Painlevé property of the modified C-KdV equation and its exact solutions*

Wang Hui(王 惠)†, Dong Huan-He(董焕河),
Wang Yun-Hu(王云虎), and Wang Xin-Zeng(王新赠)

College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao 266510, *China*

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In this paper, the Painlevé properties of the modified C-KdV equation are verified by using the W-K algorithm. Then some exact soliton solutions are obtained by applying the standard truncated expansion method and the nonstandard truncated expansion method with the help of Maple software, respectively.

Keywords: Painlevé property, standard truncated expansion, nonstandard truncated expansion, exact solution

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1. Introduction

The investigation of integrability and the exact solutions of nonlinear equations is an important aspect in the study of nonlinear physical phenomena.[1−15] In the course of seeking exact solutions of nonlinear equations, the WTC method proposed by Weiss, Tabor, and Carnebale is one of the most powerful tools.[16] More methods have been proposed in succession through the efforts of researchers, such as the Conte invariable method,[17] the Pickering method,[18] the extended standard truncated expansion method[19,20] and the W-K algorithm.[21,22] With the help of these methods, some interesting results are obtained.[23−25] Though the precise equivalence of Painlevé property and Painlevé integrability remains to be determined, a connection between integrability and the Painlevé property has been noted since the work of Kowalevskaya.[26] Therefore, the integrability, the Lax pairs and Bäcklund transforms of equations can be studied.[2,27]

The W-K algorithm is a very efficient method; it makes the process of verifying whether the soliton equations possess Painlevé properties more simple. In this paper, the Painlevé properties of the modified C-KdV equation are verified by using the W-K algorithm. Then some exact soliton solutions are obtained by applying the standard truncated expansion method and the nonstandard truncated expansion method with the help of Maple software.

2. Painlevé analysis of the C-KdV equation

The modified C-KdV equation[28] has rich practical applications in mathematical physics; the study of its integrability and exact solutions plays an important role in the research of nonlinear physical phenomena. It has the following form:

$$\begin{aligned} u_t &= v_x - \frac{3}{2}uu_x + \beta u_x, \\ v_t &= \frac{1}{4}u_{xxx} - vu_x - \frac{1}{2}uv_x + \beta v_x, \end{aligned} \tag{1}$$

where u, v can be written as $u = \varphi^{\alpha}\sum_{i=0}^{\infty} u_i\varphi^j, v = \varphi^{\beta}\sum_{i=0}^{\infty} v_i\varphi^j$, $\varphi = \varphi(x,t), u_i = u_i(x,t), v_i = v_i(x,t)$. Substituting $u = u_0\varphi^{\alpha}$, $v = v_0\varphi^{\beta}$ into Eq. (1), we obtain $\alpha = -1, \beta = -2, u_0 = \varphi_x, v_0 = \frac{3}{4}\varphi_x^2$. In order to carry out the procedure more easily, let $\varphi(x,t) = x + \phi(t)$, and we have $u_0 = 1, v_0 = \frac{3}{4}$.

Substituting the above results into Eq. (1), after a series of complicated calculations, the recursion relations are found to be

$$\begin{aligned} &-\frac{3}{2}(j-2)u_j + (j-2)v_j \\ &= u_{j-2,t} + (j-2)u_{j-1}\varphi_t - v_{j-1,x} \end{aligned}$$

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†Corresponding author. E-mail: wanghui2007010124@163.com

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$$
\begin{aligned}
&+\frac{3}{2}\left(\sum_{l=0}^{j-1}u_l u_{j-l-1,x}+\sum_{l=1}^{j-1}u_l u_{j-l}(j-l-1)\right)\\
&-\beta(u_{j-2,x}+(j-2)u_{j-1}),\\
&\frac{1}{4}j(j-2)(j-4)u_j-\frac{1}{2}(j-4)v_j\\
=\;& v_{j-2,t}+(j-3)v_{j-1}\varphi_t-\frac{1}{4}(u_{j-3,xxx}\\
&+3(j-3)u_{j-2,xx}+3(j-2)(j-3)u_{j-1,x})\\
&+\sum_{l=0}^{j-1}v_l u_{j-l-1,x}+\sum_{l=1}^{j-1}v_l u_{j-l}(j-l-1)\\
&+\frac{1}{2}\left(\sum_{l=0}^{j-1}u_l v_{j-l-1,x}+\sum_{l=1}^{j-1}u_l v_{j-l}(j-l-2)\right)\\
&-\beta(v_{j-2,x}+(j-3)v_{j-1}). \qquad (2)
\end{aligned}
$$

It is found that the resonance points occur at $j=-1,2,3,4$. From Eq. (2), we have the following results.

(i) When $j=1$,

$$
\begin{aligned}
&\frac{3}{2}u_1-v_1=-u_0\varphi_t+\beta u_0,\\
&\frac{3}{4}u_1+\frac{3}{2}v_1=-2v_0\varphi_t+\frac{1}{2}(-u_0v_1-2u_1v_0)+2\beta v_0,\\
&\Rightarrow u_1=\beta-\varphi_t,\ v_1=\frac{1}{2}(\beta-\varphi_t). \qquad (3)
\end{aligned}
$$

(ii) When $j=2$,

$$
\begin{aligned}
v_2 =\;& v_{0,t}-v_1\varphi_t+\frac{3}{4}u_{0,xx}+v_0u_{1,x}+v_1u_{0,x}\\
&+\frac{1}{2}(u_0v_{1,x}+u_1v_{0,x}-u_1v_1)+\beta v_1. \qquad (4)
\end{aligned}
$$

Obviously, the first formula is tenable. It is also easy to get $v_2=\frac{1}{4}(\beta-\varphi_t)^2$ and u_2 is an arbitrary constant. Therefore, we could set $u_2=0$.

(iii) When $j=3$,

$$
\begin{aligned}
&-\frac{3}{2}u_3+v_3\\
=\;& u_{1,t}+u_2\varphi_t-v_{2,x}+\frac{3}{2}(u_0u_{2,x}+u_1u_{1,x}+u_2u_{0,x}\\
&+2u_0u_3+u_1u_2+u_3u_0)-\beta(u_{1,x}+u_2)\\
&-\frac{3}{4}u_3+\frac{1}{2}v_3\\
=\;& v_{1,t}-\frac{1}{4}u_{0,xxx}+v_0u_{2,x}+v_1u_{1,x}+v_2u_{0,x}+2v_1u_3\\
&+v_1u_2+\frac{1}{2}(u_0v_{2,x}+u_1v_{1,x}+u_2v_{0,x}\\
&+u_1v_2-u_2v_1)-\beta v_{1,x}. \qquad (5)
\end{aligned}
$$

Applying the known results, equation (5) can be written as $-\frac{3}{2}u_3+v_3=-\varphi_{tt}$. Obviously, there must be an arbitrary function between u_3 and v_3. We can provide $u_3=0$, $v_3=-\varphi_{tt}$.

(iv) When $j=4$,

$$
\begin{aligned}
-3u_4+2v_4 =\;& u_{2,t}+2u_3\varphi_t+\frac{3}{2}(u_0u_{3,x}+u_1u_{2,x}+u_2u_{1,x}+u_3u_{0,x}+3u_0u_4+2u_1u_3+u_2^2)\\
&-v_{3,x}-\beta(u_{2,x}+2u_3),\\
0 =\;& v_{2,t}+v_3\varphi_t-\frac{1}{4}(u_{1,xxx}+3u_{2,xx}+6u_{3,x})+v_0u_{3,x}+v_1u_{2,x}+v_2u_{1,x}+v_3u_{0,x}+2v_1u_3\\
&+v_2u_2+\frac{1}{2}(u_0v_{3,x}+u_1v_{2,x}+u_2v_{1,x}+u_3v_{0,x}+u_1v_3-u_3v_1)-\beta(v_{2,x}+v_3). \qquad (6)
\end{aligned}
$$

It is not difficult to verify that the second equation of Eq. (6) is tenable for ever. In other words, there must be an arbitrary function between u_4 and v_4. The compatibility condition at $j=4$ is satisfied identically.

From the above analysis, we can have the conclusion that the modified C-KdV equation possesses the Painlevé property. It is Painlevé integrable.

3. Standard truncated expansion

In order to obtain the exact soliton solutions of Eq. (1), we truncate the Painlevé expansion at the constant-level term[29]

$$
u(x,t)=\phi^{-j}(x,t)\sum_{l=0}^{j}u_l(x,t)\phi^l(x,t),\quad v(x,t)=\phi^{-k}(x,t)\sum_{l=0}^{k}v_l(x,t)\phi^l(x,t). \qquad (7)
$$

On balancing the highest-order contribution from the linear term with the highest-order contributions from the nonlinear term, we have $j=1$ and $k=2$.

Consequently, equation (7) can be written as

$$u(x,t)=\frac{u_0(x,t)}{\phi}+u_1,\quad v(x,t)=\frac{v_0(x,t)}{\phi^2}+\frac{v_0(x,t)}{\phi}+v_2. \tag{8}$$

Substituting Eq. (8) into Eq. (1) and letting the coefficient of the like power of ϕ vanish, a set of Painlevé–Bäcklund equations are obtained

$$\begin{aligned}
&2v_0\phi_x-\frac{3}{2}u_0^2\phi_x=0,\quad -u_0\phi_t+v_1\phi_x-v_{0x}+\frac{3}{2}(u_0u_{0x}-u_0u_1\phi_x)+\beta u_0\phi_x=0,\\
&u_{0t}-v_{1x}+\frac{3}{2}(u_0u_{1x}-u_{0x}u_1)-\beta u_{0x}=0,\quad u_{1t}=v_{2x}-\frac{3}{2}u_1u_{1x}+\beta u_{1x},\quad \frac{3}{2}u_0\phi_x^3-2u_0v_0\phi_x=0,\\
&-2v_0\phi_t-\frac{3}{2}(u_{0x}\phi_x^2+u_0\phi_x\phi_{xx})+(v_0u_{0x}-u_0v_1\phi_x)+\frac{1}{2}(u_0v_{0x}-u_0v_1\phi_x-2u_1v_0\phi_x)+2\beta v_0\phi_x=0,\\
&v_{0t}-v_1\phi_t+\frac{1}{4}(3u_{0xx}\phi_x+3u_{0x}\phi_{xx}+u_0\phi_{xxx})+(v_0u_{1x}+v_1u_{0x}-u_0v_2\phi_x)\\
&\quad+\frac{1}{2}(u_0v_{1x}+u_1v_{0x}-u_1v_1\phi_x)-\beta(v_{0x}-v_1\phi_x)=0,\\
&v_{1t}-\frac{1}{4}u_{0xxx}+(v_1u_{1x}+v_2u_{0x})+\frac{1}{2}(u_0v_{2x}+u_1v_{1x})-\beta v_{1x}=0,\\
&v_{2t}=\frac{1}{4}u_{1xxx}-v_2u_{1x}-\frac{1}{2}u_1v_{2x}+\beta v_{2x}.
\end{aligned} \tag{9}$$

Case 1

In order to simplify the solutions, we may let $u_1=v_2=0$, then we have the following results.

(I) Single-soliton solutions

Now, we are able to find a family of exact analytical solutions of Eq. (1) after taking into consideration a trial solution $\phi(x,t)=1+\mathrm{e}^{kx+ct+\xi_0}$. Substituting it into Eq. (9), we obtain $c=(2\beta k-k^2)/2$.

The single soliton solutions are obtained by combining all terms

$$\begin{aligned}
u(x,t)&=\frac{k}{2}\Big(1+\tanh\Big(\frac{1}{2}\Big(kx+\Big(\beta k-\frac{k^2}{2}\Big)t+\xi_0\Big)\Big)\Big),\\
v(x,t)&=-\frac{3k^2}{16}\tanh^2\Big(\frac{1}{2}\Big(kx+\Big(\beta k-\frac{k^2}{2}\Big)t+\xi_0\Big)\Big)+\frac{k^2}{8}\tanh\Big(\frac{1}{2}\Big(kx+\Big(\beta k-\frac{k^2}{2}\Big)t+\xi_0\Big)\Big)-\frac{k^2}{16},
\end{aligned} \tag{10}$$

where k,β,ξ_0 are arbitrary constants (see Fig. 1).

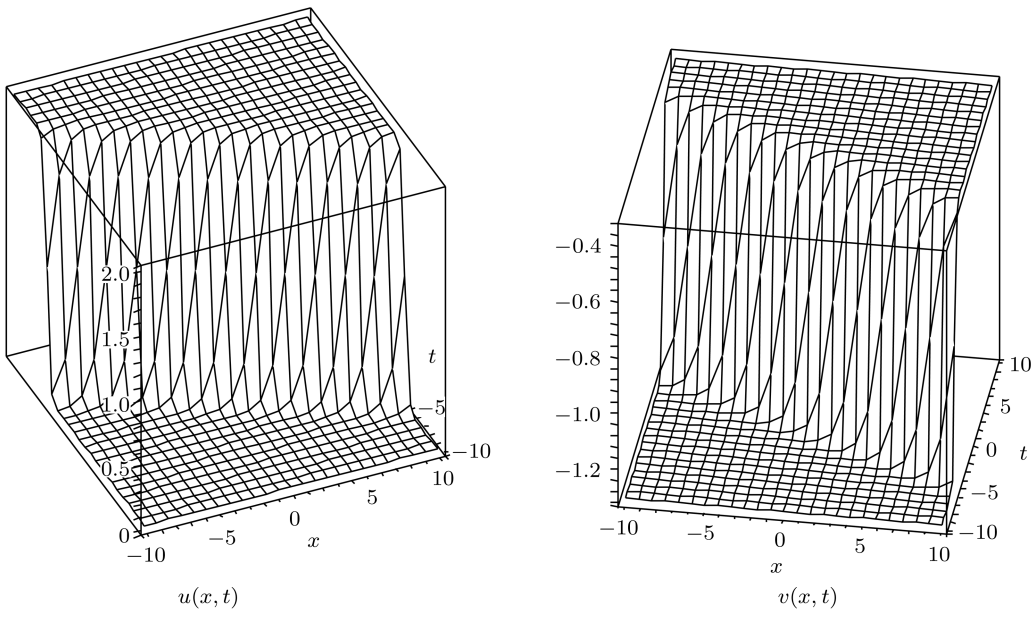


Fig. 1. Plots of $u(x,t)$ and $v(x,t)$ at $k=2,\beta=3,\xi_0=0$.

(II) Two-soliton solutions

If we set $\phi(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}$, $(\theta_i = k_i x + c_i t + \xi_{i0},\ i = 1,2)$, a family of the analytical solutions of Eq. (1) are obtained as follows:

$$u(x,t) = \frac{k_1\,\mathrm{e}^{\theta_1} + k_2\,\mathrm{e}^{\theta_2}}{1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}},$$
$$v(x,t) = \frac{-2k_1^2\,\mathrm{e}^{\theta_1} - 2k_2^2\,\mathrm{e}^{\theta_2} + k_1^2\,\mathrm{e}^{2\theta_1} + k_2^2\,\mathrm{e}^{2\theta_2} + (-2k_1^2 - 2k_2^2 + 6k_1k_2)\,\mathrm{e}^{\theta_1+\theta_2}}{4(1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_1})^2}, \tag{11}$$

where $\theta_i = k_i x + c_i t + \xi_{i0},\ c_i = (2\beta k_i - k_i^2)/2,\ i = 1,2$, and k, β, ξ_0 are arbitrary constants (see Fig. 2).

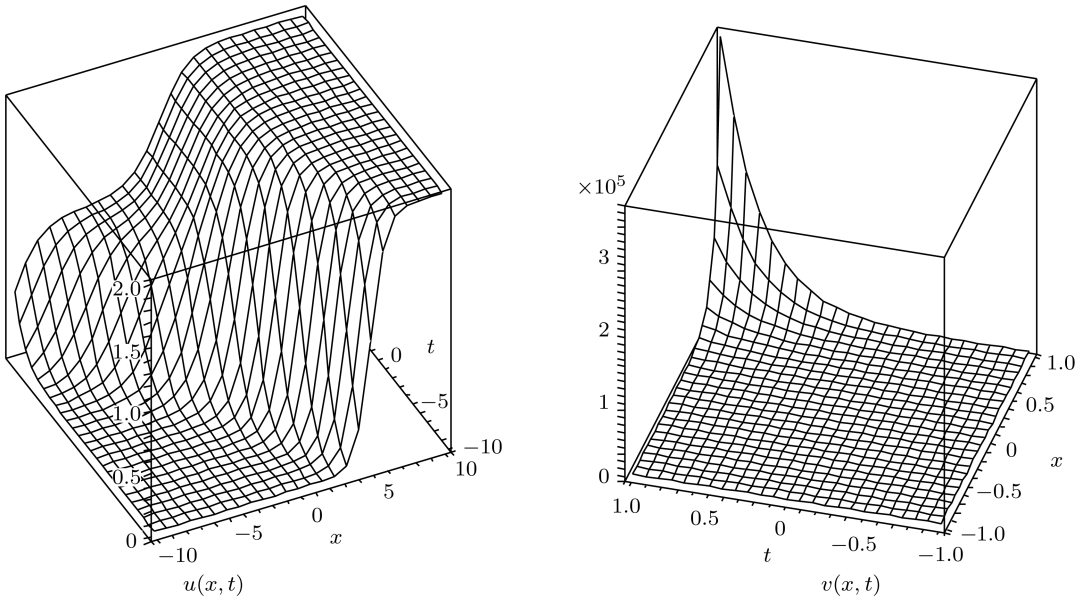


Fig. 2. Plots of $u(x,t)$ and $v(x,t)$ at $k_1 = 1, k_2 = 2, \beta = \frac{3}{2}, \xi_{10} = \xi_{20} = 0$.

(III) Three-soliton solutions

Let $\phi(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} + a_{12}a_{13}a_{23}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}$, where $\theta_i = k_i x + c_i t + \xi_{i0}, i = 1,2,3$. A family of analytical solutions of Eq. (1) is given as follows:

$$u(x,t) = \frac{k_1\,\mathrm{e}^{\theta_1} + k_2\,\mathrm{e}^{\theta_2} + k_3\,\mathrm{e}^{\theta_3}}{1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3}},$$
$$v(x,t) = \frac{-2k_1^2\,\mathrm{e}^{\theta_1} - 2k_2^2\,\mathrm{e}^{\theta_2} - 2k_3^2\,\mathrm{e}^{\theta_3} + k_1^2\,\mathrm{e}^{2\theta_1} + k_2^2\,\mathrm{e}^{2\theta_2} + k_3^2\,\mathrm{e}^{2\theta_3} + (-2k_1^2 - 2k_2^2 + 6k_1k_2)\,\mathrm{e}^{\theta_1+\theta_2}}{4(1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3})^2}$$
$$+ \frac{(-2k_1^2 - 2k_3^2 + 6k_1k_3)\,\mathrm{e}^{\theta_1+\theta_3} + (-2k_2^2 - 2k_3^2 + 6k_2k_3)\,\mathrm{e}^{\theta_1+\theta_2}}{4(1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3})^2}, \tag{12}$$

where $\theta_i = k_i x + c_i t + \xi_{i0}, i = 1,2,3$ and k, β, ξ_0 are arbitrary constants (see Fig. 3).

Case 2

Based on Eq. (9), through a complicated calculation, the following results can be obtained directly

$$u_0 = \phi_x, \quad v_0 = \frac{3}{4}\phi_x^2, \quad u_1 = \frac{-2\phi_t - \phi_{xx} + 2\beta\phi_x}{2\phi_x}, \quad v_1 = -\frac{1}{2}\phi_t - \frac{3}{4}\phi_{xx} + \frac{1}{2}\beta\phi_x,$$
$$v_2 = \frac{\frac{1}{2}\phi_x\phi_{xt} + \frac{1}{4}\phi_t^2 - \frac{1}{4}\phi_t\phi_{xx} - \frac{1}{2}\beta\phi_x\phi_{xt} + \frac{1}{4}\phi_x\phi_{xxx} - \frac{1}{4}\beta\phi_x\phi_{xx} - \frac{3}{16}\phi_{xx}^2 + \frac{1}{4}\beta^2\phi_x^2}{\phi_x^2}. \tag{13}$$

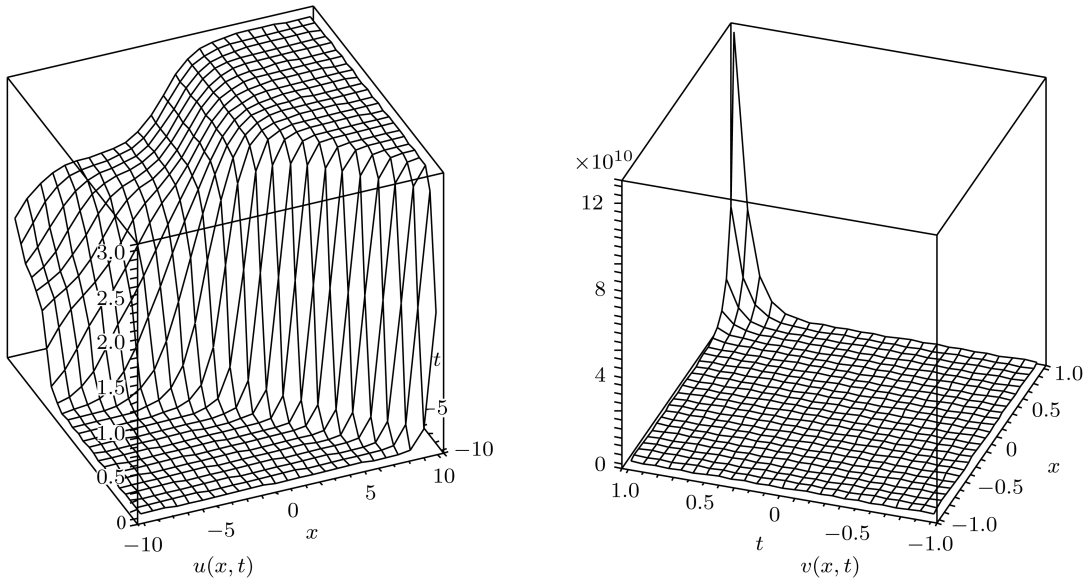


Fig. 3. Plots of $u(x,t)$ and $v(x,t)$ at $k_1 = 1, k_2 = 2, k_3 = 3, \beta = \frac{5}{2}, \xi_{i0} = 0, i = 1, 2, 3$.

Assuming $\phi(x,t) = 1 + \mathrm{e}^{kx+ct+\xi_0}$ and substituting it into Eq. (9), with the help of Eq. (13), we obtain $k = c$. Therefore, the solutions can be written as follows:

$$\begin{aligned} u(x,t) &= \frac{k}{2}\tanh\Big(\frac{1}{2}\Big(kx+kt+\xi_0\Big)\Big) - 1 + \beta, \\ v(x,t) &= \frac{3k^2}{16}\tanh^2\Big(\frac{1}{2}\Big(kx+kt+\xi_0\Big)\Big) + \Big(-\frac{k}{2} + \frac{k^2}{8} + \frac{\beta k}{2}\Big)\tanh\Big(\frac{1}{2}\Big(kx+kt+\xi_0\Big)\Big) \\ &\quad + \frac{1}{8}k^2 - \frac{1}{2}\beta k + \frac{1}{4} + \frac{1}{4}\beta^2, \end{aligned}$$

where k, β, ξ_0 are arbitrary constants (see Fig. 4).

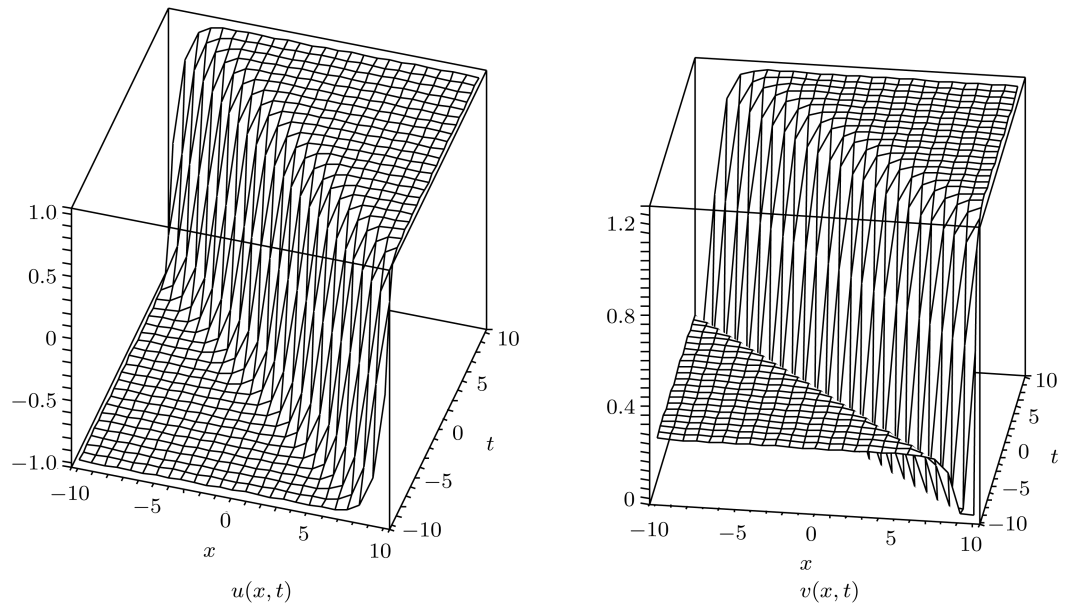


Fig. 4. Plots of $u(x,t)$ and $v(x,t)$ at $k = 2, \beta = 1, \xi_0 = 0$.

4. Non-standard truncated expansion

Applying the extended truncated expansion and obtaining the partial derivatives of variable ξ with respect to x and t respectively as $\xi_x = \sum_{j=0}^{N} S_j \xi^j$, $\xi_t = \sum_{j=0}^{N} Y_j \xi^j$, the solutions of the system can be expressed as

$u=\sum_{j=0}^{M}u_j\xi^{j-1}, v=\sum_{j=0}^{Q}v_j\xi^{j-2}$.

By using the maximum power of ϕ, we have $M=N, Q=2N$. Since S_i and Y_i $(i=0,1,2)$ are set to be constants, it is easy to verify that the results are equivalent to the soliton solutions of the standard truncated expansion when $N=1, M=1, Q=2$.

Now, let $N=2, M=2, Q=4$, then

$$\xi_x=S_0+S_1\xi+S_2\xi^2,\quad \xi_y=Y_0+Y_1\xi+Y_2\xi^2. \tag{14}$$

Because ξ satisfies the compatibility conditions, we have

$$Y_1S_2-Y_2S_1=0,\quad Y_2S_0-Y_0S_2=0,\quad Y_0S_1-Y_1S_0=0. \tag{15}$$

This is equivalent to

$$\frac{Y_0}{S_0}=\frac{Y_1}{S_1}=\frac{Y_2}{S_2}=\lambda, \tag{16}$$

where λ is a constant.

Substituting $u=\sum_{j=0}^{2}u_j\xi^{j-1}, v=\sum_{j=0}^{4}v_j\xi^{j-2}$ and Eqs. (14), (15) into Eq. (1), and setting the coefficients of ξ of the sub-power to zero, we obtain the following equations

$$\begin{aligned}
&2v_0S_0-\frac{3}{2}u_0^2S_0=0,\\
&-\frac{3}{2}u_0^2S_1+2v_0S_1+v_1S_0-v_{0x}+\beta u_0S_0-u_0Y_0-\frac{3}{2}u_0u_1S_0+\frac{3}{2}u_0u_{0x}=0,\\
&u_{0t}+2v_0S_2-\frac{3}{2}u_0^2S_2-u_0Y_1+v_1S_1+\beta u_0S_1-v_{1x}-\beta u_{0x}+\frac{3}{2}u_0u_{1x}+\frac{3}{2}u_1u_{0x}-\frac{3}{2}u_0u_1S_1=0,\\
&u_{1t}+\frac{3}{2}u_1u_{1x}+\beta u_0S_2+\frac{3}{2}u_2u_{0x}+v_1S_2-v_{2x}-\beta u_{1x}-\frac{3}{2}u_0u_1S_2-v_3S_0-\beta u_2S_0\\
&\quad+\frac{3}{2}u_1u_2S_0-u_0Y_2+u_2Y_0+\frac{3}{2}u_0u_{2x}=0,\\
&\frac{3}{2}u_2u_{1x}-v_3S_1-\beta u_2S_1+\frac{3}{2}u_1u_{2x}-v_{3x}-\beta u_{2x}+u_{2t}-2v_4S_0+\frac{3}{2}u_2^2S_0+u_2Y_1+\frac{3}{2}u_1u_2S_1=0,\\
&\frac{3}{2}u_2^2S_1+u_2Y_2-v_3S_2-\beta u_2S_2-2v_4S_1-v_{4x}+\frac{3}{2}u_2u_{2x}+\frac{3}{2}u_1u_2S_2=0,\\
&\frac{3}{2}u_2^2S_2-2v_4S_2=0,\quad \frac{3}{2}u_0S_0^3-2u_0v_0S_0=0,\\
&-\frac{3}{2}u_{0x}S_0^2-\frac{3}{2}u_0v_1S_0+3u_0S_1S_0^2+\frac{1}{2}u_0v_{0x}-2\beta v_0S_0+v_0u_{0x}-u_1v_0S_0-2u_0v_0S_1-2v_0Y_0=0,\\
&\frac{1}{2}u_0v_{1x}-2\beta v_0S_1-\frac{9}{4}u_{0x}S_0S_1-u_0v_2S_0-2u_0v_0S_2+v_0u_{1x}+\frac{7}{4}u_0S_1^2S_0+v_1u_{0x}-v_1Y_0+v_{0t}\\
&\qquad-\frac{3}{2}u_0v_1S_1-2v_0Y_1+\frac{3}{4}u_{0xx}S_0+\beta v_{0x}-\frac{1}{2}u_1v_1S_0-u_1v_0S_1+2u_0S_0^2S_2-\beta v_1S_0+\frac{1}{2}u_1v_{0x}=0,\\
&-\frac{3}{2}u_{0x}S_0S_2+\frac{1}{2}u_2v_{0x}+u_{2x}v_0+\frac{1}{2}u_1v_{1x}-\frac{1}{4}u_{0xxx}+v_{1t}-\frac{3}{4}u_{0x}S_1^2-u_0v_2S_1+\beta v_{1x}+2u_0S_0S_1S_2\\
&\qquad+\frac{1}{4}u_0S_1^3+\frac{3}{4}u_{0xx}S_1-v_1Y_1-\frac{3}{2}u_0v_1S_2-2v_0Y_2-\frac{1}{2}u_1v_1S_1+\frac{1}{2}u_2v_1S_0-\frac{1}{2}u_0v_3S_0-2\beta v_0S_2\\
&\qquad+\frac{1}{2}u_0v_{2x}-\beta v_1S_1+u_{0x}v_2+u_{1x}v_1-u_1v_0S_2=0,\\
&-\frac{1}{4}u_{1xxx}+\frac{3}{4}u_{0xx}S_2^-\frac{3}{4}u_{2xx}S_0^+\frac{1}{4}u_0S_0S_1^2-\frac{1}{4}u_2S_0S_1^2+\frac{1}{2}u_0S_0S_2^2-\frac{3}{4}u_{0x}S_1S_2-\frac{1}{2}u_2S_2S_0^2-\frac{3}{4}u_{2x}S_1S_0\\
&\qquad+\beta v_3S_0-\frac{1}{2}u_0v_3S_1-\frac{1}{2}u_1v_1S_2+\frac{1}{2}u_2v_1S_1+\frac{1}{2}u_1v_3S_0+\frac{1}{2}u_1v_{2x}+\frac{1}{2}u_2v_{1x}+\frac{1}{2}u_0v_{3x}+u_{1x}v_2\\
&\qquad+u_{0x}v_3+u_{2x}v_1+u_0v_2S_0-u_0v_2S_2-v_1Y_2+v_3Y_0+v_{2t}+\beta v_{2x}-\beta v_1S_2=0,\\
&\frac{1}{2}u_2v_{2x}+v_3u_{1x}+\frac{1}{2}u_1v_{3x}+2\beta v_4S_0+v_3Y_1+\frac{1}{2}u_1v_3S_1-\frac{3}{2}u_{2x}S_0S_2+\frac{3}{2}u_2v_3S_0+v_{3t}+v_4u_{0x}\\
&\qquad-\frac{1}{4}u_{2xxx}+v_2u_{2x}+\frac{1}{2}u_2v_1S_2-\frac{3}{4}u_{2x}S_1^2-\frac{3}{4}u_{2xx}S_1+\frac{1}{2}u_0v_{4x}+u_2v_2S_1-\frac{1}{2}u_0v_3S_2+2v_4Y_0+\beta v_{3x}\\
&\qquad-\frac{1}{4}u_2S_1^3+\beta v_3S_1+u_1v_4S_0-2u_2S_0S_1S_2=0,
\end{aligned}$$

$$\beta v_{4x} + v_3 u_{2x} + \frac{1}{2}u_1 v_{4x} - \frac{3}{4}u_{2xx}S_2 + v_3 Y_2 + v_4 u_{1x} - \frac{7}{4}u_2 S_2 S_1^2 + v_{4t} + 2v_4 Y_1 + \frac{1}{2}u_2 v_{3x} + u_1 v_4 S_1$$
$$+ 2u_2 v_4 S_0 - \frac{9}{4}u_{2x}S_1 S_2 - 2u_2 S_0 S_2^2 + \beta v_3 S_2 + 2\beta v_4 S_1 + u_2 v_2 S_2 + \frac{1}{2}u_1 v_3 S_2 + \frac{3}{2}u_2 v_3 S_1 = 0,$$
$$\frac{3}{2}u_2 v_3 S_2 + 2v_4 Y_2 + u_1 v_4 S_2 + 2\beta v_4 S_2 + \frac{1}{2}u_2 v_{4x} + 2u_2 v_4 S_1 - \frac{3}{2}u_{2x}S_2^2 - 3u_2 S_1 S_2^2 + v_4 u_{2x} = 0,$$
$$-\frac{3}{2}u_2 S_2^3 + 2u_2 v_4 S_2 = 0. \tag{17}$$

Substituting Eq. (16) into Eq. (17), and making use of Maple software, we have

$$u_0 = S_0, \quad v_0 = \frac{3}{4}S_0^2, \quad u_1 = \frac{1}{2}S_1 - \lambda, \quad v_1 = \Big(\frac{3}{4}S_1 - \frac{\lambda}{2} - \beta\Big)S_0, \quad u_2 = S_2, \quad v_4 = \frac{3}{4}S_2^2,$$
$$v_2 = \frac{1}{16}S_1^2 - \frac{1}{4}\lambda S_1 - \frac{1}{2}\beta S_1 + \lambda\beta + \frac{1}{4}\lambda^2 + \beta^2 + \frac{1}{2}S_0 S_2 v_3 = \Big(\frac{3}{4}S_1 - \frac{\lambda}{2} - \beta\Big)S_2,$$
$$\xi(x,t) = \frac{-S_1 + \tan\Big(\frac{1}{2}x\sqrt{4S_0S_2 - S_1^2} + \frac{1}{2}\Big(\lambda t \pm c\Big)\sqrt{4S_0S_2 - S_1^2}\Big)\sqrt{4S_0S_2 - S_1^2}}{2S_2}. \tag{18}$$

Therefore, one of the solutions of Eq. (1) can be written as

$$u = \frac{u_0}{\xi} + u_1 + u_2\xi, \quad v = \frac{v_0}{\xi^2} + \frac{v_1}{\xi} + v_2 + v_3\xi + v_4\xi^2, \tag{19}$$

where $\xi(x,t) = \dfrac{-S_1 + \tan\Big(\frac{1}{2}x\sqrt{4S_0S_2 - S_1^2} + \frac{1}{2}(\lambda t \pm c)\sqrt{4S_0S_2 - S_1^2}\Big)\sqrt{4S_0S_2 - S_1^2}}{2S_2}$.

5. Conclusions

In this paper, we have introduced the Painlevé property of the modified C-KdV equation in detail, then drawn the conclusion that it is Painlevé integrable. Applying the standard truncated expansion and non-standard truncated expansion methods, we obtained two types of exact soliton solutions, respectively. We can use these methods to derive more solutions of other nonlinear equations.

References

[1] Yao R X, Jiao X Y and Lou S Y 2009 *Chin. Phys.* B **18** 1827
[2] Bai C L 2001 *Chin. Phys.* **10** 1091
[3] Yan Z Y and Zhang H Q 2000 *Acta Phys. Sin.* **49** 2113 (in Chinese)
[4] Xia T C and Chen D Y 2005 *Chaos, Solitons and Fractals* **23** 1405
[5] Xia T C, Wang H and Zhang Y F 2005 *Chin. Phys.* **14** 247
[6] Xia T C and You F C 2007 *Chin. Phys.* **16** 650
[7] Mao J J and Yang J R 2007 *Acta Phys. Sin.* **56** 5049 (in Chinese)
[8] Li W and Xie F D 2004 *Chin. Phys.* **13** 1639
[9] Zhang Q F and Fan E G 2007 *Chin. Phys.* **16** 1505
[10] Dong H H and Yang H W 2009 *Chin. Phys.* B **18** 845
[11] Dong H H 2007 *Chin. Phys.* **16** 1177
[12] Dong H H and Zhang N 2006 *Chin. Phys.* **15** 1919
[13] Li Z 2009 *Chin. Phys.* B **18** 850
[14] Chen Y and Fan E G 2007 *Chin. Phys.* **16** 6
[15] Gao B, Liu S D and Liu S K 2009 *Acta Phys. Sin.* **58** 2155 (in Chinese)
[16] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
[17] Conte R 1989 *Phys. Lett.* A **140** 383
[18] Pickering A 1993 *J. Phys.* A. *Math. Gen.* **26** 4395
[19] Lou S Y 1998 *Acta Phys. Sin.* **47** 1937 (in Chinese)
[20] Lou S Y 1998 *Z. Naturforsch.* **53** 251
[21] Lou S Y, Chen C L and Tang X Y 2002 *J. Math. Phys.* **43** 4078
[22] Zhang S L, Wu B and Lou S Y 2002 *Phys. Lett.* A **300** 40
[23] Chen C L, Zhang J and Li Y S 2007 *Chin. Phys.* **16** 2167
[24] Zhang J F and Chen F Y 2001 *Acta Phys. Sin.* **50** 1648 (in Chinese)
[25] Wei C M, Xia Z Q and Tian N S 2005 *Acta Phys. Sin.* **54** 2463 (in Chinese)
[26] Kowalevskaya S 1890 *Acta Math.* **14** 81
[27] Lou S Y and Hu X B 1997 *J. Math. Phys.* **38** 6401
[28] Fan E G 2004 *Integrable Systems and Computer Algebra* (Beijing: Science Press) p. 156 (in Chinese)
[29] Zedan Hassan A 2009 *Chaos, Solitons and Fractals* **30** 550