The s-parameterized Weyl–Wigner correspondence in the entangled form and its applications*

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Based on the theory of integration within s-ordering of operators and the bipartite entangled state representation we introduce s-parameterized Weyl–Wigner correspondence in the entangled form. Some of its applications in quantum optics theory are presented as well.

Keywords: s-parameterized Wigner operator, s-parameterized Weyl–Wigner correspondence, integration within s-ordered product of operators technique

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1. Introduction

In the history of the development of quantum optics, the theory of phase space quantum mechanics has played an essential role. With respect to the general phase space description of photon-matter atom interactions, especially in nonlinear media, and their detection, a generalization of the operator-ordering (s-ordering) technique interpolating between normal and antinormal orderings is appropriate; this was the pioneer work of Cahill and Glauber,[1,2] who first introduced the s-parameterized phase-space distribution function for describing quantum states and studying ordering of photonic field operators: $s = 1$ for normal ordering, $s = 0$ for Weyl ordering,[3,4] and $s = -1$ for antinormal ordering. The normal ordering is related to the photodetection process, the antinormal ordering is related to the detection of field by emitting a stimulated photon to the field, and the Weyl ordering is related to the detection of field by means of a scattering process, in which one photon is absorbed and another one is emitted.

In a preceding paper,[5] we invented the technique of integration within the s-ordered product of operators (IWSOP) which can unify the three integration techniques (within normal-, Weyl- and antinormal-ordering of operators) as one. For example, the s-ordered operator expansion (denoted by $\circledS\cdots\circledS$) formula of density operators is

$$\rho = \frac{2}{1-s}\int \frac{\mathrm{d}^2\beta}{\pi}\,\langle -\beta|\,\rho\,|\beta\rangle\,\circledS \exp\left[\frac{2}{s-1}\left(s|\beta|^2 - \beta^* a + \beta a^\dagger - a^\dagger a\right)\right]\circledS, \tag{1}$$

where bosonic annihilation operator a and creation operator $a^\dagger$ satisfy $\left[a, a^\dagger\right] = 1$, $|\beta\rangle = \exp(-|\beta|^2/2 + \beta a^\dagger)\,|0\rangle$ is the well-known coherent state.[6,7] When $s = 0$, $\circledS\cdots\circledS$ corresponds to the Weyl ordering $\vdots\cdots\vdots$; for $s = -1$, $\circledS\cdots\circledS$ corresponds to the antinormal ordering denoted as $\vdots\ \vdots$, and for $s = 1$, it corresponds to the normal ordering $::$.

The s-parameterized quasiprobability distribution of a density operator is summarized in Schleich's book[8]

$$W(\alpha, s) = \int \frac{\mathrm{d}^2\xi}{2\pi^2}\,\mathrm{Tr}(\rho\,\mathrm{e}^{\xi a^\dagger - \xi^* a})\,\mathrm{e}^{\xi^*\alpha - \xi\alpha^* + s|\xi|^2/2} = \mathrm{Tr}\left[\rho\Delta_s(\alpha)\right]. \tag{2}$$

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From Eq. (2) we define

$$\Delta_s(\alpha) = \int \frac{\mathrm{d}^2\xi}{2\pi^2} \exp\left(\frac{s|\xi|^2}{2} + \xi a^\dagger - \xi^* a - \xi\alpha^* + \xi^*\alpha\right) \tag{3}$$

as the generalized Wigner operator for s-parameterized distributions. By performing the integration for $s < 1$ we obtain its normally ordered form

$$\Delta_s(\alpha) = \frac{1}{(1-s)\pi} {:}\exp\left[\frac{-2}{1-s}\left(a^\dagger - \alpha^*\right)(a-\alpha)\right]{:}, \tag{4}$$

and by using the completeness of coherent state we can have

$$4\pi \,\mathrm{Tr}\left[\Delta_{-s}(\alpha')\,\Delta_{-s}(\alpha)\right] = \delta^{(2)}(\alpha'-\alpha) \equiv \delta(\alpha'-\alpha)\,\delta(\alpha'^* - \alpha^*). \tag{5}$$

It then follows that the corresponding s-parameterized quantization scheme (the correspondence between the classical s-related function $\mathfrak{P}(\alpha, s)$ and the operator ρ) reads

$$\rho = 2\int \mathrm{d}^2\alpha \Delta_s(\alpha)\,\mathfrak{P}(\alpha, s). \tag{6}$$

In this paper we want to extend Eq. (3) to the two-mode entangled case, because in recent years quantum entanglement has been widely used in quantum information and quantum optics, especially the bipartite entangled state representation of continuous variables. As one can see a little later, the s-parameterized Wigner operator in the entangled form will expose the physical meaning of the Wigner function for the entangled system clearly, thus the s-parameterized Weyl–Wigner correspondence can be established.

2. The s-parameterized Wigner operator in the entangled form

Similar in spirit to the introduction of the single-mode Wigner operator in Eq. (3), we introduce the following form of Wigner operator for the two-mode case

$$\begin{aligned}\Delta_s(\sigma,\gamma) = & \int \frac{\mathrm{d}^2\eta\mathrm{d}^2\xi}{4\pi^4} \exp\left[\frac{s}{2}|\eta|^2 + \eta a^\dagger - \eta^* a + \frac{1}{2}(\xi^* - \eta)\gamma^* + \frac{1}{2}(\eta^* - \xi)\gamma\right] \\ & \times \exp\left[\frac{s}{2}|\xi|^2 + \xi b^\dagger - \xi^* b + \frac{1}{2}(\eta^* + \xi)\sigma - \frac{1}{2}(\eta + \xi^*)\sigma^*\right],\end{aligned} \tag{7}$$

where η and ξ are complex numbers, $a^\dagger$, a and $b^\dagger$, b are two-mode Boson operators. By means of the identity[9]

$$\exp A \exp B = \exp\left(A + B + \frac{1}{2}[A, B]\right), \tag{8}$$

which holds whenever the commutator $[A, B]$ commutes with both A and B, and the technique of integration within ordered product (IWOP) of operators,[10,11] we put the exponential in Eq. (7) in the normally ordered form for $s < 1$ and arrive at

$$\begin{aligned}\Delta_s(\sigma,\gamma) = & \frac{1}{(1-s)^2\pi^2} {:}\exp\left[-\frac{1}{1-s}\left(\sigma - a + b^\dagger\right)\left(\sigma^* - a^\dagger + b\right)\right. \\ & \left. - \frac{1}{1-s}\left(\gamma - a - b^\dagger\right)\left(\gamma^* - a^\dagger - b\right)\right]{:},\end{aligned} \tag{9}$$

where we have used the integral formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp\left(\zeta|z|^2 + \xi z + \eta z^*\right) = -\frac{1}{\zeta}\mathrm{e}^{-\xi\eta/\zeta}, \quad \mathrm{Re}(\zeta) < 0. \tag{10}$$

Equation (9) is just the s-parameterized Wigner operator in the entangled form.

In fact, by introducing

$$\alpha = \frac{\gamma + \sigma}{2}, \quad \beta = \frac{\gamma^* - \sigma^*}{2}, \tag{11}$$

or

$$\gamma = \alpha + \beta^*, \quad \sigma = \alpha - \beta^*, \tag{12}$$

then substituting it into Eq. (9) and comparing with Eq. (4) we see

$$\begin{aligned} \Delta_s(\sigma,\gamma) &= \frac{1}{(1-s)^2\pi^2} {:}\exp\left\{\frac{-2}{1-s}[(a^\dagger-\alpha^*)(a-\alpha) \right. \\ &\quad \left. + (b^\dagger-\beta^*)(b-\beta)]\right\}{:} \\ &= \Delta_s(\alpha)\,\Delta_s(\beta). \end{aligned} \tag{13}$$

It then follows from Eqs. (5) and (13) that

$$\begin{aligned} &\mathrm{Tr}\left[\Delta_{-s}(\sigma',\gamma')\,\Delta_s(\sigma,\gamma)\right] \\ &= \frac{1}{4\pi^2}\delta^{(2)}(\sigma'-\sigma)\,\delta^{(2)}(\gamma'-\gamma). \end{aligned} \tag{14}$$

Thus we obtain that the generalized s-parameterized Weyl–Wigner correspondence for the two-mode case is

$$\rho = \int \mathrm{d}^2\sigma \int \mathrm{d}^2\gamma \Delta_s(\sigma,\gamma)\,\mathfrak{P}(\sigma,\gamma,s) \tag{15}$$

and

$$\mathfrak{P}(\sigma,\gamma,s) = 4\pi^2\,\mathrm{Tr}\left[\rho\Delta_{-s}(\sigma,\gamma)\right]. \tag{16}$$

The merit of the form of $\Delta_s(\sigma,\gamma)$ is: when performing the integral with respect to $\mathrm{d}^2\sigma$ for $\Delta_s(\sigma,\gamma)$, we have

$$\begin{aligned} &\int \mathrm{d}^2\sigma \Delta_s(\sigma,\gamma) \\ &= \frac{1}{(1-s)\pi}{:}\exp\left[-\frac{1}{1-s}\left(\gamma - a - b^\dagger\right)\right. \\ &\quad \left.\times\left(\gamma^* - a^\dagger - b\right)\right]{:}, \end{aligned} \tag{17}$$

as $s = 0$,

$$\int \mathrm{d}^2\sigma \Delta_{s=0}(\sigma,\gamma) = \frac{1}{\pi}\left|\xi\right\rangle\left\langle\xi\right|_{\xi=\gamma}, \tag{18}$$

where

$$\begin{aligned} \left|\xi\right\rangle &= \exp\left[-\frac{1}{2}\left|\xi\right|^2 + \xi a^\dagger + \xi^* b^\dagger - a^\dagger b^\dagger\right]\left|00\right\rangle, \\ \xi &= \xi_1 + \mathrm{i}\,\xi_2 \end{aligned} \tag{19}$$

is the common eigenstate of two quadratures, $X_1 + X_2$ and $P_1 - P_2$,[12]

$$\begin{aligned} (X_1 + X_2)\left|\xi\right\rangle &= \sqrt{2}\,\xi_1\left|\xi\right\rangle, \\ (P_1 - P_2)\left|\xi\right\rangle &= \sqrt{2}\,\xi_2\left|\xi\right\rangle, \end{aligned} \tag{20}$$

where $X_1 = (a + a^\dagger)/\sqrt{2}$, $X_2 = (b + b^\dagger)/\sqrt{2}$, $P_1 = (a - a^\dagger)/\mathrm{i}\sqrt{2}$ and $P_2 = (b - b^\dagger)/\mathrm{i}\sqrt{2}$. On the other hand, carrying out the integral with respect to $\mathrm{d}^2\gamma$ for $\Delta_s(\sigma,\gamma)$ yields

$$\begin{aligned} &\int \mathrm{d}^2\gamma \Delta_s(\sigma,\gamma) \\ &= \frac{1}{(1-s)\pi}{:}\exp\left[-\frac{1}{1-s}\left(\sigma - a + b^\dagger\right)\right. \\ &\quad \left.\times\left(\sigma^* - a^\dagger + b\right)\right]{:}; \end{aligned} \tag{21}$$

as $s = 0$

$$\int \mathrm{d}^2\gamma \Delta_{s=0}(\sigma,\gamma) = \frac{1}{\pi}\left|\eta\right\rangle\left\langle\eta\right|_{\eta=\sigma}, \tag{22}$$

where

$$\begin{aligned} \left|\eta\right\rangle &= \exp\left[-\frac{1}{2}\left|\eta\right|^2 + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger\right]\left|00\right\rangle, \\ \eta &= \eta_1 + \mathrm{i}\,\eta_2 \end{aligned} \tag{23}$$

is the two-mode entangled state satisfying the eigenvector equations

$$\begin{aligned} (X_1 - X_2)\left|\eta\right\rangle &= \sqrt{2}\eta_1\left|\eta\right\rangle, \\ (P_1 + P_2)\left|\eta\right\rangle &= \sqrt{2}\eta_2\left|\eta\right\rangle. \end{aligned} \tag{24}$$

Obviously, $\left|\eta\right\rangle$ is just the simultaneous eigenvector of the two-particle relative position $X_1 - X_2$ and the total momentum $P_1 + P_2$ in Fock space.[12] Thus for $s = 0$, the marginal distributions of the s-parameterized Wigner function can expose quantum entanglement clearly for describing entangled systems.

3. The s-ordered expansion of $\Delta_s(\sigma,\gamma)$

Further, we consider the s-ordered expansion of the generalized Wigner operator itself. Using Eqs. (1) and (9), we have

$$\Delta_s(\sigma,\gamma) = \frac{4}{(1-s)^4\pi^2}\circledS \int \frac{\mathrm{d}^2\beta}{\pi}\exp\left[-\frac{(\sigma^* - \gamma^* + 2b)\,\beta^* + \left(\gamma - \sigma - 2b^\dagger\right)\beta}{s-1}\right]$$

$$
\begin{aligned}
&\times \int \frac{\mathrm{d}^2\alpha}{\pi} \exp\left[-\frac{(2a-\sigma-\gamma)\,\alpha^* + \left(\sigma^*+\gamma^*-2a^\dagger\right)\alpha}{s-1}\right] \exp\left[-\frac{\left(-\gamma\gamma^*-\sigma\sigma^*+2aa^\dagger+2bb^\dagger\right)}{s-1}\right] \text{Ⓢ} \\
&= \frac{4}{(1-s)^4} \text{Ⓢ}\, \delta\left[\frac{2}{s-1}\left(a^\dagger - \frac{\sigma^*+\gamma^*}{2}\right)\right] \delta\left[\frac{2}{s-1}\left(a-\frac{\gamma+\sigma}{2}\right)\right] \\
&\quad \times \delta\left[\frac{2}{s-1}\left(\frac{\sigma-\gamma}{2}+b^\dagger\right)\right] \delta\left[\frac{2}{s-1}\left(\frac{\sigma^*-\gamma^*}{2}+b\right)\right] \text{Ⓢ} \\
&= \frac{16}{(1-s)^4} \text{Ⓢ}\, \delta\left[\frac{2}{s-1}\left(a^\dagger - b - \sigma^*\right)\right] \delta\left[\frac{2}{s-1}\left(a-b^\dagger-\sigma\right)\right] \\
&\quad \times \delta\left[\frac{2}{s-1}\left(b+a^\dagger-\gamma^*\right)\right] \delta\left[\frac{2}{s-1}\left(a+b^\dagger-\gamma\right)\right] \text{Ⓢ}, \qquad (25)
\end{aligned}
$$

which in the case of $s=0$ becomes the Weyl ordered form of the usual Wigner operator[13]

$$
\Delta_{s=0}(\sigma,\gamma) = \vdots \delta\left(a^\dagger - b - \sigma^*\right) \delta\left(a-b^\dagger-\sigma\right) \delta\left(b+a^\dagger-\gamma^*\right) \delta\left(a+b^\dagger-\gamma\right) \vdots . \qquad (26)
$$

4. Application

We now derive the s-ordered expansion of the two-mode squeezing operator $S_2 \equiv \mathrm{e}^{\lambda(a^\dagger b^\dagger - ab)}$, where λ is a real number; a review of the two-mode squeezed state can be found in Ref. [14]. Noticing that S_2 has a neat representation in the entangled state basis,[15]

$$
S_2 = \mathrm{e}^{\lambda} \int \frac{\mathrm{d}^2\xi}{\pi} \left|\xi \mathrm{e}^{\lambda}\right\rangle \langle\xi| ,
$$

the normal ordering form of S_2 can be easily derived (without using SU(1,1) Lie algebra) by performing the following integration[15]

$$
\begin{aligned}
S_2 &= \mathrm{e}^{\lambda} \int \frac{\mathrm{d}^2\xi}{\pi} \left|\xi \mathrm{e}^{\lambda}\right\rangle \langle\xi| \\
&= \operatorname{sech}\lambda : \exp\left[a^\dagger b^\dagger \tanh\lambda + (\operatorname{sech}\lambda - 1)\left(a^\dagger a + b^\dagger b\right) - ab\tanh\lambda\right] : . \qquad (27)
\end{aligned}
$$

Substituting it into Eq. (1), and using Eq. (10) as well as the IWSOP technique, we have ($|\beta,\alpha\rangle$ is a two-mode coherent state)

$$
\begin{aligned}
S_2 &= \frac{4}{(1-s)^2} \int \frac{\mathrm{d}^2\beta \mathrm{d}^2\alpha}{\pi^2} \langle -\beta, -\alpha| S_2 |\beta,\alpha\rangle \\
&\quad \times \text{Ⓢ} \exp\left[\frac{2}{s-1}\left(s|\beta|^2 - \beta^* a + \beta a^\dagger - a^\dagger a + s|\alpha|^2 - \alpha^* b + \alpha b^\dagger - b^\dagger b\right)\right] \text{Ⓢ} \\
&= \frac{4\operatorname{sech}\lambda}{(1-s)^2} \text{Ⓢ} \int \frac{\mathrm{d}^2\beta \mathrm{d}^2\alpha}{\pi^2} \exp\left[-A|\beta|^2 + \left(\frac{2a^\dagger}{s-1} - \alpha\tanh\lambda\right)\beta + \left(\alpha^*\tanh\lambda - \frac{2a}{s-1}\right)\beta^*\right] \\
&\quad \times \exp\left[-A|\alpha|^2 - \frac{2b}{s-1}\alpha^* + \frac{2b^\dagger}{s-1}\alpha - \frac{2\left(b^\dagger b + a^\dagger a\right)}{s-1}\right] \text{Ⓢ} \\
&= \frac{4\operatorname{sech}\lambda}{(1-s)^2 A} \text{Ⓢ} \int \frac{\mathrm{d}^2\alpha}{\pi} \exp\left[-\frac{A^2+\tanh^2\lambda}{A}|\alpha|^2 + \frac{2\left(b^\dagger + a\tanh\lambda/A\right)}{(s-1)}\alpha\right] \\
&\quad \times \exp\left[\frac{2\left((a^\dagger/A)\tanh\lambda - b\right)}{(s-1)}\alpha^* - \frac{4a^\dagger a}{A(s-1)^2} - \frac{2\left(b^\dagger b + a^\dagger a\right)}{s-1}\right] \text{Ⓢ} \\
&= \frac{4\operatorname{sech}\lambda}{B} \text{Ⓢ} \exp\left[\frac{4}{B}\left(a^\dagger b^\dagger - ab\right)\tanh\lambda - \left(\frac{4A}{B} + \frac{2}{s-1}\right)\left(a^\dagger a + b^\dagger b\right)\right] \text{Ⓢ}, \qquad (28)
\end{aligned}
$$

where we have set

$$
A = \frac{1+s}{1-s} + \operatorname{sech}\lambda, \quad B = \left(A^2 + \tanh^2\lambda\right)(s-1)^2 . \qquad (29)
$$

So far as we know, the s-ordered form of the two-mode squeezing operator in Eq. (27) has not been reported in the literature before, and equation (28) is a new formula. The classical correspondence of $\mathrm{e}^{\lambda(a^\dagger b^\dagger - ab)}$ in the sense of s-parameterized quantization scheme is

$$\frac{4\,\mathrm{sech}\,\lambda}{B}\exp\left[\frac{4}{B}\left(\alpha^*\beta^*-\alpha\beta\right)\tanh\lambda-\left(\frac{4A}{B}+\frac{2}{s-1}\right)\left(\alpha^*\alpha+\beta^*\beta\right)\right]. \tag{30}$$

For $s=1$, where $\textcircled{S}\cdots\textcircled{S}\to{:}\ {:}$, leading to

$$\lim_{s\to1}\frac{4}{B}=1,\quad \lim_{s\to1}2\left(\frac{2A}{B}+\frac{1}{s-1}\right)=1-\mathrm{sech}\,\lambda, \tag{31}$$

thus equation (28) just reduces to Eq. (27) as expected; for $s=0$, we have $\textcircled{S}\cdots\textcircled{S}\to\vdots\ \ \vdots$, leading to $A=1+\mathrm{sech}\,\lambda$, $B=2\left(\mathrm{sech}\,\lambda+1\right)$, equation (28) becomes the Weyl ordering expansion,[16]

$$\mathrm{e}^{\lambda(a^\dagger b^\dagger-ab)}=\frac{2\,\mathrm{sech}\,\lambda}{2\,\mathrm{sech}\,\lambda+1}\vdots\exp\left[\frac{2\tanh\lambda}{1+\mathrm{sech}\,\lambda}\left(a^\dagger b^\dagger-ab\right)\right]\vdots, \tag{32}$$

which seems to be another new formula. For $s=-1$, we have $\textcircled{S}\cdots\textcircled{S}\to\vdots\ \vdots$, and $A=\mathrm{sech}\,\lambda$, $B=4$; equation (28) becomes antinormally ordered

$$\mathrm{e}^{\lambda(a^\dagger b^\dagger-ab)}=\mathrm{sech}\,\lambda\,\vdots\exp\left[\left(a^\dagger b^\dagger-ab\right)\tanh\lambda+\left(1-\mathrm{sech}\,\lambda\right)\left(a^\dagger a+b^\dagger b\right)\right]\vdots. \tag{33}$$

Using

$$\mathrm{e}^{fa^\dagger a}=\mathrm{e}^{-f}\vdots\exp\left[\left(1-\mathrm{e}^{-f}\right)a^\dagger a\right]\vdots, \tag{34}$$

we have

$$\mathrm{e}^{\lambda(a^\dagger b^\dagger-ab)}=\mathrm{e}^{-ab\tanh\lambda}\exp[\left(a^\dagger a+b^\dagger b\right)\ln\cosh\lambda]\,\mathrm{e}^{a^\dagger b^\dagger\tanh\lambda}, \tag{35}$$

which is also correct.[17]

Moreover, as a result of Eq. (33), the Glauber–Sudarshan P-representation of the two-mode squeezing operator is easily obtained

$$\mathrm{e}^{\lambda(a^\dagger b^\dagger-ab)}=\mathrm{sech}\,\lambda\int\frac{\mathrm{d}^2\beta\mathrm{d}^2\alpha}{\pi^2}\mathrm{e}^{(\beta^*\alpha^*-\beta\alpha)\tanh\lambda+(1-\mathrm{sech}\,\lambda)\left(|\beta|^2+|\alpha|^2\right)}\left|\beta,\alpha\right\rangle\left\langle\beta,\alpha\right|. \tag{36}$$

Now consider the case that a two-mode Fock state $|m,n\rangle$ suddenly gets squeezed and becomes $\mathrm{e}^{\lambda(a^\dagger b^\dagger-ab)}\left|m,n\right\rangle$, then in order to know the probability remaining in the original state, we should calculate $\langle m,n|\,\mathrm{e}^{\lambda(a^\dagger b^\dagger-ab)}\left|m,n\right\rangle$, using the overlap between $|m,n\rangle$ and $|\beta,\alpha\rangle$

$$\left\langle m,n\right.\left|\beta,\alpha\right\rangle=\frac{\beta^m\alpha^n}{\sqrt{m!n!}}\exp\left(-\frac{1}{2}\left|\beta\right|^2-\frac{1}{2}\left|\alpha\right|^2\right). \tag{37}$$

Thus we have

$$\begin{aligned}&\left\langle m,n\right|\mathrm{e}^{\lambda(a^\dagger b^\dagger-ab)}\left|m,n\right\rangle\\ =&\ \frac{\mathrm{sech}\,\lambda}{m!n!}\int\frac{\mathrm{d}^2\beta\mathrm{d}^2\alpha}{\pi^2}\beta^m\beta^{*m}\alpha^n\alpha^{*n}\exp\left[-\mathrm{sech}\,\lambda\left|\alpha\right|^2-\mathrm{sech}\,\lambda\left|\beta\right|^2-\alpha\beta\tanh\lambda+\alpha^*\beta^*\tanh\lambda\right]\\ =&\ \frac{n!}{m!}\left(\mathrm{sech}\,\lambda\right)^{m-n+1}\sum_{k=0}^{n}\frac{(n+m-k)!\left(-\tanh^2\lambda\right)^{n-k}}{k!\left[(n-k)!\right]^2},\end{aligned} \tag{38}$$

where we have used the following integration formulae

$$\int\frac{\mathrm{d}^2z}{\pi}z^{*m}z^n\exp\left[h\left|z\right|^2+\xi z+\eta z^*\right]=\mathrm{e}^{-\xi\eta/h}\sum_{l=0}^{\min[m,n]}\frac{m!n!\eta^{m-l}\xi^{n-l}}{l!\,(m-l)!\,(n-l)!\,(-h)^{m+n+1-l}},\quad \mathrm{Re}\,h<0, \tag{39}$$

and

$$\int \frac{\mathrm{d}^2 z}{\pi} z^{*m} z^n \exp\left[h\left|z\right|^2\right] = \delta_{m,n} m! \left(-\frac{1}{h}\right)^{m+1}. \quad (40)$$

On the other hand, let us recall the definition of the two-variable Hermite polynomial[18,19]

$$\mathrm{H}_{m,n}(\mu,\nu) = \sum_{l=0}^{\min[m,n]} \frac{(-1)^l\, m!n!}{l!\,(m-l)!\,(n-l)!} \mu^{m-l}\nu^{n-l}, \quad (41)$$

whose generating function is

$$\sum_{m,n=0}^{\infty} \frac{t^m \tau^n}{m!n!} \mathrm{H}_{m,n}(\mu,\nu) = \exp\left(-t\tau + t\mu + \tau\nu\right). \quad (42)$$

By comparing Eq. (41) with Eq. (19), the entangled state $|\xi\rangle$ can be expanded as

$$|\xi\rangle = \mathrm{e}^{-|\xi|^2/2} \sum_{m,n=0}^{\infty} \frac{\mathrm{H}_{m,n}(\xi,\xi^*)}{\sqrt{m!n!}} |m,n\rangle. \quad (43)$$

Noticing

$$\mathrm{H}_{m,n}(\xi,\xi^*) = \mathrm{e}^{\mathrm{i}(m-n)\theta} \mathrm{H}_{m,n}(r,r), \quad \xi = r\mathrm{e}^{\mathrm{i}\,\theta}, \quad (44)$$

and

$$\mathrm{H}_{m,n}(r,r) = p!\,(-1)^p\, r^l \mathrm{L}_p^l\left(r^2\right), \quad l = |m-n|, \quad p = \min(m,n), \quad (45)$$

where L_p^l is the associated Laguerre polynomial, as well as from Eqs. (27), (40), (43), and (38), we can alternately express (without loss of generality, let $m > n$)

$$\begin{aligned}
\langle m,n|\,\mathrm{e}^{\lambda(a^\dagger b^\dagger - ab)}\,|m,n\rangle &= \mathrm{e}^{\lambda} \int \frac{\mathrm{d}^2\xi}{\pi} \langle m,n\left|\xi\mathrm{e}^{\lambda}\right\rangle \langle\xi|\, m,n\rangle \\
&= \frac{2\mathrm{e}^{\lambda}}{m!n!} \int_0^{\infty} r\mathrm{d}r \exp\left(-\frac{\mathrm{e}^{2\lambda}+1}{2} r^2\right) \mathrm{H}_{m,n}(\mathrm{e}^{\lambda} r, \mathrm{e}^{\lambda} r) \mathrm{H}_{m,n}(r,r) \\
&= \frac{2\mathrm{e}^{\lambda}}{m!n!} (n!)^2 \int_0^{\infty} r\mathrm{d}r \exp\left(-\frac{\mathrm{e}^{2\lambda}+1}{2} r^2\right) \mathrm{e}^{\lambda(m-n)} r^{2(m-n)} \mathrm{L}_n^{m-n}\left(\mathrm{e}^{2\lambda} r^2\right) \mathrm{L}_n^{m-n}\left(r^2\right) \\
&= \frac{n!}{m!} (\operatorname{sech}\lambda)^{m-n+1} \sum_{k=0}^{n} \frac{(n+m-k)!\left(-\tanh^2\lambda\right)^{n-k}}{k!\left[(n-k)!\right]^2},
\end{aligned} \quad (46)$$

where the last step implies a new integration formula.

In summary, based on the theory of integration within the s-ordering of operators and the bipartite entangled state representation, we introduce the s-parameterized Wigner operator in the entangled form and reveal its physical meaning for the entangled system. The s-parameterized Weyl–Wigner correspondence in the entangled form is thus established.

References

[1] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857

[2] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882

[3] Weyl H 1927 *Z. Phys.* **46** 1

[4] Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover Publications)

[5] Fan H Y 2010 *Chin. Phys.* B **19** 050303

[6] Glauber R J 1963 *Phys. Rev.* **130** 2529

[7] Zhang W M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867

[8] Schleich W P 2001 *Quantum Optics in Phase Space* (New York: Wiley-VCH Berlin)

[9] Walls D F and Milburn G 1994 *Quantum Optics* (Berlin: Springer-Verlag)

[10] Fan H Y 2003 *J. Opt.* B: *Quantum Semiclass. Opt.* **5** R147

[11] Wűnsche A 1999 *J. Opt.* B: *Quantum Semiclass. Opt.* **1** R11

[12] Fan H Y and Klauder J R 1994 *Phys. Rev.* A **49** 704

[13] Fan H Y and Guo Q 2007 *Commun. Theor. Phys.* **48** 823

[14] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709

[15] Fan H Y and Fan Y 1996 *Phys. Rev.* A **54** 958

[16] Fan H Y and Fan Y 2002 *Int. J. Mod. Phys.* A **17** 701

[17] Fan H Y 1992 *J. Phys.* A: *Math. Gen.* **25** 1013

[18] Meng X G, Wang J S, Liang B L and Li H Q 2008 *Chin. Phys.* B **17** 1791

[19] Hu L Y and Fan H Y 2009 *Chin. Phys.* B **18** 902