Stokes parameters of optical singularities of random electromagnetic beams*

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In this paper, the relation between the spectral degree of coherence and degree of polarization of random electromagnetic beams is derived by the Stokes parameters. And the concept of polarization singularity is extended from spatially fully coherent beams to partially coherent electromagnetic beams. Theoretical analysis shows that correlation vortices are linearly polarized singularities. The results are illustrated by numerical examples.

Keywords: Stokes parameters, optical singularities, electromagnetic beams
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1. Introduction

The study of the behaviour of optical vortices has spawned a subfield of optical research, now known as singular optics.[1−7] As yet, however, an analysis of a relation between the spectral degree of coherence and degree of polarization of optical vortices does not appear to have been made. Using the Stokes parameters,[8−11] we derive the relation between spectral degree of coherence and degree of polarization of random electromagnetic beams in the present paper, and expand the research range of the polarization singularity from fully coherent, quasi-monochromatic beams to partially coherent, quasi-monochromatic beams. Finally, some results are illustrated by a radically polarized partially coherent LG (0, 1) beam.

2. Stokes parameters of optical singularities of random electromagnetic beams

We consider two stochastic electromagnetic beams which are statistically stationary, at least in the wide sense. They may be characterized by ensembles (denoted by curly brackets) of realizations, $\{E(r_1,\omega)\}$, and $\{E(r_2,\omega)\}$, in the coherence theory in the space–frequency domain (Ref. [8], Chapters 4 and 8). The superposition field at two points with position vector $\boldsymbol{r}_{\rm i}$ is represented by an ensemble $\{\boldsymbol{E}\}$. Let the components of the electric field density in two mutually perpendicular directions perpendicular to the direction of propagation of the stochastic electromagnetic field be expressed in the form

$$\begin{aligned} E_x &= A_1[\cos(\Phi_1(t)-\omega t)], \\ E_y &= A_2[\cos(\Phi_2(t)-\omega t)], \end{aligned} \tag{1}$$

where t denotes the time, A_1 and A_2 represent the amplitudes of the x and the y components of the electric field density, respectively. The Stokes parameters of the superposition field can be introduced into electromagnetic wave fields expressed as Eq. (1), which corresponds to the tradition Stokes parameters, that is

$$\begin{aligned} s_0 &= \langle A_1^2\rangle + \langle A_2^2\rangle, \\ s_1 &= \langle A_1^2\rangle - \langle A_2^2\rangle, \\ s_2 &= 2\langle A_1 A_2\cos(\Phi_1-\Phi_2)\rangle, \\ s_3 &= 2\langle A_1 A_2\sin(\Phi_1-\Phi_2)\rangle, \end{aligned} \tag{2}$$

where the sharp brackets denote the time average. In Eq. (1) only E_x (t) and $E_y(t)$ may be regarded as being uniquely associated with the wave, whereas the A's and Φ's may evidently be chosen in many ways, leading to different sets of Stokes parameters.

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Suppose that $\boldsymbol{J}$ denotes the coherency matrix of partially coherent electromagnetic beams, and it can be expressed as

$$J = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix}, \quad (3)$$

where $J_{xy} = J_{yx}^*$ and the coherency matrix is Hermitian. The trace of J represents the intensity of the incident wave; its non-diagonal elements express the correlation between the x and the y-components of the complex electric vector $\boldsymbol{E}$. And its elements in Eq. (3) satisfy the following Schwarz' inequality for integrals

$$|J_{xy}| \leq (J_{xx} J_{yy})^{1/2}, \quad |J_{yx}| \leq (J_{yy} J_{xx})^{1/2}, \quad (4)$$

hence

$$|J| = J_{xx} J_{yy} - J_{xy} J_{yx} \geq 0, \quad (5)$$

which means the discrimination of the coherency matrix is non-negative.

Let $\Psi_1(t) = \Phi_1(t) - \omega t$ and $\Psi_2(t) = \Phi_2(t) - \omega t$ be the phases of $E_x(t)$ and $E_y(t)$ respectively, then

$$E_x = A_1 \exp[\mathrm{i}\, \Psi_1(t)], \text{ and } E_y = A_2 \exp[\mathrm{i}\, \Psi_2(t)]. \quad (6)$$

So the elements of the coherency matrix are

$$J_{xx} = \langle A_1^2 \rangle, \quad J_{yy} = \langle A_2^2 \rangle, \quad (7a)$$

$$J_{xy} = \langle A_1 A_2 \exp[\mathrm{i}(\Psi_1 \Psi_2)] \rangle,$$

$$J_{yx} = \langle A_1 A_2 \exp[-\mathrm{i}(\Psi_1 \Psi_2)] \rangle. \quad (7b)$$

Now, a set of Stokes parameters of the random electromagnetic beam can be introduced by the relations

$$s_0 = J_{xx} + J_{yy}, \quad s_1 = J_{xx} - J_{yy},$$
$$s_2 = J_{xy} + J_{yx}, \quad \text{and} \quad s_3 = \mathrm{i}(J_{yx} - J_{xy}). \quad (8)$$

The degree of polarization (P) of the random stochastic electromagnetic can be written as

$$P = (s_1^2 + s_2^2 + s_3^2)^{1/2} / s_0. \quad (9)$$

In this paper, we choose the complex degree of coherence μ_{xy} of the electric vibrations along the x and the y directions to be

$$\mu_{xy} = \frac{J_{xy}}{\sqrt{J_{xx} J_{yy}}} \quad (10)$$

and Eq. (10) can be rewritten with the Stokes parameters as follows:

$$\mu_{xy} = \frac{s_2 + \mathrm{i}\, s_3}{2\sqrt{s_0^2 - s_1^2}}. \quad (11)$$

It corresponds to the complete coherence state of the random electromagnetic field when the absolute value of μ_{xy} is unity and its phase is equal to the difference in phase between the two components. According to Eqs. (9) and (11) the relation between P and $|\mu_{xy}|$ can be expressed as

$$P = \sqrt{1 - \frac{(s_0^2 - s_1^2)(s_0^2 - s_1^2 - s_2^2 - s_3^2)}{s_0^2 (s_0^2 - s_1^2 - s_2^2 + s_3^2)} [1 - |\mu_{xy}|^2]}. \quad (12)$$

Equation (12) is one of the main results in this paper. And P and μ_{xy} satisfy the following inequality:

$$P \geq |\mu_{xy}|. \quad (13)$$

Equation (13) will hold when $J_{xy} = J_{yx}$. It implies that the (time averaged) intensities of the electromagnetic field in (along) the two orthogonal directions are equal. As shown in Eq. (13), the fully coherent and full polarization beams are the special cases of the partially coherent, partially polarization beams.

The positions of the coherence vortices can be obtained from the following equations:[12]

$$\mathrm{Re}[\mu_{xy}] = 0, \quad \text{and} \quad \mathrm{Im}[\mu_{xy}] = 0, \quad (14)$$

where Re and Im denote the real and the imaginary parts of μ_{xy}, respectively. The topological charge and its sign of correlation vortex are determined by the vorticity of phase contours around singularity

Based on Eq. (14), it is found that the Stokes parameters of the coherent vortices satisfy the following equations:

$$s_2 = 0, \quad \text{and} \quad s_3 = 0. \quad (15)$$

And the degree of polarization of the coherent vortex can be written as

$$P_{\mathrm{cv}} = s_1 / s. \quad (16)$$

Based on the determination conditions of C-points of polarization singularities, the Stokes parameters of the C points satisfy

$$s_1 = 0, \quad s_2 = 0 \quad \text{and} \quad s_3 < 0 \quad \text{(right-handed } C\text{-points)}, \quad (17a)$$

$$s_1 = 0, \quad s_2 = 0 \quad \text{and} \quad s_3 > 0 \quad \text{(left-handed } C\text{-points)}, \quad (17b)$$

and they can be expressed with the elements of the mutual cross-spectral intensity as follows:

$$J_{yy} = J_{xx}, \quad \mathrm{Re}[J_{xy}] = 0 \quad \text{and} \quad \mathrm{Im}[J_{xy}] < 0 \quad \text{(right-handed } C\text{-points)}, \tag{18a}$$

$$J_{yy} = J_{xx}, \quad \mathrm{Re}[J_{xy}] = 0 \quad \text{and} \quad \mathrm{Im}[J_{xy}] < 0 \quad \text{(left-handed } C\text{-points)}. \tag{18b}$$

The Stokes parameters of linear polarization singularity (L line) satisfy

$$s_3 = 0, \quad (L\text{-line}). \tag{19}$$

Let

$$\mathrm{PU} = (1 - P^2)/(1 - |\mu_{xy}|^2), \quad \mu_{xy} \neq \pm 1 \tag{20}$$

denote the relation between polarization degree and spectral degree of coherence. It is easy to know that PU < 1, and PU can be rewritten as

$$\mathrm{PU} = \frac{(s_0^2 - s_1^2)(s_0^2 - s_1^2 - s_2^2 - s_3^2)}{s_0^2(s_0^2 - s_1^2 - s_2^2 + s_3^2 - 2\,\mathrm{i}\,s_2 s_3)}. \tag{21}$$

As mentioned above, the coherence vortices and polarization singularities of random electromagnetic beams can be determined by Eqs. (15) and (17–19). $s_2 = 0$ and $s_3 = 0$ implies that optical singularities correspond to coherence vortices. $s_1 = 0$ and $s_2 = 0$ implies that optical singularities correspond to circular polarization singularities, i.e. C points of electromagnetic beams, where $s_3 > 0$ ($s_3 < 0$) is the left- (the right-) handed C-point at which the orientations of the major and the minor axes of the polarization ellipse are undefined. $s_3 = 0$ implies that optical singularities correspond to linear polarization singularities. Additionally, it is readily seen that in the fully coherent limit the broad spectral singularities reduce to the Stokes singularities. The concept of polarization singularity is extended from spatially fully coherent beams to stochastic electromagnetic beams.

3. A typical example: polarization singularities of partially coherent LG(0,1) electromagnetic beams

In the plane $z = 0$, the elements of the BCP matrix for a radically polarized partially coherent LG (0, 1) beam read[13]

$$J_{\alpha\beta} = \frac{2E_0^2}{w_0^2}\alpha_{10}\beta_{20}\exp\left(-\frac{\boldsymbol{r}_{10}^2 + \boldsymbol{r}_{20}^2}{w_0^2}\right) \times \exp\left[-\frac{(\boldsymbol{r}_{10} - \boldsymbol{r}_{20})^2}{\sigma_{\alpha\beta}^2}\right], \quad (\alpha, \beta = x, y), \tag{22}$$

in which $r_d = (x_d, y_d)$ ($d = 1, 2$) is the position vector in the input plane, w_0 is the beam waist, $\sigma_{\alpha\beta}$ denotes the mutual coherent length (when $\alpha \neq \beta$) or auto-coherent length (when $\alpha = \beta$), and E_0 is the intensity parameter independent of position. Substituting Eq. (22) into Eqs. (15) and (23), we can have the relation between P and $|\mu_{xy}|$ of the radically polarized partially coherent LG (0, 1) beam and its PU parameters.

Now, we numerically calculate the relation between the spectral degree of coherence and degree of polarization of the radically polarized partially coherent LG(0,1) beam. The calculation parameters are $w = 0.2$ mm, spatial correlation parameters $\beta = (1 + (w_0/\sigma_{xx})^2)^{1/2} = 0.5, \sigma_{xx} = w_0/((\beta)^{-1/2} - 1)^{1/2}$, $\sigma_{xy} = \sigma_{yx} = 0.2\sigma_1$, and $\sigma_{yy} = 0.6\sigma_{xx}$.

Figure 1 shows the variations of the degree of polarization of the LG(0,1) beam with the spectral degree of coherence of the LG (0,1) electromagnetic field at positions 0.02 mm and 0.01 mm. As shown in Fig. 1, the degree of polarization P (denoted by '•') increases with the spectral degree of coherence μ_{xy} increasing, and the degree of polarization P is significantly greater than the spectral degree of coherence μ_{xy}. Their changing tendencies are consistent with those described by Eq. (16).

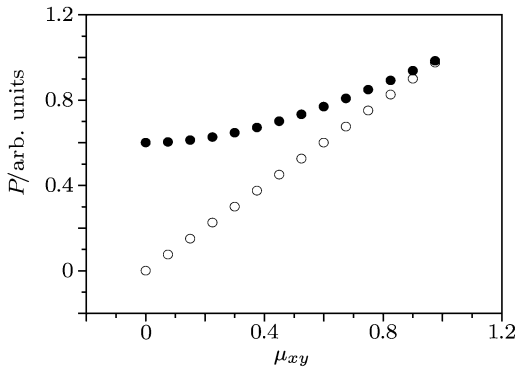


Fig. 1. Variations of degree of polarization with spectral degree of coherence of LG (0,1) electromagnetic beam at positions 0.02 mm and 0.01mm.

Figure 2 gives the distribution and the contour lines of the polarization degree of the coherent vortices.

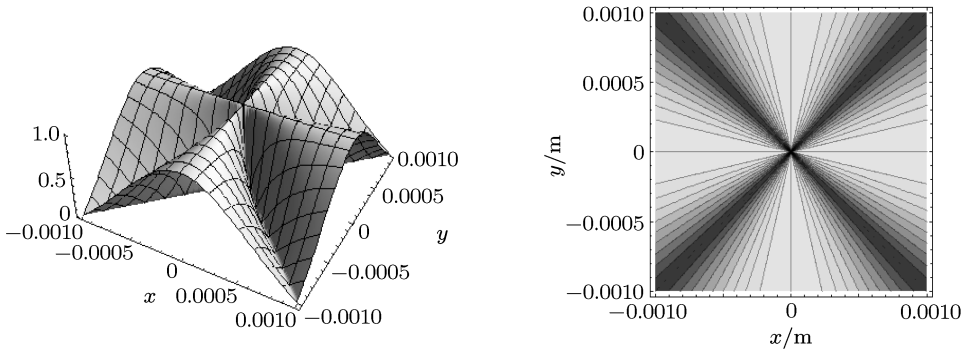


Fig. 2. Space distribution and contour of the polarization degree of the coherence vortices.

From Fig. 2 it follows that the distribution of polarization degree is Centro-symmetric distribution with the position (0, 0). The degree of polarization is maximum (equal to 1) when $x = 0$ or $y = 0$, and the minimum degree of polarization exists along the line $y = x$ (or $-x$).

Figure 3 displays the space distribution and the contour lines of PU. As shown in Fig.3, the space distribution of PU is Centro-symmetric distribution with the position (0, 0). The degree of polarization is maximum (equal to 1) when $y = x$ (or $-x$), and the minimum degree of polarization exists on the line $y = 0$ or ($x = 0$). Compared with the changing tendencies in Fig. 2, their changing tendencies are complementary each other.

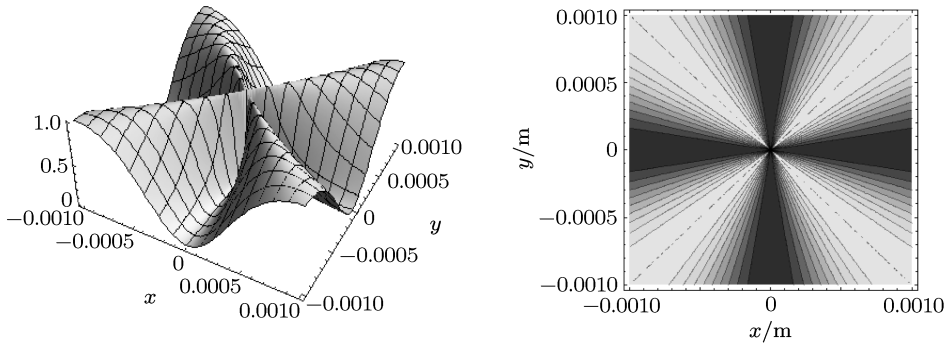


Fig. 3. Space distribution and contour of the parameter PU of the stochastic electromagnetic field.

Figure 4 indicates the space distribution of linear polarization coherence vortices (denote by '•') of radically polarized partially coherent LG (0, 1) beams. As shown in the figure, there exist five linear polarization coherence vortices, and they are of Centro-symmetric distribution with the position (0, 0).

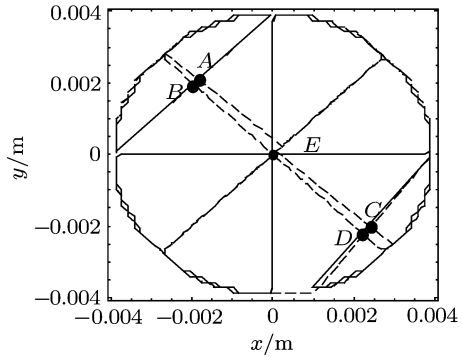


Fig. 4. Space distribution of linear polarization coherence vortices of radically polarized partially coherent LG (0,1) beams.

4. Conclusion

In this paper, the relation between the spectral degree of coherence and degree of polarization of random electromagnetic field is derived. The theory demonstrates that the degree of polarization P is bigger than the spectral degree of coherence μ_{xy}. And the concept of polarization singularity is extended from fully coherent beams to partially coherent electromag-

netic beams. Taking radically polarized partially coherent LG (0, 1) beams as a typical example, we perform the numerical analysis of space distributions and contour lines of polarization degree of the coherent vortices, and obtain the results that coherent vortices are all linearly polarized.

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