The super-classical-Boussinesq hierarchy and its super-Hamiltonian structure*

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Based on the constructed Lie superalgebra, the super-classical-Boussinesq hierarchy is obtained. Then, its super-Hamiltonian structure is obtained by making use of super-trace identity. Furthermore, the super-classical-Boussinesq hierarchy is also integrable in the sense of Liouville.

Keywords: Lie superalgebra, super-trace identity, super-integrable system, super-Hamiltonian structure

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1. Introduction

It has been an important and interesting topic in soliton theory to search for new Lax integrable or Liouville integrable systems, as many as possible, such that they are associated with certain evolution equations with physical meaning. The simple and efficient method to obtain continuous or discrete integrable systems was proposed by Tu.[1,2] Ma[3] developed it and called it the Tu scheme. By taking advantage of it, a family of integrable systems and their integrable couplings have been obtained.[4−13] For example, a basis for the Lie algebra A_1 was presented as

$$e_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad e_2=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad e_3=\begin{pmatrix}0&0\\1&0\end{pmatrix}. \tag{1}$$

The basis for loop algebra $\tilde{A}_1$ is $\{e_1(n),e_2(n),e_3(n)|n\in\mathbb{Z}\}$ with $x(n)=x\otimes\lambda^n, x\in A_1$. By making use of the Tu scheme, we can obtain the isospectral problem of the classical-Boussinesq hierarchy[14]

$$\begin{aligned}\varphi_x &= U\varphi,\\ U &= -e_1(1)-\frac{1}{4}qe_1(0)+re_2(0)-e_3(0)\\ &= \begin{pmatrix}-\lambda-\frac{1}{4}q & r\\ -1 & \lambda+\frac{1}{4}q\end{pmatrix},\\ \lambda_t &= 0.\end{aligned} \tag{2}$$

Taking $V^{(n)}=\sum_{m=0}^{n}(a_me_1(n-m)+b_me_2(n-m)+c_me_3(n-m))-c_{n+1}e_1(0)$, it admits the classical-Boussinesq hierarchy

$$\begin{aligned}u_t &= \begin{pmatrix}q\\r\end{pmatrix}_t = J\begin{pmatrix}-\frac{1}{2}a_{n+1}\\c_{n+1}\end{pmatrix}\\ &= JL^n\begin{pmatrix}0\\\alpha\end{pmatrix}=J\frac{\delta H_n}{\delta u},\end{aligned} \tag{3}$$

where

$$\begin{aligned}J &= \begin{pmatrix}0&4\partial\\4\partial&0\end{pmatrix},\\ L &= \begin{pmatrix}-\frac{1}{2}\partial-\frac{1}{4}\partial^{-1}q\partial & -\frac{1}{4}r-\frac{1}{4}\partial^{-1}r\partial\\ -2 & \frac{1}{2}\partial-\frac{1}{4}q\end{pmatrix},\\ H_n &= \int\frac{2a_{n+2}}{n+1}\mathrm{d}x, n\geq 0.\end{aligned}$$

With the development of soliton theory, super-integrable systems and their super-Hamiltonian structures have been receiving growing attention.[15−20] In 1990, Hu[15] proposed the super-trace identity first. Then the super-trace identity was mentioned and applied to establish the super-Hamiltonian structures of super-integrable systems.[16,17] Recently, Ma *et al.*[18] gave a systematic proof of super-trace identity and the expression of its constant γ. Meanwhile, he presented

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the super-Hamiltonian structures of super-AKNS hierarchy and super-Dirac hierarchy for application. The main ideas in Ref. [18] are as follows:

Let $\mathcal{A}$ be a commutative superalgebra over $\mathbb{R}$ or $\mathbb{C}$, and G be a matrix loop superalgebra over $\mathcal{A}$ with the nondegenerate killing form. Based on G, we consider the following isospectral problem

$$\begin{aligned}&\varphi_x = U(u,\lambda)\varphi,\\ &\varphi_t = V(u,\lambda)\varphi, \lambda_t = 0,\end{aligned} \tag{4}$$

where $u = (u_1, u_2, \ldots, u_q) \in \mathcal{A}^q$ is a potential consisting of commuting and anticommuting variables, λ is a spectral parameter. The compatibility of Eq. (4) gives the zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{5}$$

where $[U, V] = UV - VU$. If an equation

$$u_t = K(u) \tag{6}$$

can be worked out through Eq. (5), we call Eq. (6) a super-evolution equation. If there is a super-Hamiltonian operator J and a function $\mathcal{H}$ such that

$$u_t = K(u) = J\frac{\delta \mathcal{H}}{\delta u}, \tag{7}$$

then Eq. (6) is called a super-Hamiltonian equation. If so, we say that Eq. (6) has a super-Hamiltonian structure.

In this paper, we would like to consider the super-classical-Boussinesq hierarchy and its super-Hamiltonian structure, as there are some special techniques to deal with.

2. The super-classical-Boussinesq hierarchy

We consider the following Lie superalgebra $G = \{\sum_{i=1}^{5} \lambda_i e_i \big| \lambda_i \in \mathcal{A}, i = 1, 2, 3, 4, 5.\}$

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}&[e_2, e_1] = [e_4, e_4]_+ = -2e_2, \quad [e_3, e_1] = [e_5, e_5]_+ = 2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = [e_2, e_5] = e_4,\\ &[e_1, e_5] = [e_4, e_3] = -e_5, \quad [e_2, e_4] = [e_3, e_5] = 0, \quad [e_4, e_5]_+ = [e_5, e_4]_+ = e_1,\end{aligned} \tag{8}$$

where e_1, e_2, e_3 are even and e_4, e_5 are odd,[21] $[\cdot, \cdot]$ and $[\cdot, \cdot]_+$ denote the commutator and the anticommutator. The corresponding loop Lie superalgebra $\tilde{G}$ is given as follows:

$$\begin{aligned}&e_i(n) = e_i \otimes \lambda^n, i = 1, 2, 3, 4, 5;\\ &[e_1(m), e_2(n)] = 2e_2(m+n), [e_4(m), e_4(n)]_+ = -2e_2(m+n),\\ &[e_3(m), e_1(n)] = 2e_3(m+n), [e_5(m), e_5(n)]_+ = 2e_3(m+n),\\ &[e_2(m), e_3(n)] = e_1(m+n), [e_1(m), e_4(n)] = e_4(m+n),\\ &[e_2(m), e_5(n)] = e_4(m+n), [e_1(m), e_5(n)] = -e_5(m+n),\\ &[e_3(m), e_4(n)] = e_5(m+n), [e_2(m), e_4(n)] = [e_3(m), e_5(n)] = 0,\\ &[e_4(m), e_5(n)]_+ = [e_5(m), e_4(n)]_+ = e_1(m+n), m, n \in \mathbb{Z}.\end{aligned} \tag{9}$$

Consider the super-isospectral problem:

$$\varphi_x = U\varphi, U = -e_1(1) - \frac{1}{4}qe_1(0) + re_2(0) - e_3(0) + u_1e_4(0) + u_2e_5(0), \lambda_t = 0. \tag{10}$$

Denoting

$$V = \sum_{m\geq 0} a_m e_1(-m) + b_m e_2(-m) + c_m e_3(-m) + d_m e_4(-m) + f_m e_5(-m) = ae_1 + be_2 + ce_3 + de_4 + fe_5,$$

solving the stationary zero curvature equation

$$V_x = [U, V], \tag{11}$$

we have

$$\begin{aligned}
&a_{mx} = b_m + rc_m + u_2 d_m + u_1 f_m,\\
&b_{mx} = -2b_{m+1} - 2ra_m - \frac{1}{2}qb_m - 2u_1 d_m,\\
&c_{mx} = 2c_{m+1} - 2a_m + \frac{1}{2}qc_m + 2u_2 f_m,\\
&d_{mx} = -d_{m+1} - u_1 a_m - u_2 b_m - \frac{1}{4}qd_m + rf_m,\\
&f_{mx} = f_{m+1} + u_2 a_m - u_1 c_m - d_m + \frac{1}{4}qf_m,\\
&b_0 = c_0 = d_0 = f_0 = 0, a_0 = \alpha = \text{constant} \neq 0,\\
&a_1 = -2\partial^{-1}u_1 u_2, b_1 = -\alpha r, c_1 = \alpha, d_1 = -\alpha u_1, f_1 = -\alpha u_2.
\end{aligned} \tag{12}$$

From Eqs. (12), we can obtain the following recursion relations for $-\frac{1}{2}a_{m+1}, c_{m+1}, -2f_{m+1}$ and $2d_{m+1}$,

$$\begin{pmatrix} -\frac{1}{2}a_{m+1} \\ c_{m+1} \\ -2f_{m+1} \\ 2d_{m+1} \end{pmatrix} = L \begin{pmatrix} -\frac{1}{2}a_m \\ c_m \\ -2f_m \\ 2d_m \end{pmatrix} \tag{13}$$

with

$$L = \begin{pmatrix} -\frac{1}{2}\partial - \frac{1}{4}\partial^{-1}q\partial - 2\partial^{-1}u_1u_2 - \partial^{-1}u_2^2\partial & L_{12} & L_{13} & L_{14} \\ -2 & \frac{1}{2}\partial - \frac{1}{4}q & \frac{1}{2}u_2 & 0 \\ -4u_2 & -2u_1 & \partial - \frac{1}{4}q & -1 \\ 4u_1 + 4u_2\partial & 2ru_2 & -r - u_1u_2 & -\partial - \frac{1}{4}q + u_2^2 \end{pmatrix}, \tag{14}$$

where $L_{12} = -\frac{1}{4}r - \frac{1}{4}\partial^{-1}r\partial - \frac{1}{2}\partial^{-1}(ru_2^2 + u_1^2)$, $L_{13} = \frac{1}{8}u_1 + \frac{1}{4}\partial^{-1}u_1\partial + \frac{1}{4}\partial^{-1}u_1u_2^2$, $L_{14} = -\frac{1}{8}u_2 - \frac{1}{4}\partial^{-1}u_2^3 + \frac{1}{4}\partial^{-1}u_2\partial$.

Denoting

$$V_+^{(n)} = \sum_{m=0}^{n} a_m e_1(n-m) + b_m e_2(n-m) + c_m e_3(n-m) + d_m e_4(n-m) + f_m e_5(n-m), V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

a direct calculation reads

$$-V_{+x}^{(n)} + [U, V_+^n] = 2b_{n+1}e_2(0) - 2c_{n+1}e_3(0) + d_{n+1}e_4(0) - f_{n+1}e_5(0). \tag{15}$$

Considering

$$V^{(n)} = V_+^{(n)} + \Delta_n, \Delta_n = -c_{n+1}e_1(0), \tag{16}$$

from the zero curvature equation $U_t - V_x^{(n)} + [U, V^{(n)}] = 0$, we obtain a new super-integrable system

$$U_t - V_{+x}^{(n)} + [U, V_+^{(n)}] - \Delta_{nx} + [U, \Delta_n] = 0. \tag{17}$$

Substituting Eq. (16) into Eq. (17), we have the following super-integrable system

$$u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 0 & 4\partial & 0 & 0 \\ 4\partial & 0 & -u_1 & u_2 \\ 0 & -u_1 & 0 & -\frac{1}{2} \\ 0 & u_2 & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}a_{n+1} \\ c_{n+1} \\ -2f_{n+1} \\ 2d_{n+1} \end{pmatrix} = J \begin{pmatrix} -\frac{1}{2}a_{n+1} \\ c_{n+1} \\ -2f_{n+1} \\ 2d_{n+1} \end{pmatrix} = JP_{n+1}, \tag{18}$$

where P_{n+1} meets the following: $P_{n+1} = LP_n$. If taking $u_1 = u_2 = 0$ in system (18), it can be reduced to the classical-Boussinesq hierarchy (3). So we call system (18) the super-classical-Boussinesq hierarchy.

3. The super-Hamiltonian structure of the super-classical-Boussinesq hierarchy

Let a spectral matrix $\boldsymbol{U}$ be defined by

$$\begin{aligned}\boldsymbol{U} &= \boldsymbol{U}(u,\lambda)\\ &= e_0(\lambda)+u_1e_1(\lambda)+\ldots+u_qe_q(\lambda),\\ &\quad u_i\in\mathcal{A},1\le i\le q,\end{aligned}\tag{19}$$

where $\mathcal{A}$ is a commutative superalgebra over $\mathbb{R}$ or $\mathbb{C}$, G is a matrix loop superalgebra over $\mathcal{A}$ with the nondegenerate Killing form, and $e_i\in G, 1\le i\le q$, are $\mathcal{A}$ linearly independent. We define $\mathrm{rank}(U)=\mathrm{rank}(\partial/\partial x)=$ constant. If two solutions $V_1,V_2\in G$ of the stationary zero curvature equation $V_x=[U,V]$ possess the same rank, then they are $\mathcal{A}$ linearly dependent on each other: $V_1=\gamma V_2,\gamma=$ constant. Thus we have the following two theorems:

Theorem 1 (The super-trace identity)[18] Let $U=U(u,\lambda)\in G$ be homogeneous in rank. Assume that the stationary zero curvature equation has a unique solution $V\in G$ of a fixed rank up to a constant multiplier. Then, there is a constant γ such that

$$\begin{aligned}&\frac{\delta}{\delta u}\int \mathrm{str}(\mathrm{ad}_V\mathrm{ad}_{U_\lambda})\mathrm{d}x\\ &=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_{\partial U/\partial u})\end{aligned}\tag{20}$$

holds for any solution $V\in G$ of stationary zero curvature equation, being homogeneous in rank.

Theorem 2[18] Let V be a solution to the stationary zero curvature equation. Then the constant in the super-trace identity is given by

$$\gamma=-\frac{\lambda}{2}\ln\big|\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_V)\big|,\tag{21}$$

if $\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_V)\neq 0$.

Based on Lie superalgebra G in Eqs. (8) and associated corresponding loop superalgebra $\tilde{G}$, a direct calculation gives

$$\mathrm{ad_a}=\begin{pmatrix}0&-a_3&a_2&a_5&a_4\\ -2a_2&2a_1&0&-2a_4&0\\ 2a_3&0&-2a_1&0&2a_5\\ -a_4&-a_5&0&a_1&a_2\\ a_5&0&-a_4&a_3&-a_1\end{pmatrix}\tag{22}$$

for $a=a_1e_1+a_2e_2+a_3e_3+a_4e_4+a_5e_5\in\tilde{G}$, where $\mathrm{ad}_ab=[a,b],a,b\in\tilde{G}$, the bracket $[\cdot,\cdot]$ is the Lie superbracket of $\tilde{G}$. If we define the compact super-trace as follows:

$$\begin{aligned}&\mathrm{str}(c)=c_{11}+c_{22}-c_{33},c=ab,a,b\in\tilde{G};\\ &\mathrm{str}(P)=p_{11}+p_{22}+p_{33}-p_{44}-p_{55},\end{aligned}\tag{23}$$

where $c=(c_{ij})_{3\times3},P=(p_{ij})_{5\times5}$ and ab is the matrix product of a and b, then we have

$$\mathrm{str}(\mathrm{ad}_a\mathrm{ad}_b)=3\mathrm{str}(ab).\tag{24}$$

It is easy to compute that

$$\begin{aligned}&\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_{U_\lambda})=-6a,\\ &\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_{\partial U/\partial q})=-\frac{3}{2}a,\\ &\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_{\partial U/\partial r})=3c,\\ &\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_{\partial U/\partial u_1})=-6f,\\ &\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_{\partial U/\partial u_2})=6d.\end{aligned}\tag{25}$$

By use of the super-trace identity (20), we have

$$\begin{aligned}&\frac{\delta}{\delta u}\int(-6a)\mathrm{d}x\\ &=\lambda^{-\gamma}\frac{\delta}{\delta\lambda}\lambda^{\gamma}\left(-\frac{3}{2}a,3c,-6f,6d\right)^{\mathrm{T}}.\end{aligned}\tag{26}$$

Comparing the coefficient of λ^{-n-1} yields

$$\begin{aligned}&\frac{\delta}{\delta u}\int(-6a_{n+1})\mathrm{d}x\\ &=(\gamma-n)\left(-\frac{3}{2}a,3c_n,-6f_n,6d_n\right)^{\mathrm{T}}.\end{aligned}\tag{27}$$

Since $\mathrm{str}(\mathrm{ad}_V\mathrm{ad}_V)=6\alpha^2=$ constant $\neq 0$, by using the computation formula (21), we obtain $\gamma=0$. Therefore, we conclude that

$$P_{n+1}=\frac{\delta\mathcal{H}_n}{\delta u},\quad \mathcal{H}_n=\int\frac{2a_{n+2}}{n+1}\mathrm{d}x.\tag{28}$$

Hence, the super-classical-Boussinesq hierarchy (18) has the following super-Hamiltonian structure

$$u_t=JP_{n+1}=J\frac{\delta\mathcal{H}_n}{\delta u},n\ge0.\tag{29}$$

Specially, the super-classical-Boussinesq hierarchy (18) has the following super-bi-Hamiltonian structure

$$u_t=JP_{n+1}=J\frac{\delta\mathcal{H}_n}{\delta u}=M\frac{\delta\mathcal{H}_{n-1}}{\delta u},n\ge1,\tag{30}$$

where the second compatible super-Hamiltonian operator M reads

$$M = JL = \begin{pmatrix} -8\partial & 2\partial^2 - \partial q & 2\partial u_2 & 0 \\ -2\partial^2 - q\partial & -\partial r - r\partial & \frac{1}{2}\partial u_1 + \frac{1}{4}qu_1 - ru_2 & -\frac{1}{2}\partial u_2 + u_1 - \frac{1}{4}qu_2 \\ -2u_2\partial & -\frac{1}{2}u_1\partial + \frac{1}{4}qu_1 - ru_2 & \frac{1}{2}r & \frac{1}{2}\partial + \frac{1}{8}q - \frac{1}{2}u_2^2 \\ 0 & \frac{1}{2}u_2\partial + u_1 - \frac{1}{4}qu_2 & -\frac{1}{2}\partial + \frac{1}{8}q + \frac{1}{2}u_2^2 & \frac{1}{2} \end{pmatrix}.$$

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