Variational iteration method for solving compressible Euler equations*

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This paper applies the variational iteration method to obtain approximate analytic solutions of compressible Euler equations in gas dynamics. This method is based on the use of Lagrange multiplier for identification of optimal values of parameters in a functional. Using this method, a rapid convergent sequence is produced which converges to the exact solutions of the problem. Numerical results and comparison with other two numerical solutions verify that this method is very convenient and efficient.

Keywords: variational iteration method, compressible Euler equations, approximate analytic solutions, Lagrange multiplier

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1. Introduction

Time-dependent Euler equations are the governing equations for the dynamics of a compressible material, such as gases or liquids, along with closure conditions in the form of equations of state. Equations of state are statements about the nature of the material in question and require some notions from thermodynamics. The Euler equations result from neglecting the effects of viscosity, heat conduction and body forces on a compressible medium. These equations are a system of hyperbolic conservation laws. Investigation of new methods for solving these equations has become an interesting task for the physicists and the mathematicians. During the last few decades, the construction of exact solutions for some nonlinear equations has been perfprmeo by many researchers. The auxiliary equation method, the Hirota's bilinear method, the Jacobian elliptic function, the powerful inverse scattering transform, the Bäcklund transformations technique, the Painlevé analysis, the Lie group theoretical methods, the direct algebraic method, the tangent hyperbolic method, the variable separation method and the improved mapping approach have been used in the construction of exact solutions of some nonlinear equations.[1−15] Compared with other nonlinear equations, we find that the solutions of Euler equations not only depend on the prescribed initial condition but also rely on the equations of state. They are much more difficult to tackle and some high resolution numerical methods are needed, such as the finite element method, finite volume method, finite difference method and spectral method.[16−19] Most high resolution methods are more difficult to code and need much central processing unit time. Based on the general Lagrange multiplier method, the variational iteration method (VIM) was first envisioned by He[20,21] and has been successfully applied to many nonlinear equations.[22−30] Its advantages are easy to code and getting semi-analytical solutions. So in this paper, we extend the VIM to one and two-dimensional Euler equations and some approximate analytic solutions are obtained. Comparison with other numerical methods shows that the VIM is also one of the most efficient methods for studying some Euler equations.

2. The VIM

To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation:

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$$Lu + Nu = f(x), \tag{1}$$

where L is a linear operator, N is a nonlinear operator and $f(x)$ is an inhomogeneous term. According to the VIM, we can construct a correction functional as follows:

$$u_{n+1} = u_n + \int_0^x \lambda[Lu_n(\tau) + N\widetilde{u_n}(\tau) - f(x)]\,\mathrm{d}\tau, \quad (n \geq 0), \tag{2}$$

where λ is a general Lagrange multiplier which can be identified optimally via the variation theory, the subscript n denotes the nth approximation, $\widetilde{u_n}(\tau)$ is considered as a restricted variation, i.e. $\delta\widetilde{u_n}(\tau) = 0$. The iterative formula is

$$u_{n+1} = u_n + \int_0^x \lambda[Lu_n(\tau) + Nu_n(\tau) - f(x)]\,\mathrm{d}\tau, (n \geq 0). \tag{3}$$

It is shown that this method is very effective and reliable. It can solve a large class of nonlinear problems. For the linear problem, its exact solution can be obtained by only one iteration, because the Lagrange multiplier λ can be exactly identified. For thenonlinear problem, the exact solution is given by

$$u = \lim_{n\to\infty} u_n.$$

3. The VIM applied to compressible Euler equations

In this section, we describe the implementation of the VIM for one and two-dimensional Euler equations with initial conditions and periodic boundary conditions in detail.

3.1. One-dimensional case

Consider the one-dimensional Euler equations

$$u_t + f(u)_x = 0, \tag{4}$$

$$u(x, 0) = u_0(x), \tag{5}$$

with

$$u = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}, \quad f(u) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E + p) \end{pmatrix}, \tag{6}$$

here ρ is the density, v is the velocity, E is the total energy, p is the pressure, which is related to the total energy by $E = p/(\gamma - 1) + \rho v^2/2$ with $\gamma = 1.4$. To solve Eq. (4) with the initial condition Eq. (5), according to the VIM, the correction functionals in t-direction can be expressed as follows:

$$u_{n+1} = u_n + \int_0^t \lambda \left(\frac{\partial u_n(x, s)}{\partial s} + \frac{\partial f(\widetilde{u_n}(x, s))}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0). \tag{7}$$

Making the above correction functionals stationary, we obtain the following stationary conditions:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = 0, \quad 1 + \lambda|_{\tau=t} = 0. \tag{8}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda(\tau) = -1. \tag{9}$$

As a result, we obtain the following iteration formula:

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u_n(x, s)}{\partial s} + \frac{\partial f(u_n(x, s))}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0). \tag{10}$$

For compressible Euler equations, the above iterative formula can not be used directly because there are four variables but only three equations. To deal with this difficulty, we use the formula $E = p/(\gamma - 1) + \rho v^2/2$ to eliminate one variable. In the following, we denote

$$u = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} = \begin{pmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{pmatrix}. \tag{11}$$

Through careful calculation, we get the following iterative formulas:

$$u_{n+1}^{(1)} = u_n^{(1)} - \int_0^t \left(\frac{\partial u_n^{(1)}(x, s)}{\partial s} + \frac{\partial u_n^{(2)}(x, s)}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0), \tag{12}$$

$$u_{n+1}^{(2)} = u_n^{(2)} - \int_0^t \left(\frac{\partial u_n^{(2)}(x, s)}{\partial s} + \frac{\partial [(u_n^{(2)})^2/u_n^{(1)} + (u_n^{(3)} - 0.5(u_n^{(2)})^2/u_n^{(1)})(\gamma - 1)]}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0), \tag{13}$$

$$u_{n+1}^{(3)} = u_n^{(3)} - \int_0^t \left(\frac{\partial u_n^{(3)}(x, s)}{\partial s} + \frac{\partial [(u_n^{(2)})^2/u_n^{(1)}(u_n^{(3)} + (u_n^{(3)} - 0.5(u_n^{(2)})^2/u_n^{(1)}))(\gamma - 1))]}{\partial x} \right) \mathrm{d}s, \ (n \geq 0). \tag{14}$$

Considering the initial value conditions, we can obtain a sequence of approximate analytic solutions.

3.2. Two-dimensional case

Consider the two-dimensional Euler equations

$$u_t + f(u)_x + g(u)_y = 0, \tag{15}$$

$$u(x, y, 0) = u_0(x, y), \tag{16}$$

with

$$u = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad f(u) = \begin{pmatrix} \rho v \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix},$$

$$g(u) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}, \tag{17}$$

here ρ is the density, (u, v) is the velocity, E is the total energy, p is the pressure, which is related to the total energy by $E = p/(\gamma - 1) + \rho(u^2 + v^2)/2$ with $\gamma = 1.4$. To solve Eq. (15) with the initial condition Eq. (16), according to the VIM, we can express the correction functionals in t-direction expressed as follows

$$u_{n+1} = u_n + \int_0^t \lambda\left(\frac{\partial u_n(x,y,s)}{\partial s} + \frac{\partial f(\widetilde{u_n}(x,y,s))}{\partial x} + \frac{\partial g(\widetilde{u_n}(x,y,s))}{\partial y}\right) \mathrm{d}s, \quad (n \geq 0). \tag{18}$$

Making the above correction functionals stationary, we obtain the following stationary conditions:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = 0, \quad 1 + \lambda|_{\tau=t} = 0. \tag{19}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda(\tau) = -1. \tag{20}$$

As a result, we obtain the following iteration formula:

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u_n(x,y,s)}{\partial s} + \frac{\partial f(u_n(x,y,s))}{\partial x} + \frac{\partial g(u_n(x,y,s))}{\partial y}\right) \mathrm{d}s, \quad (n \geq 0). \tag{21}$$

For compressible Euler equations, the above iterative formula can not be used directly because there are five variables but only four equations. To deal with this difficulty, we use the formula $E = p/(\gamma - 1) + \rho(u^2 + v^2)/2$ to eliminate one variable. In the following, we denote

$$u = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} = \begin{pmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \\ u^{(4)} \end{pmatrix}. \tag{22}$$

Let

$$p_n = \left(u_n^{(4)} - \frac{1}{2}\left(\frac{(u_n^{(2)})^2 + (u_n^{(3)})^2}{u_n^{(1)}}\right)\right)(\gamma - 1). \tag{23}$$

Through careful calculation, we get the following iterative formulas:

$$u_{n+1}^{(1)} = u_n^{(1)} - \int_0^t \left(\frac{\partial u_n^{(1)}(x,y,s)}{\partial s} + \frac{\partial u_n^{(2)}(x,y,s)}{\partial x} + \frac{\partial u_n^{(3)}(x,y,s)}{\partial y}\right) \mathrm{d}s, \quad (n \geq 0), \tag{24}$$

$$u_{n+1}^{(2)} = u_n^{(2)} - \int_0^t \left(\frac{\partial u_n^{(2)}(x,y,s)}{\partial s} + \frac{\partial [(u_n^{(2)})^2/u_n^{(1)} + p_n]}{\partial x} + \frac{\partial [u_n^{(2)} u_n^{(3)}]}{\partial y}\right) \mathrm{d}s, \quad (n \geq 0), \tag{25}$$

$$u_{n+1}^{(3)} = u_n^{(3)} - \int_0^t \left(\frac{\partial u_n^{(3)}(x,y,s)}{\partial s} + \frac{\partial [(u_n^{(3)})^2/u_n^{(1)} + p_n]}{\partial y} + \frac{\partial [u_n^{(2)} u_n^{(3)}]}{\partial x}\right) \mathrm{d}s, \quad (n \geq 0), \tag{26}$$

$$u_{n+1}^{(4)} = u_n^{(4)} - \int_0^t \left(\frac{\partial u_n^{(4)}(x,y,s)}{\partial s} + \frac{\partial [u_n^{(2)}/u_n^{(1)}(u_n^{(4)} + p_n)]}{\partial x} + \frac{\partial [u_n^{(3)}/u_n^{(1)}(u_n^{(4)} + p_n)]}{\partial y}\right) \mathrm{d}s, \quad (n \geq 0). \tag{27}$$

Considering the initial value conditions, we can obtain a sequence of approximate analytic solutions.

4. Numerical results

To illustrate the method, some examples are considered here.

Example 1 Consider the one-dimensional Euler equations with initial conditions

$$\rho(x,0) = 1 + 0.2\sin(\pi x), \quad v(x,0) = 1, \quad p(x,0) = 1,$$

and periodic boundary conditions. The exact solutions are

$$\rho(x,t) = 1 + 0.2\sin(\pi(x-t)), \quad v(x,t) = 1, \quad p(x,t) = 1.$$

Beginning with

$$u = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \\ u_0^{(3)} \end{pmatrix} = \begin{pmatrix} 1 + 0.2\sin(\pi x) \\ 1 + 0.2\sin(\pi x) \\ \dfrac{1}{\gamma - 1} + \dfrac{1}{2}(1 + 0.2\sin(\pi x)) \end{pmatrix}, \tag{28}$$

we illustrate the first three approximations as follows:

$$\begin{aligned}
u_1^{(1)} &= 1 - 0.628319t\cos(\pi x) + 0.2\sin(\pi x), \\
u_1^{(2)} &= 1 - 0.628319t\cos(\pi x) + 0.2\sin(\pi x), \\
u_1^{(3)} &= 2.5 - 0.314159t\cos(\pi x) + 0.5(1 + 0.2\sin(\pi x)), \\
u_2^{(1)} &= 1 - 0.628319t\cos(\pi x) + 0.2\sin(\pi x) - 0.98696t^2\sin(\pi x), \\
u_2^{(2)} &= 1 - 0.628319t\cos(\pi x) + 0.2\sin(\pi x) - 0.98696t^2\sin(\pi x), \\
u_2^{(3)} &= 2.5 - 0.314159t\cos(\pi x) + 0.5(1 + 0.2\sin(\pi x)) - 0.49348t^2\sin(\pi x), \\
u_3^{(1)} &= 1 - 0.628319t\cos(\pi x) + 0.2\sin(\pi x) - 0.98696t^2\sin(\pi x) + 1.03354t^3\cos(\pi x), \\
u_3^{(2)} &= 1 - 0.628319t\cos(\pi x) + 0.2\sin(\pi x) - 0.98696t^2\sin(\pi x) + 1.03354t^3\cos(\pi x), \\
u_3^{(3)} &= 2.5 - 0.314159t\cos(\pi x) + 0.5(1 + 0.2\sin(\pi x)) - 0.49348t^2\sin(\pi x) + 0.516771t^3\cos(\pi x), \\
&\vdots
\end{aligned}$$

From the above iterations, we can easily calculate the wanted physical variables. The errors for ρ, v and p after 35th iterations are illustrated in Figs. 1–3. It is very clear that the approximate analytic solutions are in excellent agreements with the accurate solutions.

Example 2 Consider the two-dimensional Euler equations with initial conditions

$$\rho(x,y,0) = 1 + 0.2\sin(\pi(x+y)), \quad u(x,y,0) = 0.7, \quad v(x,y,0) = 0.3, \quad p(x,y,0) = 1,$$

and periodic boundary conditions. The exact solutions are

$$\rho(x,y,t) = 1 + 0.2\sin(\pi(x-y) - (u+v)t), \quad u(x,y,0) = 0.7, \quad v(x,y,t) = 0.3, \quad p(x,y,t) = 1.$$

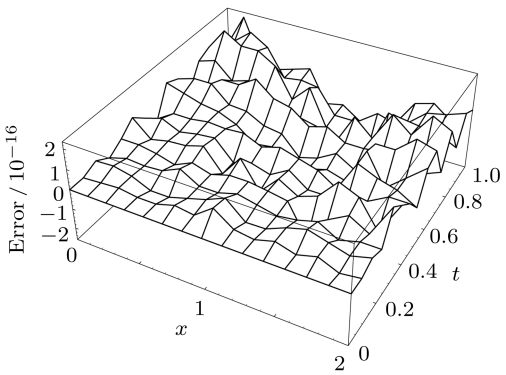


Fig. 1. Error for $\rho(x,t)$ obtained by using the VIM after 35th iterations.

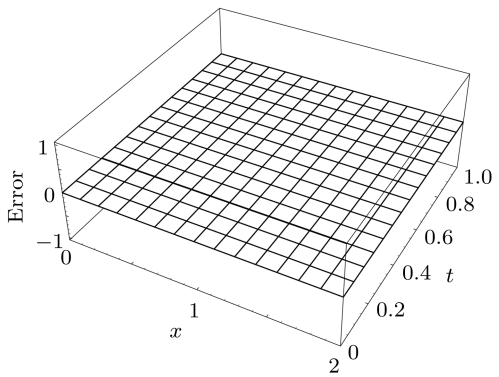


Fig. 2. Error for $v(x,t)$ obtained by using the VIM after 35th iterations.

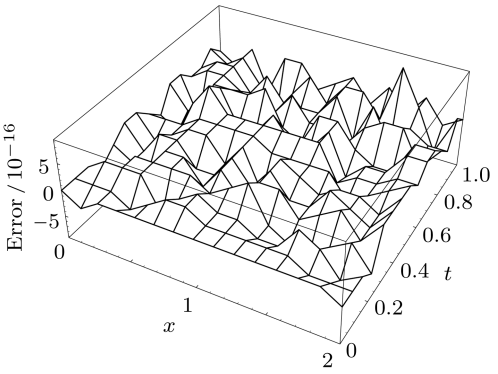


Fig. 3. Error for $p(x,t)$ obtained by using the VIM after 35th iterations.

Beginning with

$$u = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \\ u_0^{(3)} \\ u_0^{(4)} \end{pmatrix} = \begin{pmatrix} 1 + 0.2\sin(\pi(x+y)) \\ 0.7(1 + 0.2\sin(\pi(x+y))) \\ 0.3(1 + 0.2\sin(\pi(x+y))) \\ \dfrac{1}{\gamma - 1} + 0.29(1 + 0.2\sin(\pi(x+y))) \end{pmatrix}, \tag{29}$$

we illustrate some approximations as follows

$$\begin{aligned} u_1^{(1)} &= 1 - 0.628319t\cos(\pi(x+y)) + 0.2\sin(\pi(x+y)), \\ u_1^{(2)} &= -0.439823t\cos(\pi(x+y)) + 0.7[1 + 0.2\sin(\pi(x+y))], \\ u_1^{(3)} &= -0.188496t\cos(\pi(x+y)) + 0.3[1 + 0.2\sin(\pi(x+y))], \\ u_1^{(4)} &= 2.5 - 0.182212t\cos(\pi(x+y)) + 0.29[1 + 0.2\sin(\pi(x+y))], \\ &\vdots \end{aligned}$$

From the above iterations, we can easily calculate the wanted physical variables. The errors at time 0.1 for ρ, u, v and p after second iterations are illustrated in Figs. 4–7. It is very clear that the approximate analytic solutions are also in excellent agreement with the accurate solutions.

In order to compare the numerical results of the VIM with other numerical methods, we also do the numerical simulation by the conservative finite difference schemes. Here we use the first order Lax–Friedrichs scheme and the first order centred (FORCE) scheme. For the derivations of these two schemes, we refer to Ref. [31]. In all calculations, the computation domain is set to be $[0,2]$ and periodic boundary conditions

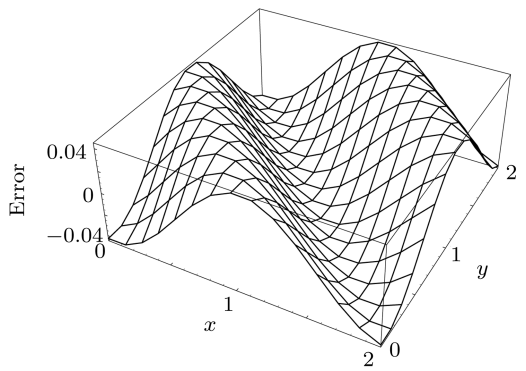


Fig. 4. Error for $\rho(x,y,t)$ obtained by using the VIM after second iterations at time 0.1.

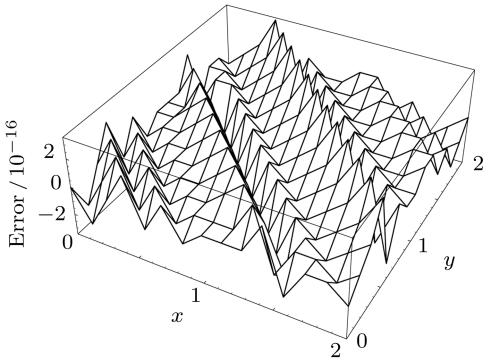


Fig. 5. Error for $u(x,y,t)$ obtained by using the VIM after second iterations at time 0.1.

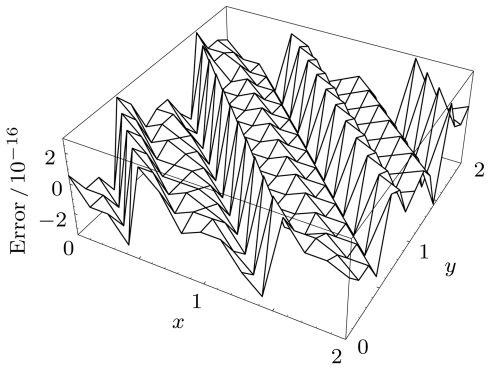


Fig. 6. Error for $v(x, y, t)$ obtained by using the VIM after second iterations at time 0.1.

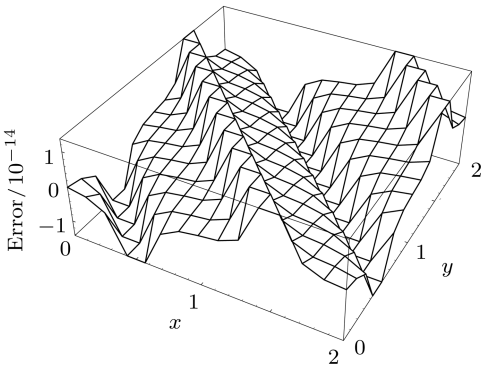


Fig. 7. Error for $p(x, y, t)$ obtained by using the VIM after second iterations at time 0.1.

are used. The computation domain is equally divided into 100 cells. Considering the time step restriction, we choose the Courant–Friedrichs–Lewy condition to be 0.9. We illustrate the numerical results at time 0.5 for density in Figs. 8 and 9. We can see that

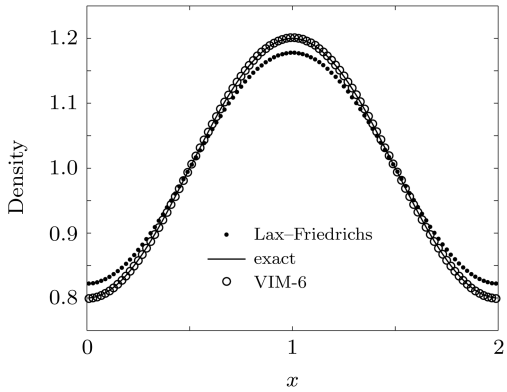


Fig. 8. Comparison between Lax–Friedrichs scheme and VIM after sixth iterations at time 0.5.

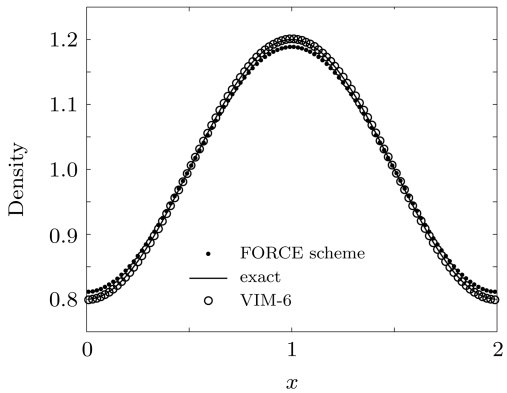


Fig. 9. Comparison between FORCE scheme and VIM after sixth iterations at time 0.5.

the numerical results using the VIM after sixth iteration are almost the same with the exact solutions. The numerical results obtained by the Lax–Friedrichs scheme and FORCE scheme are inferior to the VIM. Furthermore, by using the above two finite difference schemes we only get the numerical solution on some discrete points. It is more difficult to do some analytical manipulation than the numerical solutions by the VIM. When extending the above two schemes to two dimensional Euler equations, in general, a dimension splitting technique has to be used. It is more complex than the same extension of the VIM.

5. Conclusions

In this work, we have achieved an approximation for solutions of compressible Euler equations by applying the VIM. To the best of our knowledge, this is the first attempt to extend the use of the VIM to such equations. The numerical results show that the VIM is one of the most reliable and efficient methods in solving some compressible Euler equations. Compared with most methods used now, we find that the VIM is very reliable and easy to code. We believe that the VIM may be widely used to find the approximate analytic solutions of other nonlinear equations.

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