A generalized Collins formula derived by virtue of the displacement-squeezing related squeezed coherent state representation*

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Based on the displacement-squeezing related squeezed coherent state representation $|z\rangle_g$ and using the technique of integration within an ordered product of operators, this paper finds a generalized Fresnel operator, whose matrix element in the coordinate representation leads to a generalized Collins formula (Huygens–Fresnel integration transformation describing optical diffraction). The generalized Fresnel operator is derived by a quantum mechanical mapping from z to $sz - rz^*$ in the $|z\rangle_g$ representation, while $|z\rangle_g$ in phase space is graphically denoted by an ellipse.

Keywords: generalized Fresnel operator, generalized Collins formula, phase space, IWOP technique, squeezed coherent state

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1. Introduction

According to Huygens–Fresnel diffraction theory the formula describing the light propagation in an optical system characterized by the $[A, B; C, D]$ ray transfer matrix is the Collins diffraction integral formula (2-dimensional case),[1]

$$U_2(x_2, y_2) = \frac{k \exp(\mathrm{i}kz)}{2\pi B\,\mathrm{i}} \iint \mathrm{d}x_1 \mathrm{d}y_1 U_1(x_1, y_1) \times \exp\left\{ \frac{\mathrm{i}k}{2B} [A(x_1^2 + y_1^2) - 2(x_1 x_2 + y_1 y_2) + D(x_2^2 + y_2^2)] \right\}, \quad (1)$$

where $AD - BC = 1$, if the system is lossless. In Refs. [2] and [3], based on the coherent state[4,5] representation and using the technique of integration within an ordered product (IWOP)[6,7] of operators, the Fresnel operator, whose matrix element in the coordinate representation can lead to the optical Collins formula, was constructed. The obtained Fresnel operator and Collins diffraction integral formula have been widely applied to the discussions in the relationships between classical optics and quantum optics and so on.[8–16]

In this paper we want to derive a new generalized Collins diffraction integral formula which in quantum statistical theory graphically corresponds to a movement of an ellipse which represents a squeezed state in phase space (we will emphasize this point of view later). We shall derive the formula via the approach of quantum optics theory, to be more concrete, we shall find a generalized Fresnel operator (GFO) based on the (displacement-squeezing related) squeezed coherent state $|z\rangle_g$ representation and by using the technique of IWOP operators, we shall then show that the matrix elements of GFO in the coordinate basis lead to the generalized Collins formula.

Our work is arranged as follows: In Section 2 we propose the displacement-squeezing related squeezed coherent state, and by using the IWOP technique prove its completeness relation. In Section 3 we derive the GFO by mapping from $|z\rangle_g \to |sz - rz^*\rangle_g$ and explain its physical meaning in phase space. In Section 4 we derive the new generalized Collins diffraction integral formula.

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2. The displacement squeezing related squeezed coherent state

Since the seventies of the last century, increasing attention in the field of quantum optics has been paid to squeezed states of light[17−19] because of their potential uses in optical communication and interferometers. In this section we construct a squeezed coherent state

$$|z\rangle_g = \exp\left[-\frac{|z|^2}{2} + (fz + gz^*)\, a^\dagger - fga^{\dagger 2}\right]|0\rangle\,, \quad (2)$$

where $|0\rangle$ is the vacuum state in Fock space, $[a, a^\dagger] = 1$, f, g are the displacement parameter with the constraint $ff^* + gg^* = 1$. The fg is a squeezing parameter, so in this state the displacement and squeezing are related. This state is complete, since

$$\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\ {}_{gg}\langle z| = 1, \quad (3)$$

which can be verified with the use of the normal ordering form $|0\rangle\langle 0| =: \mathrm{e}^{-a^\dagger a} :$ and the IWOP technique

$$\begin{aligned} &\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\ {}_{gg}\langle z| \\ &= \int \frac{\mathrm{d}^2 z}{\pi} :\exp[-|z|^2 + (fz + gz^*)\, a^\dagger - fga^{\dagger 2} \\ &\quad + (f^* z^* + g^* z)\, a - f^* g^* a^2 - a^\dagger a] :\,. \end{aligned} \quad (4)$$

Moreover, using the completeness relation of coherent state $|\beta\rangle = \exp\left[-|\beta|^2/2 + \beta a^\dagger\right]|0\rangle$,

$$\int \frac{\mathrm{d}^2 \beta}{\pi} |\beta\rangle\langle\beta| = 1 \quad (5)$$

and the overlap

$$\begin{aligned} {}_g\langle \beta|\ z\rangle_g = \exp\Bigg[&-\frac{|\beta_1|^2 + |z|^2}{2} + (fz + gz^*)\,\beta^* \\ &- fg\beta^{*2}\Bigg], \end{aligned} \quad (6)$$

we have

$$\begin{aligned} {}_g\langle z|\ z\rangle_g &= {}_g\langle z| \int \frac{\mathrm{d}^2 \beta}{\pi} |\beta\rangle\langle\beta|\ z\rangle_g \\ &= \mathrm{e}^{-|z|^2} \int \frac{\mathrm{d}^2 \beta}{\pi} \exp[-|\beta|^2 + (fz + gz^*)\,\beta^* \\ &\quad - fg\beta^{*2} + (f^* z^* + g^* z)\,\beta - f^* g^* \beta^2] \\ &= \frac{1}{\sqrt{1 - 4\,|fg|^2}}, \end{aligned} \quad (7)$$

where we have used the formula

$$\begin{aligned} &\int \frac{\mathrm{d}^2 z}{\pi} \exp\{\zeta\,|z|^2 + \xi z + \eta z^* + fz^2 + gz^{*2}\} \\ &= \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left\{\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right\}, \end{aligned} \quad (8)$$

so

$$\sqrt{1 - 4\,|fg|^2}\ {}_g\langle z|\ z\rangle_g = 1. \quad (9)$$

Due to

$$a\,|z\rangle_g = \left[(fz + gz^*) - 2fga^\dagger\right]|z\rangle_g\,, \quad (10)$$

$|z\rangle_g$ obeys the eigenvector equation

$$\frac{a + 2fga^\dagger}{\sqrt{1 - 4\,|fg|^2}}\,|z\rangle_g = (fz + gz^*)\,|z\rangle_g\,. \quad (11)$$

If we let $(a + 2fga^\dagger)/\sqrt{1 - 4\,|fg|^2} \equiv a'$, then $[a',\ a'^\dagger] = 1$, $|z\rangle_g$ is a squeezed state indeed. While a coherent state in phase space graphically occupies a round of area $\hbar/2$ (as a coherent state takes the minimum uncertainty relation), the squeezed state $|z\rangle_g$ graphically occupies an ellipse of area $\hbar/2$.

3. The GFO

We now consider the ellipse in phase space, which denotes the quantum state $|z\rangle_g$ and has a movement characteristic of $z \to sz - rz^*$. Correspondingly, we construct the following ket–bra integration operator,

$$U_g(r, s) = \sqrt{ss^*} \int \frac{\mathrm{d}^2 z}{\pi} |sz - rz^*\rangle\ {}_{gg}\langle z|\,, \quad (12)$$

where s and r are complex and satisfy

$$|s|^2 - |r|^2 = 1, \quad (13)$$

which indicates that the movement is symplectic. Using Eq. (8) and the IWOP technique we can perform the integration in Eq. (12) and obtain

$$\begin{aligned} U_g(r, s) &= \sqrt{ss^*} \int \frac{\mathrm{d}^2 z}{\pi} |sz - rz^*\rangle\ {}_{gg}\langle z| \\ &= \sqrt{ss^*} \int \frac{\mathrm{d}^2 z}{\pi} :\exp\Bigg[-\frac{1}{2}|sz - rz^*|^2 - \frac{|z|^2}{2} \\ &\quad + \left(f\,(sz - rz^*) + g\,(sz - rz^*)^*\right) a^\dagger - fga^{\dagger 2} \\ &\quad + (f^* z^* + g^* z)\, a - f^* g^* a^2 - a^\dagger a\Bigg] : \\ &= \exp\left[\left(-\frac{f^2 r}{2s^*} - \frac{g^2 r^*}{2s}\right) a^{\dagger 2}\right] \\ &\quad \times :\exp\left[\left(\frac{|f|^2}{s^*} + \frac{|g|^2}{s} - 1\right) a^\dagger a\right] : \\ &\quad \times :\exp\left[\left(\frac{f^{*2} r^*}{2s^*} + \frac{g^{*2} r}{2s}\right) a^2\right]. \end{aligned} \quad (14)$$

Then with the help of the operator identity

$$\exp\left(\lambda a^{\dagger}a\right) =: \exp[\left(e^{\lambda}-1\right)a^{\dagger}a]:, \tag{15}$$

we have

$$\begin{aligned} U_g(r,s) = & \exp\left[\left(-\frac{f^2r}{2s^*}-\frac{g^2r^*}{2s}\right)a^{\dagger 2}\right] \\ & \times\exp\left[a^{\dagger}a\ln\left(\frac{|f|^2}{s^*}+\frac{|g|^2}{s}\right)\right] \\ & \times\exp\left[\left(\frac{f^{*2}r^*}{2s^*}+\frac{g^{*2}r}{2s}\right)a^2\right]. \end{aligned} \tag{16}$$

We name it the generalized Fresnel operator (GFO), since in particular, when $g=0$ and $f=1$, equation (16) reduces to

$$U_{g=0}(r,s) = \sqrt{s^*}U(r,s), \tag{17}$$

where

$$\begin{aligned} U(r,s) = & \sqrt{s}\int\frac{\mathrm{d}^2z}{\pi}|sz-rz^*\rangle\langle z| \\ = & \exp\left[\left(-\frac{r}{2s^*}\right)a^{\dagger 2}\right]\exp\left[\left(a^{\dagger}a+\frac{1}{2}\right)\ln\frac{1}{s^*}\right] \\ & \times\exp\left[\left(\frac{r^*}{2s^*}\right)a^2\right] \end{aligned} \tag{18}$$

is the Fresnel operator in Ref. [2]; when $g=1$, $f=0$,

$$U_{g=1}(r,s) = \sqrt{s}U(r^*,s^*). \tag{19}$$

Further, using the IWOP technique and Eqs. (2) and (6) we can verify

$$U_g^{\dagger}(r,s) = U_g^{-1}(r,s), \tag{20}$$

so $U_g(r,s)$ is a unitary operator.

4. A generalized Collins formula

Using the normally ordered form of $U_g(r,s)$ in Eq. (14) we take its coherent state matrix elements and immediately have

$$\begin{aligned} & \langle z|U_g(r,s)|z'\rangle \\ = & \exp\left[\left(-\frac{f^2r}{2s^*}-\frac{g^2r^*}{2s}\right)z^{*2}\right] \\ & \times\exp\left[\left(\frac{|f|^2}{s^*}+\frac{|g|^2}{s}-1\right)z^*z'\right] \\ & \times\exp\left[\left(\frac{f^{*2}r^*}{2s^*}+\frac{g^{*2}r}{2s}\right)z'^2\right]\langle z|z'\rangle. \end{aligned} \tag{21}$$

Letting $|x_1\rangle$ be the coordinate eigenstate, using its overlap with the coherent state $\langle z'|$,

$$\langle z'|x_1\rangle = \exp\left\{-\frac{x_1^2}{2}+\sqrt{2}x_1z'^*-\frac{z'^{*2}}{2}-\frac{|z'|^2}{2}\right\}, \tag{22}$$

and the completeness relation of the coherent state, we calculate the coordinate eigenstate matrix elements of $U_g(r,s)$,

$$\begin{aligned} \langle x_2|U_g(r,s)|x_1\rangle = & \int\frac{\mathrm{d}^2z\mathrm{d}^2z'}{\pi^2}\langle x_2|z\rangle\langle z|U_g(r,s)|z'\rangle\langle z'|x_1\rangle \\ = & \pi^{-1/2}\int\frac{\mathrm{d}^2z\mathrm{d}^2z'}{\pi^2}\exp\left\{-\frac{x_1^2+x_2^2}{2}+\sqrt{2}x_2z+\sqrt{2}x_1z'^*-|z|^2-|z'|^2\right. \\ & \left.+\frac{V}{2}z'^2-\frac{Y}{2}z^{*2}+Kz^*z'-\frac{z^2}{2}-\frac{z'^{*2}}{2}\right\} \\ = & \pi^{-1/2}\frac{1}{\sqrt{W}}\exp\left\{\frac{2x_1x_2}{W}K-\frac{x_2^2}{W}\left[K^2+Y(1+V)\right]-\frac{x_1^2}{W}\left[K^2-V(1-Y)\right]-\frac{x_1^2+x_2^2}{2}\right\} \\ = & \pi^{-1/2}\frac{1}{\sqrt{W}}\exp\left\{-\frac{x_1^2}{2W}\left[K^2+(1-V)(1-Y)\right]+\frac{2x_1x_2}{W}K\right. \\ & \left.-\frac{x_2^2}{2W}\left[K^2+(1+V)(1+Y)\right]\right\}, \end{aligned} \tag{23}$$

where we have defined

$$K\equiv\frac{|f|^2}{s^*}+\frac{|g|^2}{s},\quad V\equiv\frac{f^{*2}r^*}{s^*}+\frac{g^{*2}r}{s},\quad Y\equiv\frac{f^2r}{s^*}+\frac{g^2r^*}{s}, \tag{24}$$

$$W\equiv(1-Y)(1+V)-K^2. \tag{25}$$

By introducing

$$\begin{aligned} A = & \frac{1}{2}\left[K+K^{-1}(1-V)(1-Y)\right], \quad B = \frac{\mathrm{i}}{2}\left[K-K^{-1}(1+V)(1-Y)\right], \\ C = & -\frac{\mathrm{i}}{2}\left[K-K^{-1}(1-V)(1+Y)\right], \quad D = \frac{1}{2}\left[K+K^{-1}(1+V)(1+Y)\right], \end{aligned} \tag{26}$$

which just obeys

$$AD - BC = 1, \tag{27}$$

then we can rewrite Eq. (23) as

$$\begin{aligned} &\langle x_2| U_g(r,s) |x_1\rangle \\ &= \frac{1}{\sqrt{2\pi \mathrm{i} BK}} \exp\left[\frac{\mathrm{i}}{2B}\left(Ax_1^2 - 2x_1x_2 + Dx_2^2\right)\right]. \end{aligned} \tag{28}$$

In particular, if let $g = 0$ and $f = 1$, from Eqs. (24)–(26), we see

$$\begin{aligned} A &= \frac{1}{2}(s - r + s^* - r^*), \\ B &= \frac{\mathrm{i}}{2}(s + r - s^* - r^*), \\ C &= -\frac{\mathrm{i}}{2}(s - r - s^* + r^*), \\ D &= \frac{1}{2}(s + r + s^* + r^*), \end{aligned} \tag{29}$$

in this case equation (28) reduces to

$$\begin{aligned} &\langle x_2| U_{g=0}(r,s) |x_1\rangle \\ &= \frac{\sqrt{s^*}}{\sqrt{2\pi \mathrm{i} B}} \exp\left\{\frac{\mathrm{i}}{2B}\left(Ax_1^2 - 2x_1x_2 + Dx_2^2\right)\right\} \\ &= \sqrt{s^*}\,\langle x_2| U(r,s) |x_1\rangle. \end{aligned} \tag{30}$$

If let $g = 1$ and $f = 0$, from Eqs. (24)–(26), we see

$$\begin{aligned} A &= \frac{1}{2}(s^* - r^* + s - r), \\ B &= \frac{\mathrm{i}}{2}(s^* + r^* - s - r), \\ C &= -\frac{\mathrm{i}}{2}(s^* - r^* - s + r), \\ D &= \frac{1}{2}(s^* + r^* + s + r), \end{aligned} \tag{31}$$

then equation (28) reduces to

$$\begin{aligned} &\langle x_2| U_{g=1}(r,s) |x_1\rangle \\ &= \frac{\sqrt{s}}{\sqrt{2\pi \mathrm{i} B}} \exp\left\{\frac{\mathrm{i}}{2B}\left(Ax_1^2 - 2x_1x_2 + Dx_2^2\right)\right\} \\ &= \sqrt{s}\,\langle x_2| U(r^*,s^*) |x_1\rangle, \end{aligned} \tag{32}$$

where

$$\begin{aligned} &\langle x_2| U(r,s) |x_1\rangle \\ &= \frac{1}{\sqrt{2\pi \mathrm{i} B}} \exp\left\{\frac{\mathrm{i}}{2B}\left(Ax_1^2 - 2x_1x_2 + Dx_2^2\right)\right\} \end{aligned} \tag{33}$$

is the one-dimensional Collins integration (see Eq. (1)). So, equation (28) is really a new generalized Collins diffraction integral formula.

In summary, based on the squeezed coherent state representation and using the technique of IWOP operators we find a GFO, whose matrix elements in the coordinate representation lead to a generalized optical Collins formula (describing Huygens–Fresnel diffraction). The GFO corresponds to an ellipse (denoting the squeezed state $|z\rangle_g$) movement in phase space, this is a new example that there exists formal connection between classical optics with quantum statistics. At this point we recall that in the history of quantum mechanics Schrödinger considered that classical dynamics of a point particle should be the 'geometrical optics' approximation of a linear wave equation, in the same way as ray optics is a limiting approximation of wave optics. We expect that the generalized optical Collins formula may be realistically used in optical ray diffraction for propagation of light emitted from some complicated objects.

References

[1] Collins S A 1970 *J. Opt. Soc. Am.* **60** 1168
[2] Fan H Y and Lu H L 2006 *Opt. Commun.* **258** 51
[3] Fan H Y and Lu H L 2005 *Phys. Lett.* A **334** 132
[4] Klauder J R and Skagerstam B S 1995 *Coherent States* (Singapore: World Scientific)
[5] Glauber R J 1963 *Phys. Rev.* **131** 2766
[6] Fan H Y, Hu L Y and Fan Y 2006 *Ann. Phys.* **321** 480
[7] Fan H Y and Wünsche A 2000 *J. Opt.* B: *Quantum Semiclass. Opt.* **2** 464
[8] Fan H Y and Lu H L 2009 *Chin. Phys.* B **18** 611
[9] Fan H Y and Lu H L 2007 *Chin. Phys. Lett.* **24** 2238
[10] Fan H Y and Lu H L 2006 *Opt. Lett.* **31** 2622
[11] Fan H Y and Hu L Y 2008 *Chin. Phys.* B **17** 1640
[12] Hu L Y and Fan H Y 2008 *J. Mod. Opt.* **55** 242
[13] Hu L Y and Fan H Y 2008 *J. Mod. Opt.* **55** 835
[14] Fan H Y and Hu L Y 2007 *Chin. Phys. Lett.* **24** 2238
[15] Fan H Y and Hu L Y 2008 *Chin. Phys. Lett.* **25** 513
[16] Fan H Y, Li H Q and Xu X L 2009 *Acta Phys. Sin.* **58** 918 (in Chinese)
[17] Walls D F 1983 *Nature* **306** 141
[18] Yuen H P 1976 *Phys. Rev.* A **13** 2226
[19] Loudon R and Knight P L J 1987 *Mod. Opt.* **34** 709