Determination of the exact range of the value of the parameter corresponding to chaos based on the Silnikov criterion*

Li Wei-Yi(李伟义), Zhang Qi-Chang(张琪昌)†, and Wang Wei(王 炜)

Department of Mechanics, School of Mechanical Engineering, Tianjin University, Tianjin 300072, *China and State Key Laboratory of Engines, Tianjin University, Tianjin* 300072, *China*

(Received 28 July 2009; revised manuscript received 25 December 2009)

Based on the Silnikov criterion, this paper studies a chaotic system of cubic polynomial ordinary differential equations in three dimensions. Using the Cardano formula, it obtains the exact range of the value of the parameter corresponding to chaos by means of the centre manifold theory and the method of multiple scales combined with Floque theory. By calculating the manifold near the equilibrium point, the series expression of the homoclinic orbit is also obtained. The space trajectory and Lyapunov exponent are investigated via numerical simulation, which shows that there is a route to chaos through period-doubling bifurcation and that chaotic attractors exist in the system. The results obtained here mean that chaos occurred in the exact range given in this paper. Numerical simulations also verify the analytical results.

Keywords: Silnikov criterion, chaos, homoclinic orbit, period-doubling bifurcation
PACC: 0545, 4660D

1. Introduction

At present, research on the motion characteristics[1−3] and construction[4−6] of chaotic systems is receiving more and more attention. Among them, chaos in the Silnikov sense has attracted the attention of many investigators. Continuous efforts have been devoted to seeking methods to determine the existence of Smale horseshoe chaos. In this endeavour, besides Sprott's[7−9] exhaustive search for chaos via a computer program, Xu and Jing[10] found a method to determine the existences of Smale horseshoe chaos in a coupled Duffing's system by using Melnikov's method and geometrical singular perturbation theory. Zhou and Chen[11] and Zhu[12] constructed a new chaotic system of quadratic polynomial ordinary differential equations in three dimensions based on the Silnikov criterion. Zhang and Tian[13] studied the eigenvalue problem and existence of the homoclinic orbit respectively.

It is known that period-doubling bifurcation is a route to chaos. The asymptotic solution and the bifurcation value of period-doubling bifurcation were obtained by Rand.[14] Nayfeh and Balachandran[15] obtained the first order asymptotic solution of period-doubling bifurcation. Though the existence of chaos in Silnikov's sense can be determined based on the Silnikov criterion, it is difficult to find the exact range of the parameter's value corresponding to chaos. Other than a lucky encounter of chaos in numerical simulation, it is still a task with significant impact on both basic research and engineering applications.

In this paper, a chaotic system of cubic polynomial ordinary differential equations in three dimensions is analysed based on the Silnikov criterion.[16] Using the Cardano formula, we obtain a rude range of the parameter's value corresponding to chaos. The series expression of the homoclinic orbit of the system is also obtained by calculating the manifold near the equilibrium point. The system is reduced into a two-dimensional system by the centre manifold theory.[17] The critical value of parameters corresponding to the first period-doubling bifurcation is found by the method of multiple scales[18] combined with Floque theory.[19] Then the exact range of parameter's value, which is easier for people to find the chaos of the system, is obtained by an intersecting calculation with the rude range. Finally it is proved that the system has Smale horseshoes chaos and that there is a route

*Project supported by the National Natural Science Foundation of China (Grant No. 10872141) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20060056005).

†Corresponding author. E-mail: qzhang@tju.edu.cn

© 2010 Chinese Physical Society and IOP Publishing Ltd

to chaos through period-doubling bifurcation by numerical simulation.

2. Silnikov criterion

The Silnikov criterion[16] is a very useful tool for providing chaos in autonomous systems. Consider the three-dimensional case:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \quad t \in R, x \in R^3, \tag{1}$$

where $f : R^3 \to R^3$ belongs to C^r $(r \geq 2)$, and $x_0 \in R^3$ is a hyperbolic equilibrium point. If the eigenvalues of the Jacobian $A = Df(x_0)$ satisfy the following:

(i) The A has three eigenvalues: a, $b \pm c\,\mathrm{i}$, where $a < 0$, $b > 0$, $|a| > b$. The a, b, and c are real constants;

(ii) There exists a homoclinic orbit based on x_0, then the system has horseshoe chaos.

Based on the Silnikov criterion, we study a system which is improved by the Proportion Integration Differentiation controller system:[20]

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -\beta y - \alpha z - \gamma x + \delta x^2 + \kappa x^2 z + \mu x^3. \end{cases} \tag{2}$$

The fixed points of Eq. (2) are

$$X_0(0,0,0), \quad X_1\left(\frac{-\delta - \sqrt{\delta^2 + 4\gamma\mu}}{2\mu}, 0, 0\right), \quad X_2\left(\frac{-\delta + \sqrt{\delta^2 + 4\gamma\mu}}{2\mu}, 0, 0\right).$$

Here we consider just $X_0(0,0,0)$. The linearized system on this point is

$$\dot{X} = AX, \tag{3}$$

where $X = [x, y, z]^{\mathrm{T}}$, A is the Jacobian matrix of Eq. (2) and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma & -\beta & -\alpha \end{bmatrix}. \tag{4}$$

The characteristic equation of Eq. (3) is

$$\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma = 0. \tag{5}$$

Letting $\lambda = \mu - \alpha/3$, we obtain

$$\mu^3 + p\mu + q = 0, \tag{6}$$

where

$$p = \beta - \frac{\alpha^2}{3}, \quad q = \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27}. \tag{7}$$

According to Cardano's formula, if

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0,$$
$$\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} - \sqrt[3]{\frac{q}{2} + \sqrt{\Delta}} < \min\left(\frac{\alpha}{3}, -\frac{2\alpha}{3}\right), \tag{8}$$

then equation (5) has a real root m, along with a dual conjugate complex root $u \pm \mathrm{i}\,\omega$. In addition, $m < 0$, $u > 0$, $|m| > u > 0$.

To this end, letting $\alpha = 0.45$, $\gamma = 2.1$, we obtain

$$0.0675 \leq \beta < 4.66667. \tag{9}$$

To meet all the conditions stated in the Silnikov theorem, one more condition is required to prove that system (2) has a homoclinic orbit jointing the equilibrium point $X_0(0,0,0)$. The homoclinic orbit is doubly asymptotic with respect to time t, so, assume for $t > 0$ that

$$\begin{cases} x(t) = \sum\limits_{k=1}^{+\infty} a_k \mathrm{e}^{kmt}, \\ y(t) = \sum\limits_{k=1}^{+\infty} b_k \mathrm{e}^{kmt}, \\ z(t) = \sum\limits_{k=1}^{+\infty} c_k \mathrm{e}^{kmt}, \end{cases} \tag{10}$$

where a_k, b_k, c_k are undetermined coefficients, m is a real root of Eq. (5), $k \in N$.

Substituting Eq. (10) into Eq. (2) and then comparing coefficients of the like terms in e^{kmt}, when $k = 1$, we obtain

$$[mI - A] \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = 0. \tag{11}$$

If $(a_1, b_1, c_1) = (0,0,0)$, then we can conclude $(a_k, b_k, c_k) = (0,0,0)$ for $k > 1$. Thus, $(a_1, b_1, c_1) \neq (0,0,0)$. So, we can determine a_1, b_1 and c_1 with a free parameter denoted by ξ.

For $k > 1$, we have

$$[kmI - A]\begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \delta \sum\limits_{i+j=1}^{k} a_i a_j + \kappa \sum\limits_{i+j+m=1}^{k} a_i a_j c_m + \mu \sum\limits_{i+j+m=1}^{k} a_i a_j a_m \end{bmatrix}. \tag{12}$$

So

$$\begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} = [kmI - A]^{-1}\begin{bmatrix} 0 \\ 0 \\ \delta \sum\limits_{i+j=1}^{k} a_i a_j + \kappa \sum\limits_{i+j+m=1}^{k} a_i a_j c_m + \mu \sum\limits_{i+j+m=1}^{k} a_i a_j a_m \end{bmatrix}. \tag{13}$$

Therefore, $x(t)$, $y(t)$, $z(t)$ have been uniquely determined for $t > 0$.

Then we make use of the transformation $\tau = -t$ with $t > 0$. So, equation (2) becomes

$$\begin{cases} x' = -y, \\ y' = -z, \\ z' = \beta y + \alpha z + \gamma x - \delta x^2 - \kappa x^2 z - \mu x^3, \end{cases} \tag{14}$$

where $x' = \mathrm{d}x/\mathrm{d}\tau$, $y' = \mathrm{d}y/\mathrm{d}\tau$, $z' = \mathrm{d}z/\mathrm{d}\tau$.

Completely similar to the case of $t > 0$, let

$$\begin{cases} x(\tau) = \sum\limits_{k=1}^{+\infty} a'_k \mathrm{e}^{-kn\tau}, \\ y(\tau) = \sum\limits_{k=1}^{+\infty} b'_k \mathrm{e}^{-kn\tau}, \\ z(\tau) = \sum\limits_{k=1}^{+\infty} c'_k \mathrm{e}^{-kn\tau}, \end{cases} \tag{15}$$

where a'_k, b'_k, c'_k are undetermined coefficients, and n is undetermined negative real. Substituting Eq. (15) into Eq. (14) and then comparing coefficients of the like terms in $\mathrm{e}^{-kn\tau}$, if $k = 1$, we have

$$[nI - A]\begin{bmatrix} a'_1 \\ b'_1 \\ c'_1 \end{bmatrix} = 0, \tag{16}$$

if $k > 1$, we have

$$[knI - A]\begin{bmatrix} a'_k \\ b'_k \\ c'_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \delta \sum\limits_{i+j=1}^{k} a_i a_j + \kappa \sum\limits_{i+j+m=1}^{k} a_i a_j c_m + \mu \sum\limits_{i+j+m=1}^{k} a_i a_j a_m \end{bmatrix}. \tag{17}$$

Then, letting $m = n$, the nonzero solution (a'_1, b'_1, c'_1) can be determined with a free parameter denoted by η. The other coefficients a'_k, b'_k, c'_k can be also uniquely determined; if $k > 1$ we have

$$\begin{bmatrix} a'_k \\ b'_k \\ c'_k \end{bmatrix} = [knI - A]^{-1}\begin{bmatrix} 0 \\ 0 \\ \delta \sum\limits_{i+j=1}^{k} a_i a_j + \kappa \sum\limits_{i+j+m=1}^{k} a_i a_j c_m + \mu \sum\limits_{i+j+m=1}^{k} a_i a_j a_m \end{bmatrix}. \tag{18}$$

Thus, we obtain the expression of the homoclinic orbit at the equilibrium point $X_0(0, 0, 0)$ as follows:

$$\begin{cases} x(t) = \begin{cases} \sum\limits_{k=1}^{+\infty} a_k \mathrm{e}^{kmt}, & t \geq 0, \\ \sum\limits_{k=1}^{+\infty} a'_k \mathrm{e}^{-kmt}, & t < 0, \end{cases} \\ y(t) = \begin{cases} \sum\limits_{k=1}^{+\infty} b_k \mathrm{e}^{kmt}, & t \geq 0, \\ \sum\limits_{k=1}^{+\infty} b'_k \mathrm{e}^{-kmt}, & t < 0, \end{cases} \\ z(t) = \begin{cases} \sum\limits_{k=1}^{+\infty} c_k \mathrm{e}^{kmt}, & t \geq 0, \\ \sum\limits_{k=1}^{+\infty} c'_k \mathrm{e}^{-kmt}, & t < 0. \end{cases} \end{cases} \tag{19}$$

Where ξ and η are determined by

$$\begin{aligned} &\sum_{k=1}^{+\infty} a_k = \sum_{k=1}^{+\infty} a'_k, \quad \sum_{k=1}^{+\infty} b_k = \sum_{k=1}^{+\infty} b'_k, \\ &\sum_{k=1}^{+\infty} c_k = \sum_{k=1}^{+\infty} c'_k. \end{aligned} \tag{20}$$

Next we construct the one-dimensional and two-dimensional manifolds corresponding to the eigenvalues of the equilibrium point $X_0(0,0,0)$. We suppose that the eigenvalues of the system (2) are λ_1, λ_2, λ_3 at the origin, λ_1 is a real number, λ_2 and λ_3 are conjugate complex roots. The vector representation of system (2) is:

$$\mathrm{d}X(t)/\mathrm{d}t = AX + B(X)X, \tag{21}$$

where

$$X \in R^3, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma & -\beta & -\alpha \end{bmatrix},$$

$$B(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta x + \mu x^2 & 0 & \kappa x^2 \end{pmatrix}.$$

We suppose that the series expression of the one-dimensional manifold corresponding to λ_1 is

$$X(t) = \sum_{k=0}^{\infty} d_k \mathrm{e}^{k\lambda_1 t}, \tag{22}$$

where $d_k \in R^3$,$d_k = (a_k, b_k, c_k)$.

The differential quotient of Eq. (22) is

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \sum_{k=1}^{\infty} k\lambda_1 d_k \mathrm{e}^{k\lambda_1 t}. \tag{23}$$

Substituting Eq. (23) into Eq. (21), then comparing the result with Eq. (23), we get

$$Ad_0 + B(d_0)d_0 = 0, \quad (k=0), \tag{24a}$$

$$[\lambda_1 I - A - 2B(d_0)]\, d_1 = 0, \quad (k=1), \tag{24b}$$

$$[k\lambda_1 I - A - 2B(d_0)]\, d_k = \sum_{i=1}^{k-1} B(d_i) d_{k-i}, \ (k>1), \tag{24c}$$

From Eq. (24a), we know that d_0 is the equilibrium point $X_0(0,0,0)$, since it is hyperbolic; equation (24b) has the real solutions:

$$d_1 = \mu_{\lambda_1} d^1, \quad \mu_{\lambda_1} \in [-k_{\lambda_1}, \quad k_{\lambda_1}], \tag{25}$$

where d^1 is the eigenvectors of the eigenvalue λ_1, k_{λ_1} is the maximum constant to keep Eq. (22) convergent.[21] If $k>1$, $\mathrm{Det}\,[k\lambda_1 I - A - 2B(d_0)] \neq 0$, we can obtain:

$$d_k = [k\lambda_1 I - A - 2B(d_0)]^{-1} \sum_{i=1}^{k-1} B(d_i) d_{k-i}. \tag{26}$$

Since the two-dimensional manifold corresponds to λ_2, λ_3, we assume

$$X(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_{ij} \mathrm{e}^{(i\lambda_2 + j\lambda_3)t}, \tag{27}$$

where $d_{ij} \in R^3$, $i,j \in N$.

The differential quotient of Eq. (27) is

$$\begin{aligned} \frac{\mathrm{d}X(t)}{\mathrm{d}t} = &\sum_{i=0}^{\infty} i\lambda_2 d_{i0} \mathrm{e}^{i\lambda_2 t} + \sum_{j=0}^{\infty} j\lambda_3 d_{0j} \mathrm{e}^{j\lambda_3 t} \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (i\lambda_2 + j\lambda_3) d_{ij} \mathrm{e}^{(i\lambda_2 + j\lambda_3)t}. \end{aligned} \tag{28}$$

Substituting Eq. (27) into Eq. (21), then comparing the result with Eq. (28), we get

$$Ad_{00} + B(d_{00})d_{00} = 0, \quad (i=0, j=0), \tag{29a}$$

$$[i\lambda_2 - A - 2B(d_{00})]\, d_{i0} = 0, \quad (i>0, j=0), \tag{29b}$$

$$[\, j\lambda_3 - A - 2B(d_{00})]\, d_{0j} = 0 \quad (i=0, j>0), \tag{29c}$$

$$[i\lambda_2 + j\lambda_3 - A - 2B(d_{00})]\, d_{ij} = \sum_{i_1=0}^{i} \sum_{j_1=0}^{i} B(d_{i_1 j_1}) d_{i-i_1 j-j_1}, \quad (i>0, j>0). \tag{29d}$$

From Eq. (29a) we know that d_{00} is the equilibrium point $X_0(0,0,0)$, and from Eqs. (29b) and (29c) we get

$$d_{i0} = \begin{cases} \mu_{\lambda_2} d(\lambda_2) \quad \mu_{\lambda_2} \in [-k_{\lambda_2}, k_{\lambda_2}] & (i=1), \\ [i\lambda_2 I - A - 2B(d_0)]^{-1} \sum_{k=1}^{i-1} B(d_k) d_{i-k} & (i>1), \end{cases} \tag{30}$$

$$d_{0j} = \begin{cases} \mu_{\lambda_3} d(\lambda_3) \quad \mu_{\lambda_2} \in [-k_{\lambda_3}, k_{\lambda_3}] & (j=1), \\ [j\lambda_3 I - A - 2B(d_0)]^{-1} \sum_{k=1}^{j-1} B(d_k) d_{j-k} & (j>1), \end{cases} \tag{31}$$

where $d(\lambda_2)$ and $d(\lambda_3)$ are the eigenvectors of λ_2 and λ_3, respectively, k_{λ_2}, k_{λ_3} are the maximum constants to keep Eq. (27) convergent.[21] From Eq. (29d), if $i>0, j>0$, we obtain

$$d_{ij} = [(i\lambda_2 + j\lambda_3)I - A - 2B(d_{00})]^{-1} \times \sum_{i_1=0}^{i} \sum_{j_1=0}^{j} B(d_{i_1 j_1}) d_{i-i_1 j-j_1}. \tag{32}$$

By this method, we obtain the series expression of one-dimensional and two-dimensional manifolds of the homoclinic orbit near $X_0(0,0,0)$ and the homoclinic orbit of Silnikov's type, as shown in Fig. 1.

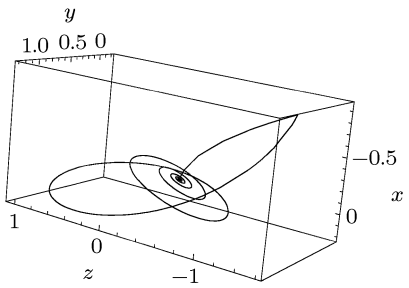


Fig. 1. The homoclinic orbit of system (2).

The one-dimensional manifold of the homoclinic orbit is

$$\begin{aligned} x_1[t] &= -0.662768\mathrm{e}^{-0.857633t} - 0.243887\mathrm{e}^{-1.71527t} - 0.0457519\mathrm{e}^{-2.5729t} - 0.00640864\mathrm{e}^{-3.43053t} \\ &\quad - 0.000779778\mathrm{e}^{-4.28817t} - 0.0000877899\mathrm{e}^{-5.1458t} - 9.38244\times10^{-6}\mathrm{e}^{-6.00343t}, \\ y_1[t] &= 0.568412\mathrm{e}^{-0.857633t} + 0.418331\mathrm{e}^{-1.71527t} - 0.117715\mathrm{e}^{-2.5729t} + 0.0219851\mathrm{e}^{-3.43053t} \\ &\quad - 0.00334382\mathrm{e}^{-4.28817t} + 0.000451749\mathrm{e}^{-5.1458t} - 0.0000563269\mathrm{e}^{-6.00343t}, \\ z_1[t] &= -0.487489\mathrm{e}^{-0.857633t} - 0.717548\mathrm{e}^{-1.71527t} - 0.302869\mathrm{e}^{-2.5729t} - 0.0754204\mathrm{e}^{-3.43053t} \\ &\quad - 0.0143388\mathrm{e}^{-4.28817t} - 0.00232461\mathrm{e}^{-5.1458t} - 0.000338154\mathrm{e}^{-6.00343t}. \end{aligned} \tag{33}$$

The two-dimensional manifold of the homoclinic orbit is

$$\begin{aligned} x_2[t] &= 2.1702\times10^{-6}\mathrm{e}^{0.815268t}\cos[6.20588t] - 3.77704\times10^{-6}\mathrm{e}^{0.815268t}\sin[6.20588t] \\ &\quad + 0.0000895194\mathrm{e}^{0.611451t}\cos[4.65441t] + 0.000318028\mathrm{e}^{0.611451t}\sin[4.65441t] \\ &\quad - 0.0206048\mathrm{e}^{0.407634t}\cos[3.10294t] - 0.0108796\mathrm{e}^{0.407634t}\sin[3.10294t] \\ &\quad + 0.628716\mathrm{e}^{0.203817t}\cos[1.55147t] - 0.16809\mathrm{e}^{0.203817t}\sin[1.55147t] + 0.189399\mathrm{e}^{0.407634t} \\ &\quad - 1.00239\mathrm{e}^{0.611451t}\cos[1.55147t] + 0.257116\mathrm{e}^{0.611451t}\sin[1.55147t] - 0.032343\mathrm{e}^{0.815268t} \\ &\quad + 0.0116725\mathrm{e}^{0.815268t}\cos[3.10294t] - 0.0159559\mathrm{e}^{0.815268t}\sin[3.10294t] \\ &\quad - 0.000454358\mathrm{e}^{1.01909t}\cos[4.65441t] + 0.000143133\mathrm{e}^{1.01909t}\sin[4.65441t] \\ &\quad - 0.7332536\mathrm{e}^{1.01909t}\cos[1.55147t] - 0.0560748\mathrm{e}^{1.01909t}\sin[1.55147t], \\ y_2[t] &= -0.0000216706\mathrm{e}^{0.815268t}\cos[6.20588t] - 0.0000165474\mathrm{e}^{0.815268t}\sin[6.20588t] \\ &\quad + 0.00153497\mathrm{e}^{0.611451t}\cos[4.65441t] - 0.000222202\mathrm{e}^{0.611451t}\sin[4.65441t] \\ &\quad - 0.042158\mathrm{e}^{0.407634t}\cos[3.10294t] + 0.0595008\mathrm{e}^{0.407634t}\sin[3.10294t] \\ &\quad - 0.132643\mathrm{e}^{0.203817t}\cos[1.55147t] - 1.00968\mathrm{e}^{0.203817t}\sin[1.55147t] + 0.9108055\mathrm{e}^{0.407634t} \end{aligned}$$

$$
\begin{aligned}
&+ 0.337618\mathrm{e}^{0.611451t}\cos[1.55147t] + 0.312732\mathrm{e}^{0.611451t}\sin[1.55147t] - 0.0263682\mathrm{e}^{0.815268t}\\
&- 0.0399938\mathrm{e}^{0.815268t}\cos[3.10294t] - 0.0492272\mathrm{e}^{0.815268t}\sin[3.10294t]\\
&+ 0.00020317\mathrm{e}^{1.01909t}\cos[4.65441t] + 0.00226064\mathrm{e}^{1.01909t}\sin[4.65441t]\\
&- 0.120886\mathrm{e}^{1.01909t}\cos[1.55147t] - 0.00555294\mathrm{e}^{1.01909t}\sin[1.55147t],\\
z_2[t] = &-0.000120358\mathrm{e}^{0.815268t}\cos[6.20588t] + 0.000120994\mathrm{e}^{0.815268t}\sin[6.20588t]\\
&- 0.0000956576\mathrm{e}^{0.611451t}\cos[4.65441t] - 0.00728024\mathrm{e}^{0.611451t}\sin[4.65441t]\\
&+ 0.167442\mathrm{e}^{0.407634t}\cos[3.10294t] + 0.155068\mathrm{e}^{0.407634t}\sin[3.10294t]\\
&- 1.59354\mathrm{e}^{0.203817t}\cos[1.55147t] + 0.0314716\mathrm{e}^{0.407634t} + 0.69163\mathrm{e}^{0.611451t}\cos[1.55147t]\\
&- 0.332582\mathrm{e}^{0.611451t}\sin[1.55147t] - 0.0214971\mathrm{e}^{0.815268t} - 0.185355\mathrm{e}^{0.815268t}\cos[3.10294t]\\
&+ 0.0839652\mathrm{e}^{0.815268t}\sin[3.10294t] + 0.010729\mathrm{e}^{1.01909t}\cos[4.65441t]\\
&+ 0.00135814\mathrm{e}^{1.01909t}\sin[4.65441t] - 0.7011809\mathrm{e}^{1.01909t}\cos[1.55147t]\\
&+ 0.181893\mathrm{e}^{1.01909t}\sin[1.55147t]. \qquad (34)
\end{aligned}
$$

3. Period-doubling bifurcation

Based on the Silnikov criterion, we know that system (2) has Smale horseshoes. From Eq. (9), it is difficult for people to find the accurate range of the parameter's value corresponding to chaos. It is known that period-doubling bifurcation is a route to chaos. So the next task is to suggest an analytic approach which will yield a critical value of β corresponding to first period-doubling bifurcation, which makes it easier for people to find the exact range of the parameter's value corresponding to chaos.

The idea of the method is to use centre manifold theory to the upflow equation of the centre manifold, then the method of multiple scales combined with Floque theory is used to determine the value of β for the first period-doubling bifurcation.

In order to approximate the centre manifold, we set

$$
\begin{aligned}
z = &\, c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2 + c_6x^3\\
&+ c_7x^2y + c_8xy^2 + c_9y^3 + O(4). \qquad (35)
\end{aligned}
$$

Differentiating Eq. (35) with respect to time t and simultaneously considering Eq. (2), we obtain

$$
\begin{aligned}
&- \beta y - \alpha z - \gamma x + \delta x^2 + \kappa x^2 z + \mu x^3\\
&= (c_1 + 2c_3x + c_4y + 3c_6x^2 + 2c_7xy + c_8y^2)x'\\
&\quad + (c_2 + c_4x + 2c_5y + c_7x^2\\
&\quad + 2c_8xy + 3c_9y^2)y' + O(4). \qquad (36)
\end{aligned}
$$

Comparing coefficients of x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, we have

$$
\begin{aligned}
&\gamma + \alpha c_1 + c_1c_2 = 0,\\
&\beta + c_1 + \alpha c_2 + c_2^2 = 0,\\
&-\delta + \alpha c_3 + c_2c_3 + c_1c_4 = 0,\\
&c_4 + \alpha c_5 + 3c_2c_5 = 0,\\
&2c_3 + \alpha c_4 + 2c_2c_4 + 2c_1c_5 = 0,\\
&-\mu - \kappa c_1 + c_3c_4 + \alpha c_6 + c_2c_6 + c_1c_7 = 0,\\
&2c_5^2 + c_8 + \alpha c_9 + 4c_2c_9 = 0,\\
&-\kappa c_2 + c_4^2 + 2c_3c_5 + 3c_6 + \alpha c_7 + 2c_2c_7 + 2c_1c_8 = 0,\\
&3c_4c_5 + 2c_7 + \alpha c_8 + 3c_2c_8 + 3c_1c_9 = 0. \qquad (37)
\end{aligned}
$$

Letting $\alpha = 0.45$, $\gamma = 2.1$, $\delta = 3.3$, $\kappa = 0.78$, $\mu = 0.03$, solving for c_1–c_9 and expanding the result for small β, we obtain

$$
\begin{aligned}
c_1 &= -1.59493 - 0.0336589\beta - 0.223648\beta^2\\
&\quad + 0.0240119\beta^3 - 0.00101602\beta^4,\\
c_2 &= 1.05779 - 0.376656\beta + 0.0318751\beta^2,\\
c_3 &= 1.67686 - 0.336425\beta + 0.149429\beta^2,\\
c_4 &= -0.935603 - 0.0101501\beta + 0.0408121\beta^2,\\
c_5 &= 0.230362 + 0.116961\beta - 0.0025709\beta^2,\\
c_6 &= -0.717175 + 0.471962\beta - 0.283089\beta^2,\\
c_7 &= -0.421127 + 0.304737\beta - 0.0474081\beta^2,\\
c_8 &= 0.362494 - 0.0717868\beta - 0.0681536\beta^2,\\
c_9 &= -0.0330858 - 0.125389\beta + 0.02981\beta^2. \qquad (38)
\end{aligned}
$$

Thus, we obtain the upflow equation of the centre

manifold of system (2)

$$\begin{cases} \dot{x} = y, \\ \dot{y} = c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 \\ \qquad + c_6 x^3 + c_7 x^2 y + c_8 x y^2 + c_9 y^3. \end{cases} \tag{39}$$

Using the method of multiple scales, we get the average equation of Eq. (39)

$$\dot{a} = \frac{a c_2}{2} + \frac{a^3 c_7}{8} + \frac{3}{8} a^3 c_1^2 c_9, \tag{40a}$$

$$\dot{\theta} = \frac{3a^2 c_6}{8c_1} + \frac{1}{8} a^2 c_1 c_8. \tag{40b}$$

From Eq. (40a), we get the stationary solution a^*,

$$a^* = 2.54488 - 0.923697\beta + 0.109126\beta^2, \tag{41}$$

and the following approximation for the limit cycle

$$\begin{cases} x_0 = a^* \cos(\omega t + \theta), \\ y_0 = a^* \sin(\omega t + \theta), \end{cases} \tag{42}$$

where $\omega = -c_1$,

$$\theta = \left(\frac{3(a^*)^2 c_6}{8c_1} + \frac{1}{8}(a^*)^2 c_1 c_8 \right) t + \theta_0.$$

The third formula of Eq. (2) governs the stability of the limit cycle and of the centre manifold which contains it. Note that the limit cycle cannot period-double as long as it lies in the centre manifold, since the latter is two-dimensional and trajectories cannot self-intersect. Thus substitution of Eq. (42) into Eq. (35) gives an expression for z on the limit cycle, we call it z_0. To investigate its stability, we set

$$z = z_0 + \sigma z, \tag{43}$$

and linearize the variation σz; we have

$$\sigma \dot{z} = (\kappa(a^* \cos(\omega t + \theta))^2 - \alpha)\sigma z. \tag{44}$$

Equation (44) has the general solution

$$\sigma z(t) = \sigma z(0) \exp \int_0^t (\kappa(a^* \cos(\omega t + \theta))^2 - \alpha) \mathrm{d}t. \tag{45}$$

The transition from stable to unstable occurs when $\sigma z(T) = \sigma z(0)$, where T is the period of the limit cycle oscillation. Thus β is given by

$$\int_0^t (\kappa(a^* \cos(\omega t + \theta))^2 - \alpha) \mathrm{d}t = 0. \tag{46}$$

So the critical value of β corresponding to the first period-doubling bifurcation is

$$\beta = 2.26791. \tag{47}$$

Linearization of system (2) about the origin shows that it is stable for $\beta > 4.66667$ and unstable for $\beta < 4.66667$. The Hopf bifurcation occurs at $\beta = 4.66667$ in a centre manifold which is tangential to the x–y plane at the origin. So we can conclude that the range of β corresponding to chaos is

$$\beta < 2.26791. \tag{48}$$

Thus we obtain the exact range of β corresponding to chaos by intersecting Eq. (48) with Eq. (9)

$$0.0675 < \beta < 2.26791. \tag{49}$$

4. Numerical simulation of chaos

Results from the theoretical analysis give us an exact range of the parameter's value. In order to prove that the chaos occurs in the exact range obtained by us, the space trajectory and Lyapunov exponent are given by numerical simulation. Let $\alpha = 0.45$, $\gamma = 2.1$, $\delta = 3.3$, $\kappa = 0.78$, $\mu = 0.03$, but let β gradually vary. Numerical simulation demonstrates that there is a limit cycle at $\beta = 2.8$, the limit cycle period-doubles until $\beta = 2.28$, and period-doubling again occurs at $\beta = 2.18$; finally, it enters the chaos phase at $\beta = 2.099$, as shown in Fig. 2.

In order to further prove that there is an attractor existing in the system, we give the Lyapunov characteristic exponent spectrum[22] as shown in Fig. 3. The maximum Lyapunov exponent is greater than zero ($l_1 = 0.148513$), and the Hauadorff dimension is 2.32375.

Results from the numerical simulation show that the limit cycle period-doubles until $\beta = 2.28$. Considering the result of theoretical analysis at $\beta = 2.26791$, the numerical simulation results agree well. It is proved that the chaos occurs in the exact range which we obtained from the results from the theoretical analysis.

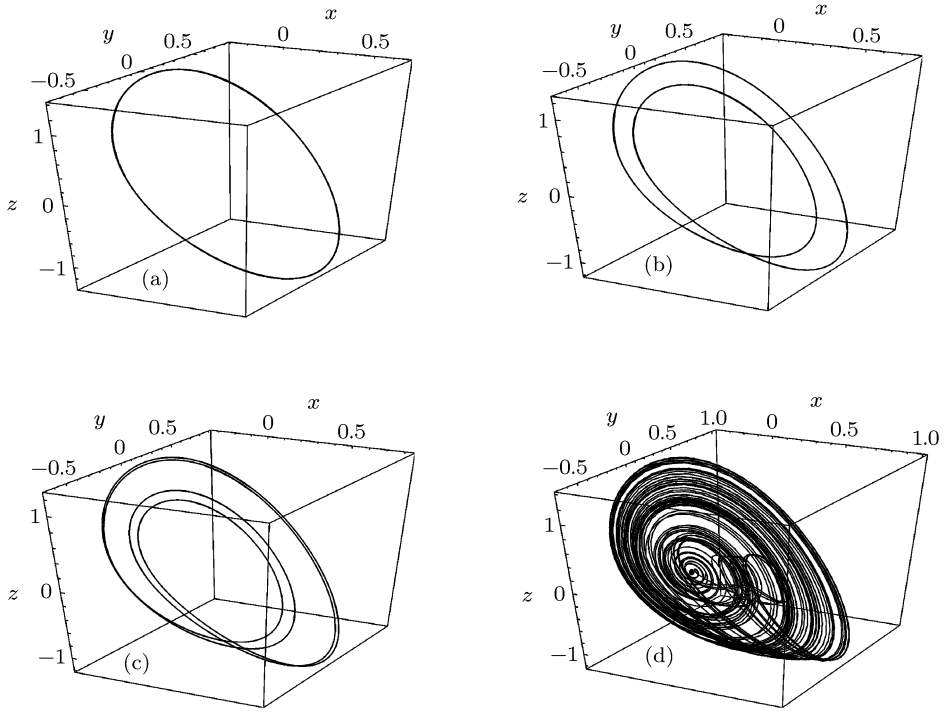


Fig. 2. The process of chaos: (a) $\beta = 2.8$; (b) $\beta = 2.28$; (c) $\beta = 2.18$; (d) $\beta = 2.099$.

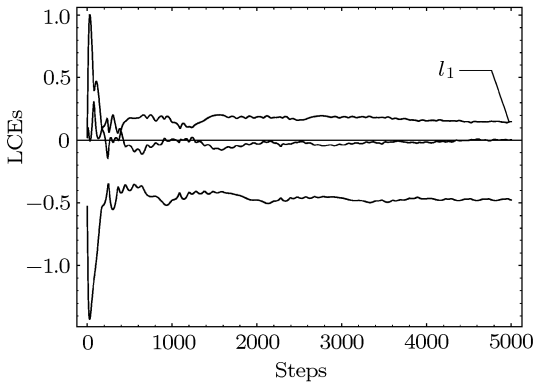


Fig. 3. The LCE spectrum of the system.

5. Conclusions

Based on the Silnikov criterion, the strategy of obtaining the exact range of the parameter's value corresponding to chaos is presented by using the centre manifold theory and multiple scales method combined with Floque theory. The series expression of the homoclinic orbit of the system is provided by calculating the manifold near the equilibrium point. In contrast to the result of the numerical simulation, it reveals a good prediction for the critical value of β corresponding to the first period-doubling bifurcation. This helps people find the chaos of a system more easily, and also facilitates research into chaotic behaviour.

The strategy presented in this work is sufficiently general, so it would be possible to apply the method to more complicated three-dimensional autonomous systems and other non-autonomous systems with excitations. In addition, the critical value of the parameter corresponding to the second period-doubling bifurcation will greatly improve the precision of the range by this method; these will be the topics of further research.

References

[1] Hu J B, Han Y and Zhao L D 2008 *Acta Phys. Sin.* **57** 7522 (in Chinese)

[2] Lü L, Li G and Chai Y 2008 *Acta Phys. Sin.* **57** 7517 (in Chinese)

[3] Yang C Y and Tang G N 2009 *Acta Phys. Sin.* **58** 143 (in Chinese)

[4] Zhang Q C, Wang W and Liu F H 2008 *Chin. Phys.* B **17** 4123

[5] Liu Z C 2009 *Chin. Phys.* B **18** 636
[6] Zhang C X and Yu S M 2009 *Acta Phys. Sin.* **58** 120 (in Chinese)
[7] Sprott J C 1994 *Phys. Rev.* E **50** R647
[8] Sprott J C 2000 *Am. J. Phys.* **68** 858
[9] Sprott J C 2000 *Phys. Lett.* A **266** 19
[10] Xu P C and Jing Z J 2000 *Chaos, Solitons and Fractals* **11** 853
[11] Tian S Z, Chen G R and Yang Q G 2004 *Chaos, Solitons and Fractals* **19** 985
[12] Zhu S Q, Yang M, Zhang X H and Min L Q 2005 *J. Univ. Sci. Technol.* Beijing **27** 635 (in Chinese)
[13] Zhang Q C, Tian R L and Wang W 2008 *Acta Phys. Sin* **57** 2799 (in Chinese)
[14] Rand R H 1989 *Mech. Res. Commun.* **16** 11
[15] Nayfeh A H and Balachandran B 1990 *Mech. Res. Commun.* **17** 191
[16] Shilnikov L P 1965 *Sov. Phys. Dokl.* **6** 163
[17] Chen Y S 1993 *Bifurcation and Chaos Theory of Nonlinear Vibration Systems* (Tianjin: Higher Education Press) p. 223 (in Chinese)
[18] Wang H L and Zhang Q C 2002 *Theory and Application of the Nonlinear Dynamics* (Tianjin: Tianjin Technology Press) p. 100 (in Chinese)
[19] Nayfeh A H and Mook D T 1979 *Nonlinear Oscillations* (New York: Wiley) p. 258
[20] Cui F S, Chew C H, Xu J X and Cai Y L 1997 *Nonlinear Dynam.* **12** 251
[21] Li Y H and Zhu S M 2006 *Nonlinear Dynam.* **29** 1155
[22] Chen Z M, Djidjeli K and Price W G 2006 *Math. Appl. Comput.* **174** 982