Multi-agent coordination in directed moving neighbourhood random networks

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This paper considers the consensus problem of dynamical multiple agents that communicate via a directed moving neighbourhood random network. Each agent performs random walk on a weighted directed network. Agents interact with each other through random unidirectional information flow when they coincide in the underlying network at a given instant. For such a framework, we present sufficient conditions for almost sure asymptotic consensus. Numerical examples are taken to show the effectiveness of the obtained results.

Keywords: multi-agent system, consensus problem, stochastic stability
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1. Introduction

Synchronisation and consensus problems in multi-agent systems have a long history.[1] Over the last few years, considerable research attention has been paid to cooperative group coordination, which has broad applications in a variety of areas including vehicle formation, distributed sensor networks, coupled oscillator systems, animal aggregation and epidemics.[2−5] In a consensus problem, a group of autonomous agents seeks to agree on some quantities of interest via a process of making a distributed decision based upon local information interactions. A consensus is reached when all agents in the system tend to attain agreement on the quantities of interest as time evolves, that is, they converge to a common value.

The ability to achieve coherent behaviour (or consensus) of the multi-agent system is intimately related with the coupling topology of the underlying communication graph associated with the agents' positions. For example, Jadbabaie *et al.*[6] analysed a simplified model of flocking by Vicsek *et al.*[7] showing that all agents will synchronise eventually, provided that the communication graph switching deterministically over time is periodically jointly connected. More recently, there has been some interest in stochastic consensus problems. Reference [8] deals with the agreement problem among a group of static agents with the communication graph presented as the Erdös–Rényi random graph. The results are further extended by Ref. [9] to include weighted and directed information flows. In Ref. [10], the authors who reveal this problem can be reduced to weak ergodicity of a sequence of stochastic matrices. However, the agents are static in these works, in other words, the switching of communication graph is merely due to the adjunctive random mechanism rather than the motion of agents.

The moving neighbourhood network proposed in Ref. [11] is meant to treat stochastic consensus among dynamical agents (in social interaction and epidemics originally), where each agent carries an oscillator and diffuses in the environment. The computer simulation in the aforementioned paper shows that synchronisation is possible even when the communication network is spatially disconnected in general at any frozen time instant. This is in sharp contrast with the behaviour of the Vicsek model mentioned above. Analytical formalisations are provided by authors of Refs. [12]–[14] with different consensus protocols (continuous-time with dwelt time or discrete-time) through fast switching techniques[12] and stochastic stability theory.[15] In particular, they specify the moving neighbourhood network model as a group of n identical agents implementing simple random walks on a fixed finite connected graph. The vertices of the moving neighbourhood network are represented by the agents, and the edges are determined by their locations in the underlying graph, i.e. a link between two agents appears if and only if they reside in the same node simultaneously. The time evolution of the moving neighbourhood graph governed by the random walk of agents is called network dynamics, whereas the time evolution

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of the corresponding oscillator system is called system dynamics, which is coupled together with the network dynamics.

The purpose of this note is to generalise the above results and go a step further in the direction of Ref. [14]. Our extension is threefold. First, we consider the underlying fixed graph as a weighted directed graph. In other words, random walks on the weighted directed graph are tackled. Secondly, further randomness is involved in the moving neighbourhood graph. A probability is associated (not necessarily independent) to the link between two agents when they occupy the same node, which can be interpreted as communication links and may be unreliable due to disturbance. We allow different probabilities and unidirectional communications. Therefore the topology of the moving neighbourhood graph is no longer strictly separated clique-like. Finally, we study a discrete-time consensus protocol with inhomogeneous weights, which quantifies the way by which the agents influence each other. By these modifications, some pertinent known results can be recovered (see the remarks in Sections 2 and 4), and a number of introduced parameters enhance more flexibility.

The rest of this paper is organised as follows. In Section 2, we elaborate on our formal model under consideration. Section 3 contains some mathematical preliminaries and the statement of main result. Proofs are given in Section 4. Numerical simulation is presented in Section 5 and the conclusion drawn from the present study is given in Section 6.

2. Problem setup

Let $\bar{G} = (\bar{V}, \bar{E}, \boldsymbol{W})$ be a weighted digraph with vertex set $\bar{V} = [m]$, here $[m] := \{1, 2, \ldots, m\}$. The elements in $\bar{V}$ are referred to as nodes. The $\bar{E}$ is a set of ordered pairs of elements of $\bar{V}$. The $\boldsymbol{W} = (w_{ij})$ is the weight matrix, where $w_{ij} > 0$ if $(i, j) \in \bar{E}$, and $w_{ij} = 0$ otherwise. Notice that we do not exclude loops. We impose the following condition on the graph $\bar{G}$.

Assumption 1 The underlying graph $\bar{G}$ is strongly connected and the GCD (greatest common divisor) of all cycle lengths in $\bar{G}$ is 1.

This assumption will be elucidated in the next section. Let $t \in \mathbb{N} \cup \{0\}$ be the discrete time step. We consider n identical agents signified by $\{v_1, v_2, \ldots, v_n\}$ as random walkers meandering on $\bar{G}$, moving randomly to a neighbour of their current location in $\bar{G}$ at each time step. For each agent, the random chosen neighbour is not affected by the agent's previous trajectory. The n random walk processes are mutually independent. The time-varying moving neighbourhood graph, $G = (V, E, \boldsymbol{P})$, is constructed as follows. Let $V = V(t) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set. The $\boldsymbol{P} = (p_{ij})$ is an $n \times n$ matrix with entries $0 < p_{ij} \leq 1$ for $i \neq j$, and $p_{ii} = 0$ for $1 \leq i \leq n$. The $\boldsymbol{P}$ is dubbed as the linkage probability matrix. Given v_i and v_j met at the same node simultaneously, an arc (v_i, v_j) originating from v_i and terminating in v_j appears with probability p_{ij}. The arc set is denoted by $E = E(t)$. We emphasise here that the selections of arcs are independent with the random walk processes, but are not required to be independent with each other. By definition, $G = G(t)$ has no loops with probability 1.

Let $X_i(t) \in \mathbb{R}$ be the state (attitude, heading, opinion etc.) of agent v_i at time t. The consensus protocol can be expressed as

$$X_i(t+1) = X_i(t) + \varepsilon \sum_{j \in N_i(t)} b_{ij}(X_j(t) - X_i(t)), \quad (1)$$

where $\varepsilon > 0$ and $N_i(t)$ is the indexical set of neighbours of agent v_i at instant t, i.e. $N_i(t) := \{j | (v_i, v_j) \in E(t)\}$. Here, weighting factor $b_{ij} > 0$ for $i \neq j$, and $b_{ii} = 0$ for $1 \leq i \leq n$. Let $\boldsymbol{A}(t) = (a_{ij}(t))$ be the adjacency matrix of the moving neighbourhood graph $G(t)$, whose entries are given by

$$a_{ij}(t) = \begin{cases} b_{ij}, & (v_i, v_j) \in E(t) \\ 0, & \text{otherwise} \end{cases}$$

for $1 \leq i, j \leq n$.

Assumption 2 Suppose that $\sum_{j=1}^{n} a_{ij}(t) = \sum_{j=1}^{n} a_{ji}(t)$, for $1 \leq i \leq n$.

The implication of Assumption 2 is that the out-degrees are equal to in-degrees for every $v \in V(t)$. A similar definition used in Ref. [13] is referred to as 'balanced' digraph. Note that in the context of Ref. [13], the underlying communication network is static, while in the current case, we dictate that each trajectory of $G(t)$ should be 'balanced'. However, this requirement is not very stringent after all. For example, if we postulate that $G(t)$ can only have bidirectional (or undirected) edges, that is, $b_{ij} = b_{ji}$ and $p_{ij} = p_{ji}$ for $1 \leq i, j \leq n$, then Assumption 2 is satisfied automatically. Also, if $b_{ij} \equiv b$ for $i \neq j$ and suppose for each $v \in V(t)$, the numbers of arcs going out and coming in are equal, then Assumption 2 is satisfied. It is worth noting that since the selections of arcs can be arbitrarily dependent with each other as mentioned before, the latter of the two examples above can actually occur.

Next, let $\Delta := \max_{1\le i\le n}(\sum_{j=1}^{n} b_{ij})$, and we further assume $\varepsilon \in (0, 1/\Delta)$. This assumption is natural; see Section 4.

As usual, denote the linear subspace $\mathcal{M} := \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n |\ x_i = x_j\ \forall\ 1 \le i, j \le n\}$ as the synchronisation manifold. A consensus is said to be reached if the states of agents converge into $\mathcal{M}$ as t approaches infinity. We aim at showing that the n agents in V reach consensus almost surely.

We end this section with two remarks.

Remark 1 Suppose that the linkage probabilities $p_{ij}(t)$ and the weighting factors $b_{ij}(t)$ are time-dependent, then the main result in this paper holds verbatim as long as $p_{ij}(t)$ and $b_{ij}(t)$ converge.

Remark 2 If we take the underlying graph $\bar{G}$ to be a single node, then our framework somewhat reduces to that of Refs. [8]–[10].

3. Preliminaries and main results

Let $\{Y_i(t), t \in \mathbb{N} \cup 0\}$ denote the random walk process performed by agent v_i such that $Y_i(t) \in \bar{V}$ designates the position of v_i in the underlying graph $\bar{G}$ at instant t. For $1 \le i \le n$, let $\pi_i(t) = (\pi_{i1}(t), \ldots, \pi_{im}(t))^{\mathrm{T}}$ be the probability distribution of $Y_i(t)$ at time t, that is, $\mathrm{P}(Y_i(t) = j) = \pi_{ij}(t)$ for $j \in \bar{V}$. By the construction, $\{Y_i(t), 1 \le i \le n\}$ are n independent homogeneous Markov chains with finite state space $\bar{V}$, sharing the same transition probability matrix $\boldsymbol{Q} = (q_{ij})$, whose entry is given by

$$q_{ij} = \frac{w_{ij}}{d_i}, \qquad \text{for} \quad 1 \le i, j \le m, \tag{2}$$

where $d_i = \sum_{j=1}^{m} w_{ij}$ represents the out-degree of node i. Also recall that $q_{ij} = \mathrm{P}(Y_1(t+1) = j |\ Y_1(t) = i)$. Notice that by Assumption 1 $d_i \ne 0$, and Eq. (2) is well-defined.

As is known, a finite Markov chain converges to a unique stationary distribution if it is ergodic, i.e. irreducible and aperiodic (see e.g. Theorem 4.2 in Ref. [16]). It is clear that the transition probability matrix $\boldsymbol{Q}$ is irreducible if and only if $\bar{G}$ is strongly connected. Also, it is easy to see that the GCD of all cycle lengths in $\bar{G}$ is 1 if and only if all eigenvalues of $\boldsymbol{Q}$ other than 1 have modules strictly less than 1. Actually, the above assertions may be proven by employing Perron–Frobenius theorem[16] and some other equivalent conditions in terms of the ergodicity of random walk are given in Ref. [17]. Consequently, in view of Assumption 1, $Y_i(t)$ converges to a unique stationary distribution $\pi = (\pi_1, \ldots, \pi_m)^{\mathrm{T}}$ for $1 \le i \le n$, that is, $\pi^{\mathrm{T}} = \pi^{\mathrm{T}}\boldsymbol{Q}$. For instance, if $\sum_{j=1}^{m} w_{ij} = \sum_{j=1}^{m} w_{ji}$ for $1 \le i \le m$, then we have $\pi_i = d_i / \sum_{i,j=1}^{m} w_{ij}$ for $1 \le i \le m$, see e.g. Ref. [18]. As for the rate of convergence, we have the following lemma (see Ref. [19] Theorem 8.9 for a proof).

Lemma 1[19] For a finite ergodic Markov chain with transition probability matrix $\boldsymbol{P}$, let $p_{ji}^{(t)}$ be the (j, i) entry of $\boldsymbol{P}^t$, $t \in \mathbb{N} \cup 0$, then there is a unique stationary distribution $\{p_i\}$ and

$$|p_{ji}^{(t)} - p_i| \le \lambda\rho^t,$$

where $\lambda \ge 0$, $0 \le \rho < 1$.

Next, let $\boldsymbol{D}(t) = \mathrm{diag}(d_{11}(t), \ldots, d_{nn}(t))$ be the out-degree diagonal matrix of the moving neighbourhood graph $G(t)$. To be precise, $d_{ii}(t) = \sum_{j=1}^{n} a_{ij}(t)$, for $1 \le i \le n$, where $a_{ij}(t)$ are the entries of $\boldsymbol{A}(t)$ defined in Section 2. Notice that $\boldsymbol{A}(t)$ and $\boldsymbol{D}(t)$ are non-negative time-dependent random matrices. Since $a_{ij}(t)$ is a two-point distribution, its expectation is shown to be given by

$$\mathrm{E}(a_{ij}(t)) = b_{ij}p_{ij}\sum_{k=1}^{m} \pi_{ik}(t)\pi_{jk}(t), \quad \text{for} \quad i \ne j, \tag{3}$$

and $a_{ii}(t) = 0$. The diagonal entries of $\boldsymbol{D}(t)$ can be written as

$$d_{ii}(t) = \sum_{k=1}^{m}\sum_{j=1, j\ne i}^{n} b_{ij}1_{[Y_i(t)=k,\ Y_j(t)=k,\ (v_i,v_j)\in E(t)]}.$$

Thereby, $\mathrm{E}(d_{ii}(t)) = \sum_{k=1}^{m}\sum_{j=1}^{n} b_{ij}p_{ij}\pi_{ik}(t)\pi_{jk}(t)$, for $1 \le i \le n$. Now let $\boldsymbol{L}(t) = \boldsymbol{D}(t) - \boldsymbol{A}(t)$ denote the Laplacian matrix[17] of $G(t)$, and its expectation at time t is $\mathrm{E}\boldsymbol{L}(t) = \mathrm{E}\boldsymbol{D}(t) - \mathrm{E}\boldsymbol{A}(t)$.

For a sequence of random elements $Z(t)$, the ergodic limit, $\mathrm{E}^*(Z)$, is defined by $\mathrm{E}^*(Z) = \lim_{t\to\infty}\mathrm{E}(Z(t))$, whenever the limit exists. The $\mathrm{E}^*(Z)$ describes the long-run average with respect to its stationary distribution.[20] By the above discussion, the ergodic limit of Eq. (3) is $\mathrm{E}^*(a_{ij}) = b_{ij}p_{ij}\sum_{k=1}^{m}\pi_k^2$ for $i \ne j$ and $\mathrm{E}^*(a_{ii}) = 0$ for $1 \le i \le n$. Likewise, $\mathrm{E}^*(d_{ii}) = \sum_{k=1}^{m}\pi_k^2\sum_{j=1}^{n} b_{ij}p_{ij}$ for $1 \le i \le n$.

Now we make a reference to the Schur product of two real matrices, which is also known as Hadamard product.[21] Let $\boldsymbol{C} = (c_{ij})$, $\boldsymbol{E} = (e_{ij})$ be two $n \times m$ real matrices, then the Schur product, $\boldsymbol{C} \circ \boldsymbol{E} = (c_{ij}e_{ij})$, is simply the product of corresponding entries of $\boldsymbol{C}$ and $\boldsymbol{E}$. Obviously, $\boldsymbol{C} \circ \boldsymbol{E} = \boldsymbol{E} \circ \boldsymbol{C}$. Utilising this notation, we obtain $\mathrm{E}^*(\boldsymbol{A}) = \pi^{\mathrm{T}}\pi\boldsymbol{B} \circ \boldsymbol{P}$ and $\mathrm{E}^*(\boldsymbol{D}) = \pi^{\mathrm{T}}\pi\mathrm{diag}(\sum_{j=1}^{n} b_{1j}p_{1j}, \ldots, \sum_{j=1}^{n} b_{nj}p_{nj})$. Hereby, we have

$$\mathrm{E}^*(\boldsymbol{L})$$

$$= \pi^{\mathrm{T}}\pi\Bigg(\mathrm{diag}\Bigg(\sum_{j=1}^{n} b_{1j}p_{1j},\ldots,\sum_{j=1}^{n} b_{nj}p_{nj}\Bigg) - \boldsymbol{B}\circ\boldsymbol{P}\Bigg). \quad (4)$$

The $\mathrm{E}^*(\boldsymbol{L})$ is positive semi-definite and has an eigenvalue 0 since $\mathrm{E}^*(\boldsymbol{L})\mathbf{1}=0$, where $\mathbf{1}$ is the all-1 column vector. We need the following lemma regarding Schur product.

Lemma 2 Suppose $\boldsymbol{C}$, $\boldsymbol{E}$ are two $n\times n$ real matrices and $\boldsymbol{x}$, $\boldsymbol{y}$ are two $n\times 1$ real vectors, then

$$\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{C}\circ\boldsymbol{E})\boldsymbol{x} = \mathrm{tr}\big(\mathrm{diag}(\boldsymbol{y}^{\mathrm{T}})^{\mathrm{T}}\boldsymbol{C}\mathrm{diag}(\boldsymbol{x}^{\mathrm{T}})\boldsymbol{E}^{\mathrm{T}}\big).$$

The proof will be given in the next section. An asymptotic stability result in Ref. [15] is restated below for convenience, which is an analogy of the deterministic Lyapunov stability theorem.

Lemma 3[15] Let $\{X_n\}$ be a Markov chain on state space S. Suppose that there is a non-negative function $\xi(x)$ satisfying

$$\mathrm{E}(\xi(X_1)|X_0=x)-\xi(x)=-\eta(x),$$

where $\eta(x)\geq 0$ on the set $Q_\beta := \{x:\xi(x)<\beta\}$. Then

$$\mathrm{P}\big(\sup_{n\geq 0}\xi(X_n)\geq\beta|X_0=x\big)\leq\xi(x)/\beta,$$

and accordingly, $\eta(X_n)\to 0$ almost surely for paths which remain in Q_β.

It is at this stage, we state our main result as follows.

Theorem 1 Under the circumstances and assumptions presented above, the stochastic system expressed by Eq. (1) reaches consensus almost surely.

4. Proofs

This section includes the proofs of Lemma 2 and Theorem 1.

Proof of lemma 2 Let $\mathbf{1}=(1,1,\ldots,1)^{\mathrm{T}}$ be the $n\times 1$ vector. Note that $\mathrm{diag}(\boldsymbol{y}^{\mathrm{T}})\mathbf{1}=\boldsymbol{y}$. Therefore, we get

$$\begin{aligned}\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{C}\circ\boldsymbol{E})\boldsymbol{x} &= \mathbf{1}^{\mathrm{T}}\mathrm{diag}(\boldsymbol{y}^{\mathrm{T}})^{\mathrm{T}}(\boldsymbol{C}\circ\boldsymbol{E})x\\ &= \mathbf{1}^{\mathrm{T}}(\mathrm{diag}(\boldsymbol{y}^{\mathrm{T}})^{\mathrm{T}}\boldsymbol{C}\circ\boldsymbol{E})\boldsymbol{x}\\ &= \mathrm{tr}\big(\mathrm{diag}(\boldsymbol{y}^{\mathrm{T}})^{\mathrm{T}}\boldsymbol{C}\mathrm{diag}(\boldsymbol{x}^{\mathrm{T}})\boldsymbol{E}^{\mathrm{T}}\big).\end{aligned}$$

We have exploited a basic property: $(\boldsymbol{A}\mathrm{diag}(\boldsymbol{x}^{\mathrm{T}})\boldsymbol{B}^{\mathrm{T}})(i,i)=((\boldsymbol{A}\circ\boldsymbol{B})\boldsymbol{x})(i)$ in the last equality above. Here notation $\boldsymbol{A}(i,i)$ means the (i,i) entry of matrix $\boldsymbol{A}$, and $x(i)$ is the ith element of vector $\boldsymbol{x}$. □

Proof of theorem 1 We rewrite the protocol Eq. (1) in a compact matrix form as

$$\boldsymbol{X}(t+1)=\boldsymbol{F}(t)\boldsymbol{X}(t), \quad (5)$$

where $\boldsymbol{X}(t)=(X_1(t),\ldots,X_n(t))^{\mathrm{T}}$ is the state vector of agents at instant t, and $\boldsymbol{F}(t)=\boldsymbol{I}_n-\varepsilon\boldsymbol{L}(t)$. The spectrum of $\boldsymbol{F}(t)$ satisfies $1=\lambda_n(\boldsymbol{F}(t))\geq\ldots\geq\lambda_1(\boldsymbol{F}(t))\geq 1-2\varepsilon\Delta$, since the eigenvalues of Lapacian $\boldsymbol{L}(t)$ are $2\Delta\geq\lambda_n(\boldsymbol{L}(t))\geq\ldots\geq\lambda_1(\boldsymbol{L}(t))=0$. See Ref. [17] for more about Laplacian spectrum.

Recall that we assume $\varepsilon\in(0,1/\Delta)$ in Section 2, which yields $|\lambda_i(\boldsymbol{F}(t))|<1$ for $1\leq i<n$ and causes the state transition matrix $\boldsymbol{F}(t)$ to be stable at each time step, (see e.g. Ref. [2] Lemma 3). By the Assumption 2 and iterative equation (5), we obtain

$$\begin{aligned}\mathbf{1}^{\mathrm{T}}\boldsymbol{X}(t+1) &= \mathbf{1}^{\mathrm{T}}\boldsymbol{F}(t)\boldsymbol{X}(t)\\ &= (\boldsymbol{F}(t)\boldsymbol{X}(t))^{\mathrm{T}}\mathbf{1}\\ &= \boldsymbol{X}(t)^{\mathrm{T}}\boldsymbol{F}(t)^{\mathrm{T}}\mathbf{1}\\ &= (\boldsymbol{X}(t)^{\mathrm{T}}\mathbf{1})^{\mathrm{T}}\\ &= \mathbf{1}^{\mathrm{T}}\boldsymbol{X}(t).\end{aligned}$$

From this, it is clear that the projection of the state vector $\boldsymbol{X}(t)$ on synchronisation manifold $\mathcal{M}$ is a constant. In fact, let $\alpha:=\sum_{i=1}^{n}\boldsymbol{X}_i(0)/n$, then $\mathbf{1}^{\mathrm{T}}\boldsymbol{X}(t)=n\alpha$ for all t.

Now we may decompose $\boldsymbol{X}(t)$ as

$$\boldsymbol{X}(t)=\boldsymbol{X}^{\mathrm{pa}}(t)+\boldsymbol{X}^{\mathrm{pe}}(t), \quad (6)$$

where $\boldsymbol{X}^{\mathrm{pa}}(t)\in\mathcal{M}$ and $\boldsymbol{X}^{\mathrm{pe}}(t)\perp\mathcal{M}$. The superscript 'pa' stands for 'parallel', while 'pe' for 'perpendicular'. We thereby get $||\boldsymbol{X}^{\mathrm{pa}}(t)||=\mathbf{1}^{\mathrm{T}}\boldsymbol{X}(t)/\sqrt{n}=\sqrt{n}\alpha$. Here and in the sequel, we take $||\cdot||$ as 2-norm for vectors and induced 2-norm for matrices.[21] The disagreement among the agents now can be described by $||\boldsymbol{X}^{\mathrm{pe}}(t)||^2=||\boldsymbol{X}(t)||^2-n\alpha^2$. For proving the theorem, it suffices to show $||\boldsymbol{X}^{\mathrm{pe}}(t)||\to 0$ almost surely as $t\to\infty$.

We follow Ref. [14] to introduce a Lyapunov function $\xi(\boldsymbol{x})=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}-\alpha^2 n$, and let

$$\xi(t):=\xi(\boldsymbol{X}(t))=\boldsymbol{X}(t)^{\mathrm{T}}\boldsymbol{X}(t)-\alpha^2 n.$$

Our plan is to use Lemma 3 to show $\xi(t)\to 0$ almost surely, as $t\to\infty$. Note that $\boldsymbol{X}(t)$ is indeed a Markov chain. With this in mind, we evaluate by employing Eq. (5),

$$\mathrm{E}(\xi(t+1)|\boldsymbol{X}(t))=\boldsymbol{X}(t)^{\mathrm{T}}\mathrm{E}(\boldsymbol{F}(t)^{\mathrm{T}}\boldsymbol{F}(t))\boldsymbol{X}(t)-\alpha^2 n.$$

Therefore we get

$$\mathrm{E}(\xi(t+1)|\boldsymbol{X}(t))-\xi(t)=\boldsymbol{X}(t)^{\mathrm{T}}\mathrm{E}\boldsymbol{H}(t)\boldsymbol{X}(t), \quad (7)$$

where, $\boldsymbol{H}(t) := \boldsymbol{F}(t)^{\mathrm{T}}\boldsymbol{F}(t) - \boldsymbol{I} = \varepsilon^2\boldsymbol{L}(t)^{\mathrm{T}}\boldsymbol{L}(t) - \varepsilon(\boldsymbol{L}(t)^{\mathrm{T}} + \boldsymbol{L}(t))$.

Let $\boldsymbol{L}_i$ be the ith possible realisation of $\boldsymbol{L}(t)$, with probability $p_i(t)$ at instant t, and denote Ω as the collection of all $\boldsymbol{L}_i$. We hereby may write the expectation of $\boldsymbol{L}(t)$ as $\mathrm{E}\boldsymbol{L}(t) = \sum_{i=1}^{|\Omega|}\boldsymbol{L}_i p_i(t)$. The $|\Omega|$ is the cardinality of Ω. Since $\boldsymbol{L}(t)$ inherited the property of the random walks of the agents, it is also an ergodic Markov chain with state space Ω. Let p_i be the ith component of the unique stationary probability distribution. Then we may apply Lemma 1 and interpret the corresponding meaning of notations in the current situation. By doing so, we derive

$$\begin{aligned}
&|p_i(t) - p_i| \\
&= \left|\sum_{j=1}^{|\Omega|} p_{ji}^{(t)} p_j(0) - p_i\right| \\
&= \left|\sum_{j=1}^{|\Omega|} (p_{ji}^{(t)} - p_i) p_j(0)\right| \\
&\le \sum_{j=1}^{|\Omega|} \left|p_{ji}^{(t)} - p_i\right| p_j(0) \\
&\le \lambda\rho^t \sum_{j=1}^{|\Omega|} p_j(0) \\
&= \lambda\rho^t. \qquad (8)
\end{aligned}$$

Now going back to Eq. (7), we have

$$\boldsymbol{X}(t)^{\mathrm{T}}\mathrm{E}\boldsymbol{H}(t)\boldsymbol{X}(t) = \sum_{i=1}^{|\Omega|} p_i(t)\boldsymbol{X}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}(t), \qquad (9)$$

where $\boldsymbol{H}_i = \varepsilon^2\boldsymbol{L}_i^{\mathrm{T}}\boldsymbol{L}_i - \varepsilon(\boldsymbol{L}_i^{\mathrm{T}} + \boldsymbol{L}_i)$ is the ith possible realisation of $\boldsymbol{H}(t)$ and is negative semi-definite with eigenvalues $0 = \lambda_n(\boldsymbol{H}_i) \ge \ldots \ge \lambda_1(\boldsymbol{H}_i)$. Using the decomposition Eq. (6), we get

$$\begin{aligned}
&\boldsymbol{X}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}(t) \\
&= \boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}^{\mathrm{pe}}(t) + \boldsymbol{X}^{\mathrm{pa}}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}^{\mathrm{pe}}(t) \\
&\quad + \boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}^{\mathrm{pa}}(t) + \boldsymbol{X}^{\mathrm{pa}}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}^{\mathrm{pa}}(t).
\end{aligned}$$

There are four terms on the right hand side (RHS) of the above equation, the last three terms are actually zero. To see why, we take the second term as an example,

$$\begin{aligned}
&\boldsymbol{X}^{\mathrm{pa}}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}^{\mathrm{pe}}(t) \\
&= \varepsilon^2\boldsymbol{X}^{\mathrm{pa}}(t)^{\mathrm{T}}\boldsymbol{L}_i^{\mathrm{T}}\boldsymbol{L}_i\boldsymbol{X}^{\mathrm{pe}}(t) - \varepsilon\boldsymbol{X}^{\mathrm{pa}}(t)^{\mathrm{T}}\boldsymbol{L}_i^{\mathrm{T}}\boldsymbol{X}^{\mathrm{pe}}(t) \\
&\quad - \varepsilon\boldsymbol{X}^{\mathrm{pa}}(t)^{\mathrm{T}}\boldsymbol{L}_i\boldsymbol{X}^{\mathrm{pe}}(t) \\
&= \varepsilon^2(\boldsymbol{L}_i\boldsymbol{X}^{\mathrm{pa}}(t))^{\mathrm{T}}\boldsymbol{L}_i\boldsymbol{X}^{\mathrm{pe}}(t) - \varepsilon(\boldsymbol{L}_i\boldsymbol{X}^{\mathrm{pa}}(t))^{\mathrm{T}}\boldsymbol{X}^{\mathrm{pe}}(t) \\
&\quad - \varepsilon(\boldsymbol{L}_i^{\mathrm{T}}\boldsymbol{X}^{\mathrm{pa}}(t))^{\mathrm{T}}\boldsymbol{X}^{\mathrm{pe}}(t) \\
&= 0
\end{aligned}$$

since $\boldsymbol{L}_i\boldsymbol{X}^{\mathrm{pa}}(t) = \boldsymbol{L}_i^{\mathrm{T}}\boldsymbol{X}^{\mathrm{pa}}(t) = 0$. Thereby we have $\boldsymbol{X}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}(t) = \boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\boldsymbol{H}_i\boldsymbol{X}^{\mathrm{pe}}(t)$. Plugging this equation into Eq. (9), we get

$$\begin{aligned}
&\boldsymbol{X}(t)^{\mathrm{T}}\mathrm{E}\boldsymbol{H}(t)\boldsymbol{X}(t) \\
&= \boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\left(\sum_{i=1}^{|\Omega|} p_i(t)\boldsymbol{H}_i\right)\boldsymbol{X}^{\mathrm{pe}}(t) \\
&= \boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\mathrm{E}\boldsymbol{H}(t)\boldsymbol{X}^{\mathrm{pe}}(t).
\end{aligned}$$

We have the ergodic limit, defined in Section 3, of $\boldsymbol{H}(t)$ as

$$\begin{aligned}
\mathrm{E}^*\boldsymbol{H} &= \lim_{t\to\infty}\mathrm{E}\boldsymbol{H}(t) \\
&= \lim_{t\to\infty}\mathrm{E}\big(\varepsilon^2\boldsymbol{L}(t)^{\mathrm{T}}\boldsymbol{L}(t) - \varepsilon(\boldsymbol{L}(t)^{\mathrm{T}} + \boldsymbol{L}(t))\big) \\
&= \varepsilon^2\mathrm{E}^*(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{L}) - \varepsilon(\mathrm{E}^*\boldsymbol{L}^{\mathrm{T}} + \mathrm{E}^*\boldsymbol{L}), \qquad (10)
\end{aligned}$$

since we know $\boldsymbol{L}(t)$ is an ergodic Markov chain.

Take $\boldsymbol{R}(t) := \mathrm{E}\boldsymbol{H}(t) - \mathrm{E}^*\boldsymbol{H}$ as the remainder matrix, then we have

$$\begin{aligned}
&\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\mathrm{E}\boldsymbol{H}(t)\boldsymbol{X}^{\mathrm{pe}}(t) \\
&= \boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{H})\boldsymbol{X}^{\mathrm{pe}}(t) + \boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\boldsymbol{R}(t)\boldsymbol{X}^{\mathrm{pe}}(t). \qquad (11)
\end{aligned}$$

The second term on the RHS of Eq. (11) can be calculated, noting Eq. (8), we have

$$\begin{aligned}
&\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\boldsymbol{R}(t)\boldsymbol{X}^{\mathrm{pe}}(t) \\
&= \left\|\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\boldsymbol{R}(t)\boldsymbol{X}^{\mathrm{pe}}(t)\right\| \\
&= \left\|\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\left(\sum_{i=1}^{|\Omega|}\boldsymbol{H}_i(p_i(t) - p_i)\right)\boldsymbol{X}^{\mathrm{pe}}(t)\right\| \\
&\le \left|\left|\boldsymbol{X}^{\mathrm{pe}}(t)\right|\right|^2 \cdot \left\|\sum_{i=1}^{|\Omega|}\boldsymbol{H}_i(p_i(t) - p_i)\right\| \\
&\le \xi(t)|\Omega|\gamma\lambda\rho^t, \qquad (12)
\end{aligned}$$

where $\gamma := \max_{1\le i\le|\Omega|}||\boldsymbol{H}_i||$. Wherefore, Eq. (12) tends to 0, since $0 \le \rho < 1$ and $\xi(t)$ is bounded by Eq. (1).

Next, we turn to the first term on the RHS of Eq. (11). In view of Eq. (10),

$$\begin{aligned}
&\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{H})\boldsymbol{X}^{\mathrm{pe}}(t) \\
&= \varepsilon^2\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{L}^{\mathrm{T}})(\mathrm{E}^*\boldsymbol{L})\boldsymbol{X}^{\mathrm{pe}}(t) \\
&\quad - \varepsilon\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{L}^{\mathrm{T}})\boldsymbol{X}^{\mathrm{pe}}(t) \\
&\quad - \varepsilon\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{L})\boldsymbol{X}^{\mathrm{pe}}(t). \qquad (13)
\end{aligned}$$

By using Lemma 2, we have

$$\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\boldsymbol{B}^{\mathrm{T}} \circ \boldsymbol{P}^{\mathrm{T}})\boldsymbol{X}^{\mathrm{pe}}(t)$$

$$
\begin{aligned}
&= \mathrm{tr}\big(\mathrm{diag}(\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}})\boldsymbol{B}^{\mathrm{T}}\mathrm{diag}(\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}})\boldsymbol{P}\big)\\
&= \mathrm{tr}\big(\mathrm{diag}(\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}})\boldsymbol{B}\mathrm{diag}(\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}})\boldsymbol{P}^{\mathrm{T}}\big)\\
&= \boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\boldsymbol{B}\circ\boldsymbol{P})\boldsymbol{X}^{\mathrm{pe}}(t).
\end{aligned}
$$

Then it is easy to see that the last two terms on RHS of Eq. (13) is equal, by noting Eq. (4). Thereby we obtain

$$
\begin{aligned}
&\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{H})\boldsymbol{X}^{\mathrm{pe}}(t)\\
&= \varepsilon^2\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{L})^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{L})\boldsymbol{X}^{\mathrm{pe}}(t)\\
&\quad - 2\varepsilon\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{L})\boldsymbol{X}^{\mathrm{pe}}(t)\\
&\le (2\Delta\varepsilon^2 - 2\varepsilon)\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{L})\boldsymbol{X}^{\mathrm{pe}}(t)\\
&= 2\varepsilon\pi^{\mathrm{T}}\pi(\varepsilon\Delta - 1)\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}\\
&\quad \times \left(\mathrm{diag}\left(\sum_{j=1}^{n} b_{1j}p_{1j},\ldots,\sum_{j=1}^{n} b_{nj}p_{nj}\right) - \boldsymbol{B}\circ\boldsymbol{P}\right)\\
&\quad \times \boldsymbol{X}^{\mathrm{pe}}(t)\\
&\le 2\varepsilon\pi^{\mathrm{T}}\pi\Delta(\varepsilon\Delta - 1)\xi(t). \qquad (14)
\end{aligned}
$$

The second inequality above holds because there is some $N(\boldsymbol{B}) > 0$ which depends only on $\boldsymbol{B}$, such that $(\mathrm{E}^*\boldsymbol{L})^{\mathrm{T}}(\mathrm{E}^*\boldsymbol{L}) \le 2\Delta(\mathrm{E}^*\boldsymbol{L}) + N(\boldsymbol{B})\boldsymbol{1}\boldsymbol{1}^{\mathrm{T}}$ and

$$
\begin{aligned}
&\boldsymbol{X}^{\mathrm{pe}}(t)^{\mathrm{T}}N(\boldsymbol{B})\boldsymbol{1}\boldsymbol{1}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{pe}}(t)\\
&= N(\boldsymbol{B})(\boldsymbol{1}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{pe}}(t))^{\mathrm{T}}(\boldsymbol{1}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{pe}}(t))\\
&= 0.
\end{aligned}
$$

The fourth inequality holds by noting that $\varepsilon\Delta - 1 < 0$ and

$$
\begin{aligned}
&\left(\mathrm{diag}\left(\sum_{j=1}^{n} b_{1j}p_{1j},\ldots,\sum_{j=1}^{n} b_{nj}p_{nj}\right) - \boldsymbol{B}\circ\boldsymbol{P}\right)\\
&\ge \Delta\boldsymbol{I} - M(\boldsymbol{B})11^{\mathrm{T}}
\end{aligned}
$$

for some $M(\boldsymbol{B}) > 0$ depending only on $\boldsymbol{B}$. The similar argument as above may be applied. Combining Eqs. (12) and (14), we finally get

$$
\begin{aligned}
&\mathrm{E}(\xi(t+1)|\boldsymbol{X}(t)) - \xi(t)\\
&\le \xi(t)\big(|\Omega|\gamma\lambda\rho^t + 2\varepsilon\pi^{\mathrm{T}}\pi(\varepsilon\Delta - 1)\Delta\big)\\
&:= \xi(t)\delta(t) := -\eta(t).
\end{aligned}
$$

There exists $t_0 \in \mathbb{N}$ such that $\delta(t) < 0$ for $t \ge t_0$. Furthermore, $\delta(t) \to 2\varepsilon\pi^{\mathrm{T}}\pi(\varepsilon\Delta - 1)\Delta$, as $t \to \infty$. Also, note that $\xi(t)$ is a Markov chain, since $X(t)$ is so. Applying Lemma 3, where β can take any positive number, we obtain

$$\eta(t) \to 0 \quad \text{as} \quad t \to \infty$$

almost surely, which yields $\xi(t) \to 0$ almost surely, as $t \to \infty$. The proof is completed. □

Remark 3 Theorem 1 shows an average consensus which may be regarded as a kind of stochastic version of Theorem 9 in Ref. [13].

5. Simulation examples

In this section, to illustrate our theoretical results derived in the above section, we will provide two numerical simulations.

Example 1 Figure 1 is a strongly connected weighted digraph $\bar{G} = (\bar{V}, \bar{E}, \boldsymbol{W})$ of order $m = 12$ with weights $w_{1,6} = w_{2,7} = w_{3,8} = w_{2,10} = w_{6,11} = w_{7,12} = 1$ and other weights $w_{ij} = 2$ for $(i,j) \in \bar{E}$. In $\bar{G}$, each horizontal and vertical arcs are bidirectional while the six diagonal arcs are directed. We choose the linkage probabilities $p_{ij} = 1/2$ and the weight factors $b_{ij} = 1$ for all $i \ne j$. Take $\varepsilon = 0.01$. We consider $n = 5$ agents meandering on the network $\bar{G}$ with their initial states $[-2, 2.5, 3, -1, 1.5]^{\mathrm{T}}$. The simulation result is shown in Fig. 2, from which we can see the asymptotic convergence of the states.

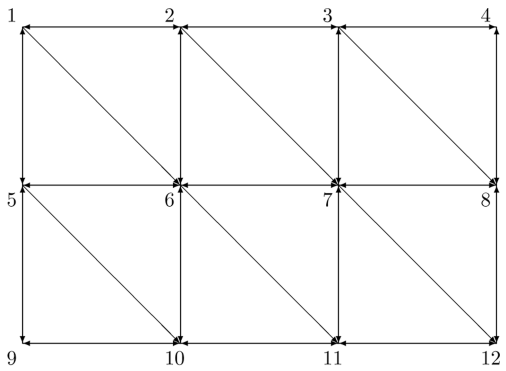


Fig. 1. Underlying network topology $\bar{G}$.

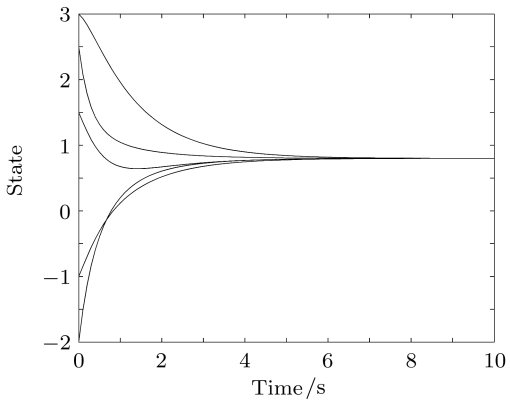


Fig. 2. Consensus for five agents.

Example 2 Consider a directed cycle $\bar{G} = (\bar{V}, \bar{E}, \boldsymbol{W})$ of order $m = 250$ with 250 directed cords. In particular, we take $\bar{V} = [m]$, $\bar{E} = \{(i, i+$

$1(\mathrm{mod}\ 250)),\ (i+2(\mathrm{mod}\ 250), i)|\ 1 \le i \le 250\}$ and unit weights for all arcs. We choose the linkage probabilities $p_{ij} = 0.8$ and the weight factors $b_{ij} = 0.35$ for all $i \neq j$. Take $\varepsilon = 0.01$. In the following, we consider three cases: (i) $n = 50$, (ii) $n = 100$ and (iii) $n = 200$. The initial states of agents are taken randomly in the interval $[-2, 2]$. We randomly choose ten agents in each of the three cases, and plot the corresponding state curves in Figs. 3, 4 and 5, respectively. Simulation (iii) costs more than five times as much as simulation (i) in personal computer time. From Figs. 3, 4 and 5 we can see that the rate of reaching consensus is reduced due to the increasing number of agents.

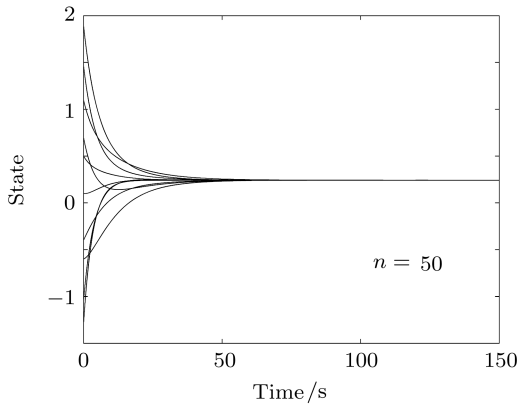


Fig. 3. Consensus for fifty agents.

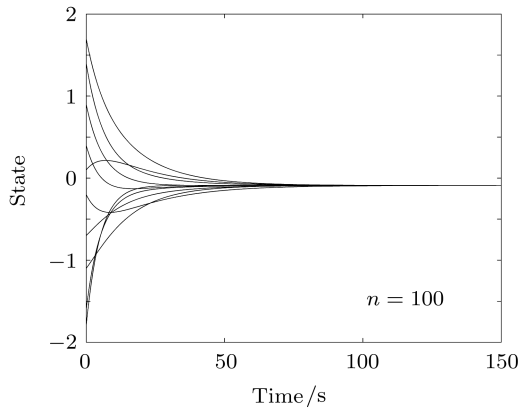


Fig. 4. Consensus for one hundred agents.

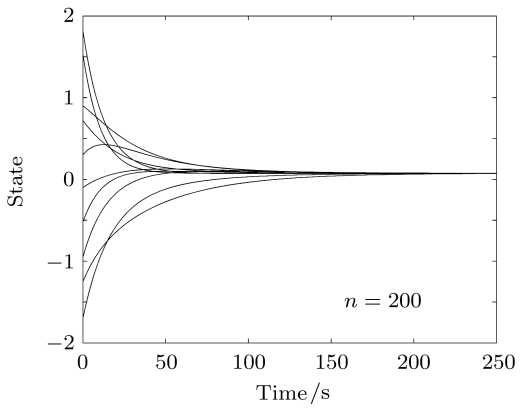


Fig. 5. Consensus for two hundred agents.

6. Conclusion

In this paper, we studied the problem of consensus among multiple agents meandering in a weighted directed network. Each agent moves to a randomly selected adjacent site at any instant independent of its past trajectory. Directed communication takes place possibly between agents when they occupy the same site. The states of agents evolve based on the agents' communication. We presented sufficient conditions for an almost sure asymptotic consensus of the agents' states. Simulation results are provided which demonstrate the effectiveness of our theoretical results and analytical tools.

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