Effects of average degree of network on an order–disorder transition in opinion dynamics*

Feng Cun-Fang(冯存芳)[a)b)], Guan Jian-Yue(关剑月)[a)], Wu Zhi-Xi(吴枝喜)[c)], and Wang Ying-Hai(汪映海)[a)†]

[a)] *Institute of Theoretical Physics, Lanzhou University, Lanzhou* 730000, *China*
[b)] *College of Science, Wuhan University of Science and Engineering, Wuhan* 430073, *China*
[c)] *Department of Physics, Umeå University,* 90187 *Umeå, Sweden*

(Received 19 September 2009; revised manuscript received 26 November 2009)

We have investigated the influence of the average degree $\langle k\rangle$ of network on the location of an order–disorder transition in opinion dynamics. For this purpose, a variant of majority rule (VMR) model is applied to Watts–Strogatz (WS) small-world networks and Barabási–Albert (BA) scale-free networks which may describe some non-trivial properties of social systems. Using Monte Carlo simulations, we find that the order–disorder transition point of the VMR model is greatly affected by the average degree $\langle k\rangle$ of the networks; a larger value of $\langle k\rangle$ results in a more ordered state of the system. Comparing WS networks with BA networks, we find WS networks have better orderliness than BA networks when the average degree $\langle k\rangle$ is small. With the increase of $\langle k\rangle$, BA networks have a more ordered state. By implementing finite-size scaling analysis, we also obtain critical exponents β/ν, γ/ν and $1/\nu$ for several values of average degree $\langle k\rangle$. Our results may be helpful to understand structural effects on order–disorder phase transition in the context of the majority rule model.

Keywords: complex networks, majority rule
PACC: 0250, 0175, 0570L

1. Introduction

In recent years, statistical physics has provided us with a powerful tool to examine and study the emergence of collective phenomena in social systems, in particular those related to the collective social behaviour of individuals, such as opinion formation,[1,2] the propagation of rumours or diseases,[3−7] language dynamics,[8] etc. Opinion dynamics has become a topic of great interest in the field of statistical physics.[9−12]

Processes of opinion formation are usually modelled as simple collective dynamics in which the individuals update their opinions following local majority[1,9] or imitation.[13] Amongst these models, individuals are located on the nodes of a graph and endowed with a finite number of available states. One simple opinion dynamics model, the majority rule (MR) model first introduced by Krapivsky,[9] has been considered as a typical model for consensus formation.[1] In Ref. [14], Lambiotte presented a variant of the MR (VMR) model through introducing a random flip probability q, and investigated the effects of degree heterogeneity on order–disorder phase transition. The VMR model on complex networks with modular structure was studied and the coexistence of opposite opinions is observed in Ref. [15]. Guan *et al.*[16] studied the effects of inhomogeneous influence of individuals on the order–disorder phase transition of the VMR model. These works revealed that the interaction structure (heterogeneous degree distribution and the presence of communities) affects the location of the order–disorder transition.

Besides studying structural effects on dynamical processes, a lot of research has focused on finding universal laws when traditional statistical models are confined to complex networks.[17−21] For instance, the majority-vote model was extensively investigated on small-world networks,[17] Voronoi–Delaunay lattices in periodic boundary conditions,[18] random graphs,[19] and (un)directed Barabási–Albert (BA) networks.[20,21] By using finite-size scaling techniques, these works studied in detail the critical exponents β/ν, γ/ν, and $1/\nu$, and found that the exponents of majority-vote (MV) on these networks are different from each other, which indicates that the phase transition on these networks belongs to different universality classes. To the best of our knowledge, little atten-

*Project supported by the National Natural Science Foundation of China (Grant No. 10775060).

†Corresponding author. E-mail: yhwang@lzu.edu.cn

© 2010 Chinese Physical Society and IOP Publishing Ltd

tion has been paid to the critical behaviour (critical exponents β/ν, γ/ν, and $1/\nu$) of the VMR model on complex networks.

In the present work, we follow the research lines of Refs. [15]–[19] to investigate the structural effects on order–disorder phase transition and the corresponding critical exponents. In particular, we mainly focus on the effects of average degree $\langle k\rangle$ of the interaction of networks on the opinion dynamics. It is well known that social networks exhibit non-trivial structures such as fat-tailed degree distributions,[22−24] community structure,[25,26] and high clustering coefficient,[27] etc. For this reason, we consider the VMR on two famous networks, Watts–Strogatz (WS) networks[27] and BA networks[22] with different average degrees. In the next section, we introduce the VMR model in detail and formulate relevant quantities used in this paper. In Section 3, simulation results and discussions are presented. Conclusions are given in Section 4.

2. Model and computational approach

The VMR model[14] can be described as follows. The population is composed of N individuals, each individual i is endowed with an opinion o_i that can be α or β. At each time step, one of the individuals is randomly selected and two processes may take place: (i) With probability q, the selected individual s randomly changes its opinion,

$$\begin{aligned} o_s &\to \alpha \quad \text{with probability } 1/2, \\ o_s &\to \beta \quad \text{with probability } 1/2. \end{aligned} \tag{1}$$

(ii) With probability $1-q$, two neighbouring individuals of the selected one s are randomly chosen and the three individuals in this triplet update their opinions according to the MR model. Therefore, the parameter q measures the competition between the individual choices and neighbouring interactions. The system will be more random as q is larger; in contrast, the opinions of individuals tend to be more homogeneous as q is smaller. In the special case $q=0$, it is well known that the system asymptotically reaches global consensus where all individuals share the same opinion.[9] In the other limiting case $q=1$, the system is purely random and the average (over the realization of the random process) number of individuals with opinion α at time t goes to $N/2$ as t is large.

To study the critical behaviour of the VMR model on networks, we define the variable m by $m = \frac{1}{N}|\sum_{i=1}^{N} o_i|$. We are interested in the magnetization M_N, the susceptibility χ_N, and Binder's fourth-order cumulant U_N,[28] defined, respectively, as

$$M_N(q) = [\langle m\rangle]_{\rm C}, \tag{2}$$

$$\chi_N(q) = N\{[\langle m^2\rangle]_{\rm C} - [\langle m\rangle]_{\rm C}^2\}, \tag{3}$$

$$U_N(q) = 1 - \frac{[\langle m^4\rangle]_{\rm C}}{3[\langle m^2\rangle]_{\rm C}^2}, \tag{4}$$

where $\langle\cdots\rangle$ denotes that the average is taken in the stationary state, and $[\cdots]_{\rm C}$ means configurational averages.

The above quantities are functions of the parameter q and satisfy the following finite-size scaling relation:[29]

$$M_N(q) = N^{-\beta/\nu}\tilde{M}(N^{1/\nu}\varepsilon), \tag{5}$$

$$\chi_N(q) = N^{\gamma/\nu}\tilde{\chi}(N^{1/\nu}\varepsilon), \tag{6}$$

$$U_N(q) = \tilde{U}(N^{1/\nu}\varepsilon), \tag{7}$$

where $\varepsilon = q - q_{\rm c}$. From the size dependence of M and χ we can obtain the exponents β/ν and γ/ν, respectively. The maximum value of susceptibility also scales as $N^{\gamma/\nu}$. Moreover, we denote the value of q for which χ has a maximum by $q_{\rm c}^{\chi_{\max}} = q_{\rm c}(N)$. It is expected to scale with the system size as

$$q_{\rm c}(N) = q_{\rm c} + bN^{-1/\nu}, \tag{8}$$

where the constant b is close to unity. The above relation is used to determine the exponent $1/\nu$. We have also checked if the calculated exponents satisfy the hyperscaling relation

$$2\beta/\nu + \gamma/\nu = D_{\rm eff} \tag{9}$$

in order to gain the effective dimensionality, $D_{\rm eff}$, for several values of $\langle k\rangle$.

3. Simulation results and discussions

In this section, we give numerical results obtained by means of Monte Carlo simulations and finite-size scaling techniques under the initial condition $a=0.2$, where a represents the average proportion of individuals with opinion α.

First, we investigate the influence of $\langle k\rangle$ on the threshold of order–disorder transition for a fixed network size N. We note that $|\delta| = |a-0.5|$ can measure the order extent of the system, which is ordered when $|\delta|$ is a positive number and becomes disordered when $|\delta|$ reduces to zero. We define the value of q as the threshold of the order–disorder transition $q_{\rm c}$

when $|\delta|$ changes from a positive number to zero. In Figs. 1(a) and 1(b), we show $|\delta|$ versus q for different values of $\langle k\rangle$ for $N = 10000$ on WS and BA networks, respectively. From Fig. 1 we note that $|\delta|$ decreases monotonously with the increase of q on both the WS and BA networks. In particular, a larger value of $\langle k\rangle$ leads to a larger value of q_c. The non-autonomous dependence of q_c on $\langle k\rangle$ can be understood as follows: When $\langle k\rangle$ is small, events of exchange of opinions among individuals are rare. In such a situation, it would be difficult to achieve consensus of opinions among the individuals. Hence, the threshold value q_c of the order–disorder transition is smaller than that in the well mixed case.[14] For large $\langle k\rangle$, each individual has more chance to interact with others. In other words, the system will be more mean-field-like with increasing $\langle k\rangle$. It is clear that for $\langle k\rangle = 20$, the threshold q_c is very close to 0.6, the mean-field solution for a completely connected graph.[14]

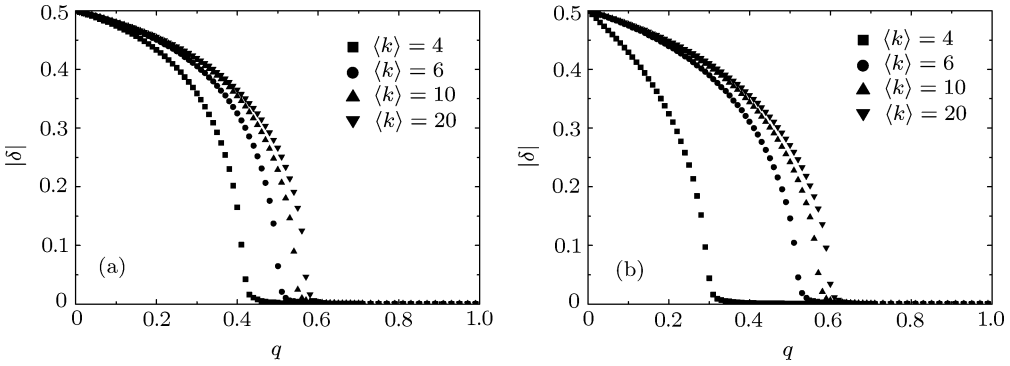


Fig. 1. Bifurcation diagram of $|\delta|$ as a function of q on the (a) WS networks and (b) BA networks for different values of $\langle k\rangle$. Network size $N = 10000$.

To obtain the threshold q_c, for each given $\langle k\rangle$ we plot Binder's fourth-order cumulant as a function of q for several system sizes $N = 250, 500, 1000, 2000, 4000$ in Fig. 2. The threshold q_c of the order–disorder transition is estimated as the point at which the curves for different network sizes N intercept each other. In Figs. 2(a) and 2(b) we plot U_N versus q for the WS networks with $\langle k\rangle = 8$ and 50, respectively. Figures 2(c) and 2(d) display U_N versus q for the BA networks with $\langle k\rangle = 6$ and 20, respectively.

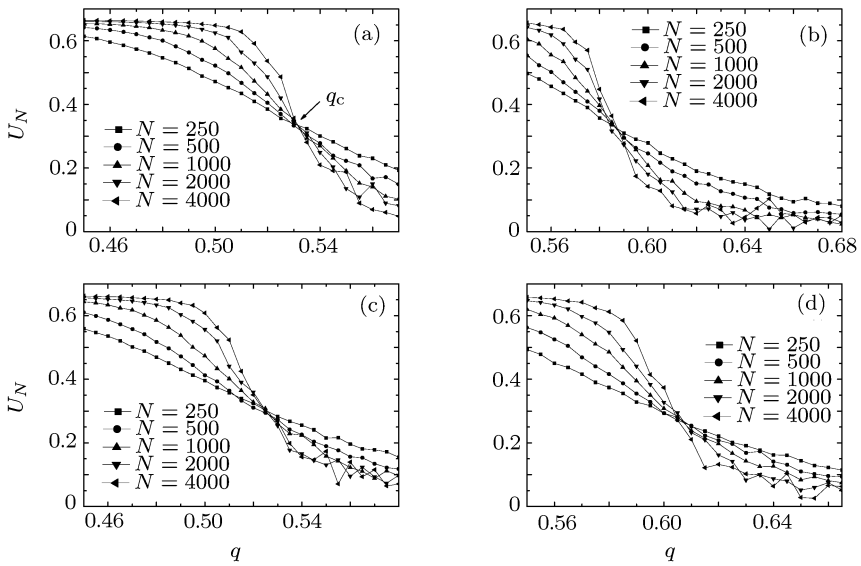


Fig. 2. Binder's fourth-order cumulant U_N as a function of q for $N = 250, 500, 1000, 2000, 4000$. In the case of the WS networks we have $\langle k\rangle = 8$ (a) and 50 (b). In the BA networks $\langle k\rangle = 6$ (c) and 20 (d).

The phase diagrams of q_c versus $\langle k\rangle$ are presented in Fig. 3(a) for the VMR on the WS networks, and 3(b) on the BA networks. It is found that the increase of q_c is more pronounced for small values of $\langle k\rangle$, which

agrees with the results given in Fig. 1. Comparing Fig. 3(a) with 3(b), we can observe that with the increase of $\langle k\rangle$ the threshold q_c for the WS networks is close to 0.6 which is the mean-field solution, while for the BA networks the threshold q_c is bigger than 0.6, which indicates that the BA networks have better orderliness than WS networks.

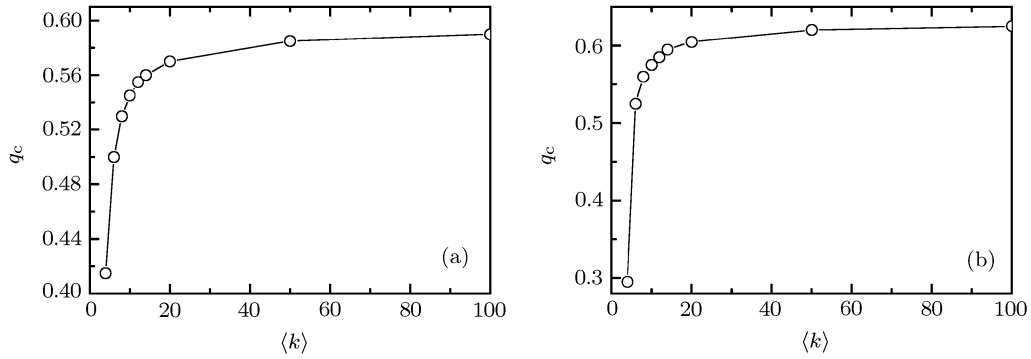


Fig. 3. Phase diagram for the threshold q_c as a function of $\langle k\rangle$, (a) on the WS networks and (b) on the BA networks.

In order to investigate the critical exponents of the VMR, in Fig. 4 we plot the size dependence of the magnetization and susceptibility at $q = q_c$ on the WS networks, while the results on the BA networks are summarized in Fig. 5. The slopes of the curves correspond to the exponents β/ν and γ/ν according to Eqs. (5) and (6). Tables 1 and 2 summarize the values of q_c, the exponents (β/ν, γ/ν, $1/\nu$) and the effective dimensionality of the system on WS networks and BA networks, respectively. In Tables 1 and 2, γ/ν^{q_c} is obtained using $\chi_N(q_c)$ and $\gamma/\nu^{q_c(N)}$ is obtained using $\chi_N(\max)$. Because there is no previous work studying the critical exponents for the VMR model on complex networks, we have compared our present results with those for the MV model on complex networks. In Ref. [21], it was pointed out that β/ν increases, and γ/ν decreases, at $q = q_c$ with increasing $\langle k\rangle$ for the MV model on undirected BA networks. By comparison, the exponents of β/ν and γ/ν for the VMR model on BA networks are not a monotonic function of $\langle k\rangle$. Instead, for all values of $\langle k\rangle$, we find that the exponents are confined to the hyperscaling relation with effective dimensionality $D_{\rm eff} = 1$, estimated using γ/ν from $\chi_N(q_c)$ in Tables 1 and 2. The same results can be derived for the VMR on the WS networks (see Table 1). We also studied the VMR model on assortative scale-free networks and obtained the same result (not shown here). Thus, we can conclude that the critical exponents β/ν and γ/ν of the VMR model on complex networks are not a monotonic function of the average degree, which is different from the MV model on BA networks[21] and random graphs.[19]

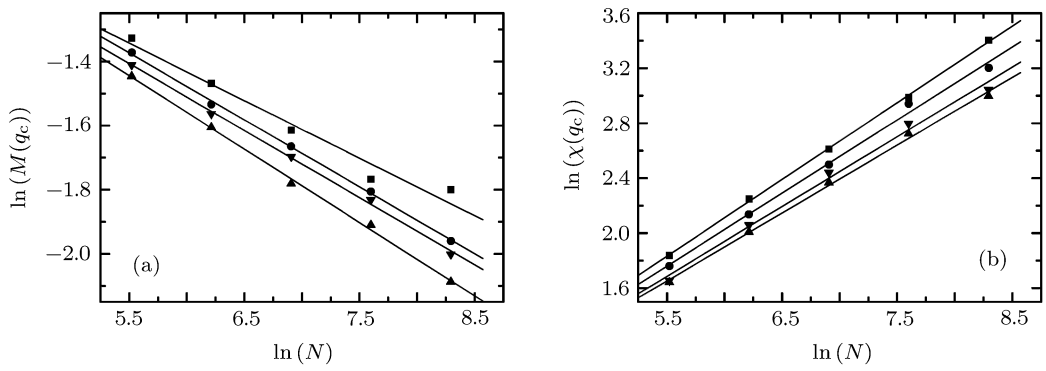


Fig. 4. Magnetization and susceptibility at $q = q_c$ versus N on the WS networks. The different data correspond to different values of $\langle k\rangle$. From top to bottom, $\langle k\rangle = 6, 8, 20, 18$. (a) Plot of the magnetization, the exponent values β/ν are 0.179(4), 0.208(6), 0.209(1), 0.229(0) from top to bottom. (b) Plot of the susceptibility, the exponent values γ/ν are 0.558(4), 0.531(4), 0.507(6), 0.493(8) from top to bottom.

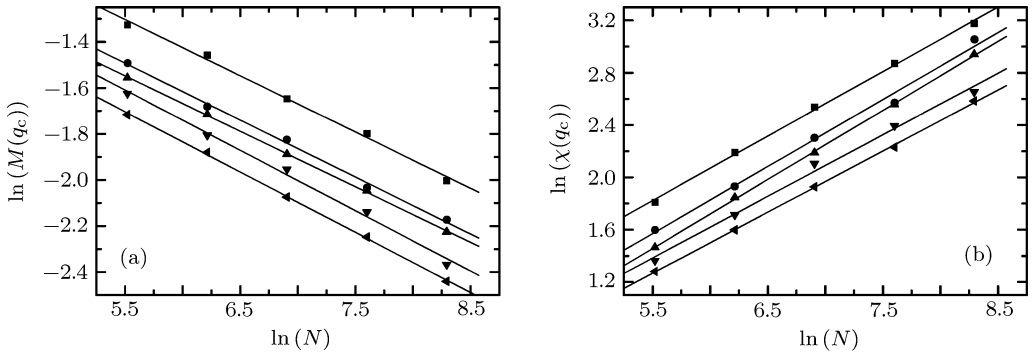


Fig. 5. Magnetization and susceptibility at $q = q_c$ versus N on the BA networks. The different data correspond to different values of $\langle k\rangle$. From top to bottom, $\langle k\rangle = 4, 6, 12, 20, 50$. (a) Plot of the magnetization, the exponent values β/ν are 0.244(3), 0.247(4), 0.241(1), 0.262(2),0.261(8) from top to bottom. (b) Plot of the susceptibility, the exponent values γ/ν are 0.492(9), 0.512(0), 0.528(1), 0.470(9),0. 467(2) from top to bottom.

Table 1. The threshold q_c, the critical exponents, and the effective dimensionality $D_{\rm eff}$, for WS networks with the average degree $\langle k\rangle$.

$\langle k\rangle$	q_c	β/ν	γ/ν^{q_c}	$\gamma/\nu^{q_c(N)}$	$1/\nu$	$D_{\rm eff}$
4	0.414(5)	0.235(0)	0.512(0)	0.496(1)	0.635(1)	0.982(2)
6	0.500(2)	0.179(4)	0.558(4)	0.540(5)	0.635(1)	0.917(3)
8	0.531(7)	0.208(6)	0.531(4)	0.463(5)	0.693(6)	0.948(7)
10	0.546(2)	0.195(1)	0.554(0)	0.556(3)	0.693(6)	0.944(3)
12	0.554(7)	0.243(4)	0.520(1)	0.493(0)	0.693(6)	1.006(9)
14	0.561(0)	0.185(1)	0.547(6)	0.499(2)	0.719(7)	0.917(8)
20	0.573(6)	0.209(1)	0.507(6)	0.464(3)	0.693(6)	0.925(9)

Table 2. The threshold q_c, the critical exponents, and the effective dimensionality $D_{\rm eff}$, for BA networks with the average degree $\langle k\rangle$.

$\langle k\rangle$	q_c	β/ν	γ/ν^{q_c}	$\gamma/\nu^{q_c(N)}$	$1/\nu$	$D_{\rm eff}$
4	0.293(7)	0.244(3)	0.492(9)	0.532(4)	0.332(1)	0.981(7)
6	0.525(9)	0.247(4)	0.512(0)	0.480(6)	0.614(1)	1.006(9)
8	0.557(2)	0.232(8)	0.475(1)	0.504(8)	0.474(8)	0.940(7)
10	0.576(2)	0.226(8)	0.517(7)	0.512(6)	0.614(1)	0.971(5)
12	0.584(1)	0.241(1)	0.528(1)	0.482(9)	0.605(8)	1.010(4)
14	0.594(5)	0.252(7)	0.480(1)	0.499(9)	0.540(4)	0.985(7)
20	0.606(7)	0.262(2)	0.470(9)	0.463(3)	0.455(6)	0.995(4)

4. Conclusion

In summary, we have studied the influence of average degree on the location of the order–disorder phase transition for a variant majority-rule model respectively on Watts–Strogatz networks and Barabási–Albert networks using Monte Carlo simulations and finite-size scaling techniques. Numerical simulations show that a larger average degree results in a larger threshold of the order–disorder transition. By means of finite-size scaling techniques, we have also analysed the critical exponents β/ν, γ/ν and $1/\nu$ of the VMR model on both the WS networks and BA networks. We found that β/ν and γ/ν are not monotonic functions of the average degree, and satisfy the hyperscaling relation with effective dimensionality $D_{\rm eff} = 1$, which is different from the MV model on the BA and random networks. These results may be helpful to understand structural effects on order–disorder phase transition in the context of majority rule.

References

[1] Galam S 1999 *Physica* A **274** 132
[2] Sznajd-Weron K and Sznajd J 2000 *Int. J. Mod. Phys.* C **11** 1157
[3] Liu M X and Ruan J 2009 *Chin. Phys.* B **18** 2115
[4] Galam S 2003 *Physica* A **320** 571
[5] Watts D J 2002 *Proc. Nat. Acad. Sci.* USA **99** 5766
[6] Boguná M, Pastor-Satorras R and Vespignani A 2003 *Phys. Rev. Lett.* **90** 028701
[7] Chen Z Q, Pei W D and Yuan Z Z 2008 *Chin. Phys.* B **17** 373
[8] Dall'Asta L, Baronchelli A, Barrat A and Loreto V 2006 *Europhys. Lett.* **73** 969
[9] Krapivsky P L and Redner S 2003 *Phys. Rev. Lett.* **90** 238701
[10] Mobilia M and Redner S 2003 *Phys. Rev.* E **68** 046106
[11] Helbing D, Farkas I and Vicsek T 2000 *Nature* (London) **407** 487
[12] Galam S 2002 *Eur. Phys. J.* B **25** 403
[13] Frachebourg L and Krapivsky P L 1996 *Phys. Rev.* E **53** R3009
[14] Lambiotte R 2007 *Europhys. Lett.* **78** 68002
[15] Lambiotte R and Ausloos M 2007 *J. Stat. Mech.* **8** 08026
[16] Guan J Y, Wu Z X and Wang Y H 2007 *Phys. Rev.* E **76** 042102
[17] Campos P R, Oliveira V M and Moreira F G B 2003 *Phys. Rev.* E **67** 026104
[18] Lima F W S, Fulco U L and Costa Filho R N 2005 *Phys. Rev.* E **71** 036105
[19] Pereira L F C and Brady Moreira F G 2005 *Phys. Rev.* E **71** 016123
[20] Lima F W S arxiv:physics/0511082v1
[21] Lima F W S arxiv:cond-mat/0607582v1
[22] Barabási A L and Albert R 1999 *Science* **286** 509
[23] Feng W W 2009 *Acta Phys. Sin.* **58** 2127 (in Chinese)
[24] Liu M X and Ruan J 2009 *Chin. Phys.* B **18** 5111
[25] Girvan M and Newman M E J 2002 *Proc. Nat. Acad. Sci.* USA **99** 7821
[26] Feng C F, Xu X J, Wu Z X and Wang Y H 2008 *Chin. Phys.* B **17** 1951
[27] Watts D J and Strogatz S H 1998 *Nature* **393** 440
[28] Binder K 1981 *Z. Phys.* B **43** 119
[29] Oliveira M J de 1992 *J. Stat. Phys.* **66** 273