frac{\mathrm{d}x\mathrm{d}y}{\pi}x^m y^r\exp[2\mathrm{i}\left(y-s\right)\left(x-t\right)]=\left(\frac{1}{\sqrt{2}}\right)^{m+r}\left(-\mathrm{i}\right)^r H_{m,r}\left(\sqrt{2}t,\mathrm{i}\sqrt{2}s\right), \tag{44}$$

where $H_{m,r}$ is the two-variable Hermite polynomials,[14,15]

$$H_{m,r}(t,s)=\sum_{l=0}^{\min(m,r)}\frac{m!r!(-1)^l}{l!(m-l)!(r-l)!}t^{m-l}s^{r-l}. \tag{45}$$

Equation (43) is a simple approach to turn $\left(a_1^\dagger-a_2\right)^n\left(a_1+a_2^\dagger\right)^m$ into its Weyl ordering. Similarly, using Eqs. (27) and (42), we see that the Weyl ordered form of $\left(a_1+a_2^\dagger\right)^m\left(a_1^\dagger-a_2\right)^n$ is

$$\begin{aligned}
\left(a_1+a_2^\dagger\right)^m\left(a_1^\dagger-a_2\right)^n&=(-1)^{m+1}2^{-(m+n)/2}\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}\\
&\quad\times \vdots H_{n-k,l}\left[\left(Q_1+Q_2\right),\mathrm{i}\left(P_1+P_2\right)\right]H_{m-l,k}\left[-\left(Q_1-Q_2\right),\mathrm{i}\left(P_1-P_2\right)\right] \vdots.
\end{aligned} \tag{46}$$

In summary, as a natural extension of our preceding paper,[7] through employing operators' Weyl ordering expansion formula we have found a new two-fold complex integration transformation about the Wigner operator $\Delta\left(\mu,\nu\right)$ (in its entangled form) in phase space quantum mechanics, which is invertible and obeys Parseval

theorem. The new complex integration transformation is compatible with $\eta-\xi$ phase space quantum mechanics which takes entangled state representations as basis. Under this transformation the relationship between the Weyl ordering and $\left(a_1^\dagger - a_2\right) \Leftrightarrow \left(a_1 + a_2^\dagger\right)$ ordering or $\left(a_1 + a_2^\dagger\right) \Leftrightarrow \left(a_1^\dagger - a_2\right)$ ordering of operators is revealed. In this way, the contents of phase space quantum mechanics[5] can be further enriched. Studies about the multi-mode case are in progress and new results will be reported later.

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