Consensus problems in multi-agent systems with double integrator model*

Gao Li-Xin(高利新)†, Yan Hui-Juan(闫慧娟), and Jin Dan(金 丹)

College of Mathematics & Information Science, Wenzhou University, Zhejiang 325027, *China*

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In this paper, we consider multi-agent consensus problems in a decentralised fashion. The interconnection topology graph among the agents is switching and undirected. The agent dynamics is expressed in the form of a double integrator model. Two different cases are considered in this study. One is the leader-following case and the other is leaderless case. Based on graph theory and common Lyapunov function method, some sufficient conditions are obtained for the consensus stability of the considered systems with the neighbour-based feedback laws in both leader-following case and leaderless case respectively. Finally, two numerical examples are given to illustrate the obtained results.

Keywords: multi-gent systems, stability analysis, consensus, double-integrator model
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1. Introduction

In recent years, the coordination problem of multiple autonomous agents has drawn increasing attention. It is because there are many applications of multi-agent systems in many areas such as the aggregation behaviour of animals, collective motion of particles, cooperative control of unmanned air vehicles, schooling for underwater vehicles, distributed sensor networks and attitude alignment for clusters of satellites in Refs. [1]–[6].

Although these practical backgrounds are different, the fundamental idea in coordinating multiple agents is that of coordinating multiple agents to achieve a goal of the whole system by local information. The investigated feedback scheme associated with those problems is inspired by the aggregates of individuals in nature. Flocks of birds and schools of fish achieve coordinated motions of large numbers of individuals without a central controlling mechanism in Ref. [4]. Reynolds first proposed a computer animation model to simulate collective behaviours of multiple agents in Ref. [3]. Several researchers in the area of statistical physics and theory of complexity have addressed flocking and schooling behaviours in systems of self-propelled particles in Refs. [1] and [7]. In real applications, the interacting topology between agents may change dynamically. For example, in the case of interaction via communications, the communication between vehicles may be unreliable due to disturbances and/or subject to a communication range limit. In the case of interaction via sensing, the agents that can be sensed by a certain agent may change over time. Based on bidirectional information exchange (modeled by undirected graphs) and a simple local rule for each agent, the consensus problem of a group of agents could be solved for a large class of switching signals under any initial conditions in Refs. [5] and [8]. Similar or generalised consensus problems have also been studied in Refs. [9]–[16].

Furthermore, with many practical applications, especially involving mechanical systems, different approaches have been proposed for the cases when each agent's dynamics is taken as a double integrator. The double integrator model is a second-order model. The interconnection topology graph among the agents is switching and undirected. There are many interesting agent-related works involved double integrator agent's dynamics such as in Ref. [10], [15], [17] and [18]. In Ref. [10], the authors pointed out that the matrix to describe this multi-agent dynamics does not satisfy the properties related to the stochastic matrix and the norm-based method proposed by Ref. [5] may fail for the double-integrator form agent dynamic. It is often to chose the Lyapunov-based approach in multi-agent systems with the double integrator model.[10,15,17,18]

With this background, we study a group of agents with double-integrator dynamics. The main purpose is to develop some decentralised control strategies to achieve global consensus of the multi-agent systems.

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†Corresponding author. E-mail: lxgao@wzu.edu.cn
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In leader-following formulation, although the interconnection topological structure among the agents keeps changing, a sufficient condition established by constructing a parameter-dependent common Lyapunov function can still guarantee global consensus stability. As is commonly known, it is not an easy task to construct a common Lyapunov function for a switching system, let alone the parameter-dependent common Lyapunov function. While in the leadless case, the common Lyapunov method can not be used directly. By using Schur orthogonal transformation, we decompose the group tracking dynamics into two subsystems: one represses the leader and the other represses a leader-following subsystem. Then, a sufficient condition is obtained based on the obtained result of leader-following case to guarantee global consensus stability.

The paper is organised as follows. In Section 2, we give a formulation of the coordination problem with the help of graph theory. Then in Section 3, the results on the consensus stability are obtained for the multi-agent system with varying interconnection topologies in the leader-following case. In Section 4, the leaderless case is studied. In Section 5, simulation examples are presented. Finally, the concluding remarks are given in Section 6.

The notation used in this paper is standard. $\mathbb{R}$ is the real number set. $\boldsymbol{I}$ is an identity matrix with compatible dimension. $\boldsymbol{A}^{\mathrm{T}}$ is denoted as transpose of a matrix $\boldsymbol{A}$. For symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} > (\geq)\boldsymbol{B}$ means $\boldsymbol{A}-\boldsymbol{B}$ is positive (semi-) definite. $\lambda(\boldsymbol{A})$ represents an eigenvalue of $\boldsymbol{A}$. For symmetric matrices $\boldsymbol{A}$, $\lambda_{\min}(\boldsymbol{A})$ and $\lambda_{\max}(\boldsymbol{A})$ represent the minimum and maximum eigenvalue of $\boldsymbol{A}$ respectively. $\|\bullet\|$ denotes Euclidean norm. $\otimes$ denotes the Kronecker product, which satisfies (1) $(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}) = (\boldsymbol{AC})\otimes(\boldsymbol{BD})$; (2) If $\boldsymbol{A}\geq 0$ and $\boldsymbol{B}\geq 0$, then $\boldsymbol{A}\otimes\boldsymbol{B}\geq 0$.

2. Problem formulation

Stability analysis of the group of agents is based on several results of algebraic graph theory. In this section, we first introduce some basic concepts and notations in graph theory that will be used throughout this paper. More details are available in Ref. [19]. Denote the graph by $\mathcal{G} = \{V, \varepsilon\}$, where $V = \{v_i, i \in \mathcal{I}\}$ is the set of vertices (with $\mathcal{I}$ an index set) and $\varepsilon \subset V \times V$ is the set of edges of the graph. Throughout this paper, we assume all the graphs have no edges (v_i, v_i) from a node to itself, and each edge is denoted by $e_k = (v_i, v_j) \in \varepsilon,\ i, j \in \mathcal{I}$. A graph is said to be undirected if $\forall\,(v_i, v_j) \in \varepsilon \Rightarrow (v_j, v_i) \in \varepsilon$. Otherwise, the graph is called a directed graph. If $(v_i, v_j) \in \varepsilon$, then v_j is said to be a neighbour of v_i and we denote the set of all neighbour vertices of vertex v_i by denoting $\mathcal{N}_i = \{j|(v_i, v_j) \in \varepsilon\}$. A path from vertex v_i to vertex v_j is a sequence of distinct vertices starting v_i and ending with v_j such that consecutive pair of vertices make an edge of graph. A graph is simple if it has no self-loops or repeated edges. If there is a path between any two vertices of a graph $\mathcal{G}$, then $\mathcal{G}$ is connected, otherwise disconnected.

The relationships between n agents can be conveniently described by a simple, undirected graph $\mathcal{G}$ with an index set $\mathcal{I} = \{1, 2, \ldots, n\}$, which is defined so that (i, j) defines one of the graph's edges if agents i and j are neighbours. Neighbouring relations reflect physical proximity between two agents, or the existence of a communication channel. This undirected interconnection graph $\mathcal{G}$ can be used straightforward in the leaderless case. Its weighted adjacency matrix $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ii} = 0$ and $a_{ij} = a_{ji} \geq 0$ with $a_{ij} > 0$ meaning that there is an edge between agent i and agent j. Its degree matrix $\boldsymbol{D} = \mathrm{diag}\{d_1, \ldots, d_n\} \in \mathbb{R}^{n\times n}$ is a diagonal matrix, where diagonal elements $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 1, \ldots, n$. Then the Laplacian of $\mathcal{G}$ is defined as

$$\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}, \tag{1}$$

which is symmetric.

In the leaderless case, we consider the graph $\mathcal{G}$, but in the leader-following case, we consider another graph $\bar{\mathcal{G}}$, associated with the system consisting of n agents and one leader. Here the state of the leader (labelled 0) is denoted by v_0, and the relationships between the agents and the leader is a directed graph; namely there are only edges from some agents to the leader at some moment t, but there are no edges from the leader to any agents. In other words, the directed graph $\bar{\mathcal{G}}$ contains $\mathcal{G}$ and v_0 with directed edges from some agents to the leader v_0.

For the multi-agent system under consideration, the relationships between neighbours can change over time and the interconnection topology may dynamically change. By that the graph of this system is connected, we mean that graph $\mathcal{G}$ for all the agents is connected (in the leaderless case), or at least one agent in each component of graph $\mathcal{G}$ associated with n agents is connected to the leader (in the leader-following case). In the leaderless case, $\mathcal{S} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{N_0}\}$ is denoted as a set of the graphs of all possible topologies, while in the leader-following case, $\bar{\mathcal{S}} =$

$\{\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \ldots, \bar{\mathcal{G}}_N\}$ is denoted as a set of the graphs of all possible topologies.

In this paper, we intend to coordinate n mobile agents with each agent expressed in the form of a double integrator:

$$\begin{cases} \dot{r}_i = p_i \\ \dot{p}_i = u_i \end{cases} \quad r_i,\ p_i,\ u_i \in \mathbb{R}^m, \quad i = 1, ..., n, \tag{2}$$

where r_i is the position of agent i, p_i is its velocity and u_i is its control input. In the leader-following case, the dynamic of the leader is taken as

$$\begin{cases} \dot{r}_0 = p_0, \\ \dot{p}_0 = a_0(t), \end{cases} \quad r_0,\ p_0,\ a_0 \in \mathbb{R}^m, \tag{3}$$

where $a_0(t)$ is a known input, which may be regarded as some given policy known to all the agents. Particularly, $a_0(t) \equiv 0$ expresses that the leader is moving in constant velocity. The leader of this considered multi-agent system is active, and its motion only depend on the known input $a_0(t)$ and do not be influenced by the following agents.

When the multi-agent system achieve consensus, it requires not only all agent not scattered, but also velocity matching. Thus, the n mobile agents are said to achieve global consensus, if under any given initial values $r_i(0)$ and $p_i(0)$ $(i = 1, 2, \ldots, n)$, $\lim\limits_{t\to\infty}(r_i(t) - r_j(t)) = 0$ and $\lim\limits_{t\to\infty}(p_i(t) - p_j(t)) = 0$ for any $i, j = 1, 2, \ldots, n$. Moreover, in the leader-following case, the multi-agent system achieve global consensus means that $\lim\limits_{t\to\infty}(r_i(t) - r_0(t)) = 0$ and $\lim\limits_{t\to\infty}(p_i(t) - p_0(t)) = 0$ for any $i = 1, 2, \ldots, n$. We say that local control law solves the consensus problem if the closed-loop feedback system achieve global consensus.

In this study, we are interesting in using neighbour control law to solve the consensus problem. To this end, the controller u_i of agent i, regarded as node i in a graph, requires state information from a subset of the agent's flockmates, called the neighbour set $\mathcal{N}_i$ defined above. For system (2), we consider the following two cases.

Leader-following Take the local control law as follows for agent i $(i = 1, \ldots, n)$, which is neighbour-based feedback law:

$$u_i = a_0(t) - \kappa\left(\sum_{j\in\mathcal{N}_i} a_{ij}(t)(r_i - r_j) + b_i(t)(r_i - r_0)\right) - \kappa\gamma\left(\sum_{j\in\mathcal{N}_i} a_{ij}(t)(p_i - p_j) + b_i(t)(p_i - p_0)\right). \tag{4}$$

Here κ is a 'control' parameter, which is positive constant and will be determined later, γ is positive weighted parameter. At time t, the $a_{ij}(t)$ and $b_i(t)$ are chose by

$$a_{ij}(t) = \begin{cases} \alpha_{ij} \text{ if agent } i \text{ is connected to agent } j, \\ 0 \quad \text{otherwise}, \end{cases} \tag{5}$$

$$b_i(t) = \begin{cases} \beta_i \text{ if agent } i \text{ is connected to the leader}, \\ 0 \quad \text{otherwise}, \end{cases} \tag{6}$$

where $\alpha_{ij} > 0$ $(i, j = 1, ..., n)$ is connection weight constant between agent i and agent j, and $\beta_i > 0$ $(i = 1, ..., n)$ is connection weight constant between agent i and leader. We also assume that $\alpha_{ij} = \alpha_{ji}$. Let $s_j = r_j - r_0$ and $q_j = p_j - p_0$, for $j = 1, \ldots, n$, and moreover, $s := \mathrm{col}(s_1, \ldots, s_n) \in \mathbb{R}^{mn}$, $q := \mathrm{col}(q_1, \ldots, q_n) \in \mathbb{R}^{mn}$, and $u := \mathrm{col}(u_1 - a_0(t), u_2 - a_0(t), \ldots, u_n - a_0(t)) \in \mathbb{R}^{mn}$. Then we can rewrite the closed-loop systems (2), (3) and (4) as

$$\dot{s} = q, \quad \dot{q} = u_\sigma, \tag{7}$$

and

$$u_\sigma = -\kappa(\boldsymbol{H}_\sigma \otimes \boldsymbol{I}_m)s - \kappa\gamma(\boldsymbol{H}_\sigma \otimes \boldsymbol{I}_m)q, \tag{8}$$

where $\boldsymbol{H}_\sigma = \boldsymbol{L}_\sigma + \boldsymbol{B}_\sigma$, $\sigma : [0, \infty) \to \mathcal{P} = \{1, 2, \ldots, N\}$ ($\mathcal{P} = \{1, 2, \ldots, N\}$ as an index set) is a piecewise constant switching signal with successive times when the topology of graph $\bar{\mathcal{G}}$ switches. $\boldsymbol{L}_\sigma$ is the Laplacian matrix of the switching graph $\mathcal{G}$ consisting of n vertices (presenting agents $1, \ldots, n$), and $\boldsymbol{B}_\sigma$ is a diagonal matrix whose i diagonal entry is $b_i(t)$.

Leaderless There is no leader dynamic (3), so we take the local control law as:

$$u_i = -\kappa\sum_{j\in\mathcal{N}_i} a_{ij}(t)(r_i - r_j) - \kappa\gamma\sum_{j\in\mathcal{N}_i} a_{ij}(t)(p_i - p_j), \quad i = 1, \ldots, n, \tag{9}$$

where the $a_{ij}(t)$ is also chose as (5). Let $r := \mathrm{col}(r_1, \ldots, r_n)$ and $p := \mathrm{col}(p_1, \ldots, p_n)$. Then we can rewrite the closed-loop system (2) and (9) as

$$\dot{r} = p, \quad \dot{p} = u_\sigma, \tag{10}$$

and

$$u_\sigma = -\kappa(\boldsymbol{L}_\sigma \otimes \boldsymbol{I}_m)r - \kappa\gamma(\boldsymbol{L}_\sigma \otimes \boldsymbol{I}_m)p, \tag{11}$$

where $\sigma : [0,\infty) \to \mathcal{P}_0 = \{1,2,\ldots,N_0\}$ ($\mathcal{P}_0 = \{1,2,\ldots,N_0\}$ as an index set) is a piecewise constant switching signal to describe the switch of the interconnection graph $\mathcal{G}$. In fact, Eq. (11) can be viewed as a special case of (8) by taking $\boldsymbol{B}_\sigma \equiv \boldsymbol{0}$.

Some well known results will be used later. The following well-known result establishes a direct relationship between the graph connectivity and its Laplacian. It can be found in Ref. [19].

Lemma 1 Let $\mathcal{G}$ be an undirected graph on n vertices with weighted Laplacian $\boldsymbol{L}$. Denote the eigenvalues of $\boldsymbol{L}$ by $\lambda_1,\ldots,\lambda_n$ satisfying $\lambda_1 \le \ldots \le \lambda_n$. Then $\lambda_1 = 0$ and $\boldsymbol{1} = (1,1,\ldots,1)^{\mathrm{T}} \in \mathbb{R}^n$ is its eigenvector. Moreover, if the graph $\mathcal{G}$ is connected, $\lambda_1 > 0$.

In what follows, we first focus on the analysis of the leader-following case, and then study the leaderless case in a similar trace.

3. Leader-following case

In this section, we concentrate on the analysis of the multi-agent systems with a leader. In this case, we consider $\bar{\mathcal{G}}$ instead of $\mathcal{G}$. Usually, it is assumed that chattering does not occur, that is, that $\sigma(t)$ switches a finite number of times in every finite time interval. Then the closed-loop system of multi-agent system has a well-defined solution. Let $t_1 = 0, t_2, \ldots$ be an infinite time sequence at which the interconnection graph of the considered multi-agent system switches. Thus in any interval $[t_j, t_{j+1})$, the Laplacian $\boldsymbol{L}_\sigma$ associated with $\mathcal{G}$ is a time-invariant matrix. Therefore, the closed-loop system (7) and (8) will be expressed in a compact form as follows:

$$\dot{x} = (\boldsymbol{F}_\sigma \otimes \boldsymbol{I}_m)x, \tag{12}$$

with

$$x = \begin{pmatrix} s \\ q \end{pmatrix}, \quad \boldsymbol{F}_\sigma = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\kappa\boldsymbol{H}_\sigma & -\kappa\gamma\boldsymbol{H}_\sigma \end{pmatrix}. \tag{13}$$

In any interval $[t_j, t_{j+1})$, the matrix $\boldsymbol{F}_\sigma$ is a time-invariant matrix. Thus, we can assume that $\sigma(t) = k, k \in \mathcal{P}, t \in [t_j, t_{j+1})$. Note that $\boldsymbol{F}_\sigma$ is not directly related to a stochastic matrix, and therefore, the method reported in Ref. [5] cannot be applied. In what follows, we will propose an approach based on common Lyapunov function for system (12).

Note that $\boldsymbol{H}_k = \boldsymbol{L}_k + \boldsymbol{B}_k \in \mathbb{R}^{n\times n}$, $k \in \mathcal{P}$ associated with graph $\bar{\mathcal{G}}$. Obviously, $\boldsymbol{H}_k$ is positive semi-definite because both $\boldsymbol{L}_k$ and $\boldsymbol{B}_k$ are positive semi-definite. Furthermore, we have (see Ref. [10])

Lemma 2 If graph $\bar{\mathcal{G}}$ is connected, then the symmetric matrix $\boldsymbol{H}_k$ is positive definite.

Based on Lemma 2 and the fact that the set $\mathcal{P}$ is finite,

$$\bar{\lambda} := \min_{p\in\mathcal{P}}\{\lambda(\boldsymbol{H}_p) | \bar{\mathcal{G}}_p \text{ is connected}\} > 0 \tag{14}$$

is fixed and depends directly on the constants α_{ij} and β_i for $i,j = 1,\ldots,n$ given in (5) and (6).

Now we give the main result of leader-following case as follows.

Theorem 1 Take the positive constants κ and γ satisfied

$$\kappa > \frac{1}{\gamma^2\bar{\lambda}}. \tag{15}$$

If the interconnection graph is connected for any interval $[t_j, t_{j+1})$, then the local control law (4) can guarantee that the multi-agent system in leader-following case achieves consensus under any given initial condition $r(0)$ and $p(0)$, i.e.,

$$\begin{cases} \lim\limits_{t\to\infty}(r_i(t) - r_0(t)) = 0, \\ \lim\limits_{t\to\infty}(p_i(t) - p_0(t)) = 0, \end{cases} \quad i = 1,...,n. \tag{16}$$

Proof To prove the theorem, we consider the dynamics in each interval at first. Note that, in any interval (say $[t_j, t_{j+1})$), the interconnection topology does not change. Therefore, $\boldsymbol{F}_{\sigma(t)}$ is a constant matrix for $t \in [t_j, t_{j+1})$ and $j \ge 0$, then the solution of equation (12) is well defined and can be expressed as $x(t) = (\mathrm{e}^{\boldsymbol{F}_k(t-t_j)} \otimes \boldsymbol{I}_m)x(t_j)$ for $t \in [t_j, t_{j+1})$. Define a symmetric matrix by

$$\boldsymbol{P} = \begin{pmatrix} 2\boldsymbol{I} & \gamma\boldsymbol{I} \\ \gamma\boldsymbol{I} & \gamma^2\boldsymbol{I} \end{pmatrix}, \tag{17}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. It is known that matrix $\boldsymbol{P}$ is positive definite by using Schur complement lemma (see Ref. [20]). Then take a common Lyapunov function $V(x) = x^{\mathrm{T}}(\boldsymbol{P} \otimes \boldsymbol{I}_m)x$, where $\boldsymbol{P}$ is defined by Eq. (17). Then, for each $k \in \mathcal{P}$ which corresponds to interval $[t_j, t_{j+1})$,

$$\begin{aligned} \dot{V}(x)|_{(12)} &= x^{\mathrm{T}}\left[(\boldsymbol{F}_k^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{F}_k) \otimes \boldsymbol{I}_m\right]x \\ &= -x^{\mathrm{T}}(\boldsymbol{Q}_k \otimes \boldsymbol{I}_m)x, \end{aligned}$$

where

$$\boldsymbol{Q}_k = \begin{pmatrix} 2\kappa\gamma\boldsymbol{H}_k & 2\kappa\gamma^2\boldsymbol{H}_k - 2\boldsymbol{I} \\ 2\kappa\gamma^2\boldsymbol{H}_k - 2\boldsymbol{I} & 2\kappa\gamma^3\boldsymbol{H}_k - 2\gamma\boldsymbol{I} \end{pmatrix}.$$

Noticing that

$$\begin{pmatrix} 2\kappa\gamma & 2\kappa\gamma^2 \\ 2\kappa\gamma^2 & 2\kappa\gamma^3 \end{pmatrix} \otimes (\boldsymbol{H}_k - \bar{\lambda}\boldsymbol{I}) \ge 0,$$

we have

$$\boldsymbol{Q}_k \geq \boldsymbol{Q} := \begin{pmatrix} 2\kappa\gamma\bar{\lambda}\boldsymbol{I} & 2\kappa\gamma^2\bar{\lambda}\boldsymbol{I} - 2\boldsymbol{I} \\ 2\kappa\gamma^2\bar{\lambda}\boldsymbol{I} - 2\boldsymbol{I} & 2\kappa\gamma^3\bar{\lambda}\boldsymbol{I} - 2\gamma\boldsymbol{I} \end{pmatrix} = \begin{pmatrix} 2\kappa\gamma\bar{\lambda} & 2\kappa\gamma^2\bar{\lambda} - 2 \\ 2\kappa\gamma^2\bar{\lambda} - 2 & 2\kappa\gamma^3\bar{\lambda} - 2\gamma \end{pmatrix} \otimes \boldsymbol{I}. \quad (18)$$

It is easy to verify that $\begin{pmatrix} 2\kappa\gamma\bar{\lambda} & 2\kappa\gamma^2\bar{\lambda} - 2 \\ 2\kappa\gamma^2\bar{\lambda} - 2 & 2\kappa\gamma^3\bar{\lambda} - 2\gamma \end{pmatrix}$ is positive definite only if Eq. (15) is satisfied. Thus $\boldsymbol{Q}$ is positive definite. It is well known that

$$\min_{x\neq 0} \frac{x^{\mathrm{T}}(\boldsymbol{Q}_k \otimes \boldsymbol{I}_m)x}{x^{\mathrm{T}}(\boldsymbol{P} \otimes \boldsymbol{I}_m)x} \geq \frac{\lambda_{\min}(\boldsymbol{Q} \otimes \boldsymbol{I}_m)}{\lambda_{\max}(\boldsymbol{P} \otimes \boldsymbol{I}_m)} = \frac{\lambda_{\min}(\boldsymbol{Q})}{\lambda_{\max}(\boldsymbol{P})}.$$

Let $\beta := \lambda_{\min}(\boldsymbol{Q})/2\lambda_{\max}(\boldsymbol{P})$. Therefore, $\dot{V}(x) \leq -2\beta V(x)$, which implies $V(x) \leq V(x(0))\,\mathrm{e}^{-2\beta t}$. Moreover, we have

$$\begin{aligned} \|x(t)\| &\leq \frac{\sqrt{V(x)}}{\sqrt{\lambda_{\min}(\boldsymbol{P})}} \\ &\leq \frac{\sqrt{V(0)}}{\sqrt{\lambda_{\min}(\boldsymbol{P})}}\,\mathrm{e}^{-\beta t} \\ &\leq \frac{\sqrt{\lambda_{\max}(\boldsymbol{P})}}{\sqrt{\lambda_{\min}(\boldsymbol{P})}}\|x(0)\|\,\mathrm{e}^{-\beta t}. \end{aligned} \quad (19)$$

Thus we have $\lim_{t\to\infty} x(t) = 0$, which means that Eq. (16) is satisfied. The proof is now completed.

Remark 1 Consider the special case that the dynamic of the group of agents with fixed interconnection topology. And then, the subscript $\sigma(t)$ in Eqs. (l2) and (13) can be dropped. If the interconnection topology graph is connected, then the matrix $\boldsymbol{H}$ is positive definite. By using Schur orthogonal transformation and simple calculations, it is not difficult to get that the eigenvalue polynomial of matrix $\boldsymbol{F}$ is expressed as follows

$$\prod_{i=1}^{n} (\lambda^2 + \kappa\gamma\lambda_i(\boldsymbol{H})\lambda + \kappa\lambda_i(\boldsymbol{H})). \quad (20)$$

Obviously, any polynomial $\lambda^2 + \kappa\gamma\lambda_i(\boldsymbol{H})\lambda + \kappa\lambda_i(\boldsymbol{H})$ is table for any $\kappa, \gamma > 0$, which means that $\boldsymbol{F}$ is stable matrix. Therefor, for fixed topology, the condition Eq. (15) of Theorem 1 can be moved.

4. Leaderless case

In this case, similar to the analysis given in last section, we consider the leaderless case. The closed-loop system (10) and (11) will be expressed in a compact form as follows:

$$\dot{x} = (\boldsymbol{F}_\sigma \otimes \boldsymbol{I}_m)x \quad (21)$$

with

$$x = \begin{pmatrix} r \\ p \end{pmatrix}, \quad \boldsymbol{F}_\sigma = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\kappa\boldsymbol{L}_\sigma & -\kappa\gamma\boldsymbol{L}_\sigma \end{pmatrix}. \quad (22)$$

Noting that $\boldsymbol{L}_k$ is symmetric semi-positive definite and vector 1 is the eigenvector corresponding to its zero eigenvalue by Lemma 1, there is an orthogonal matrix

$$\boldsymbol{U} = \begin{pmatrix} \frac{1}{\sqrt{n}} & * & \dots & * \\ \frac{1}{\sqrt{n}} & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n}} & * & \dots & * \end{pmatrix},$$

where $*$ denotes the other entries in this matrix, such that

$$\boldsymbol{U}^{\mathrm{T}}\boldsymbol{L}_k\boldsymbol{U} = \begin{pmatrix} 0 & 0 \\ \boldsymbol{0} & \bar{\boldsymbol{H}}_k \end{pmatrix}, \quad \bar{\boldsymbol{H}}_k \in \mathbb{R}^{(n-1)\times(n-1)}. \quad (23)$$

Thus, all the eigenvalues of the symmetric matrix $\bar{\boldsymbol{H}}_k$ are nonnegative and $\lambda_2(\boldsymbol{L}_k) = \lambda_{\min}(\bar{\boldsymbol{H}}_k)$. If the graph is connected, the rank of $\boldsymbol{L}_k$ is $n-1$ and therefore, all the eigenvalues of matrix $\bar{\boldsymbol{H}}_k$ are positive. Based on Lemma 1 and the fact that the set $\mathcal{P}_0$ is finite,

$$\bar{\lambda}_2 := \min_{p\in\mathcal{P}_0}\{\lambda_2(\boldsymbol{L}_p) | \mathcal{G}_p \text{ is connected}\} > 0 \quad (24)$$

is fixed and depends directly on the constants α_{ij} given in (5). Taking $\bar{s} = \boldsymbol{U}^{\mathrm{T}}r$ and $\bar{q} = \boldsymbol{U}^{\mathrm{T}}p$, the closed-loop system can be described by

$$\begin{aligned} \dot{\bar{s}} &= \bar{q}, \\ \dot{\bar{q}} &= \left[\begin{pmatrix} 0 & 0 \\ \boldsymbol{0} & -\kappa\bar{\boldsymbol{H}}_k \end{pmatrix} \otimes \boldsymbol{I}_m\right]\bar{s} \\ &\quad + \left[\begin{pmatrix} 0 & 0 \\ \boldsymbol{0} & -\kappa\gamma\bar{\boldsymbol{H}}_k \end{pmatrix} \otimes \boldsymbol{I}_m\right]\bar{q}. \end{aligned} \quad (25)$$

For convenience, take

$$\xi_0 = \bar{s}_1, \quad \psi_0 = \bar{q}_1,$$

and

$$\begin{aligned} \xi_1 &= \mathrm{col}(\bar{s}_2, \bar{s}_3, \dots, \bar{s}_n), \\ \psi_1 &= \mathrm{col}(\bar{q}_2, \bar{q}_3, \dots, \bar{q}_n), \end{aligned}$$

where the $\bar{s}_i$ and $\bar{q}_i$ are self evident. Then the whole system can be decoupled into two subsystems as follows: one is

$$\dot{\xi}_0 = \psi_0, \quad \dot{\psi}_0 = 0, \quad (26)$$

and the other one is

$$\begin{aligned}\dot{\xi}_1 &= \psi_1,\\ \dot{\psi}_1 &= -\kappa(\bar{\boldsymbol{H}}_k \otimes \boldsymbol{I}_m)\xi_1 - \kappa\gamma(\bar{\boldsymbol{H}}_k \otimes \boldsymbol{I}_m)\psi_1. \quad (27)\end{aligned}$$

Now, we present the main result of leaderless case as follows.

Theorem 2 Take the positive constants κ and γ satisfied

$$\kappa > \frac{1}{\gamma^2 \bar{\lambda}_2}. \quad (28)$$

If the interconnection graph is connected for any interval $[t_j, t_{j+1})$, then the local control law (9) can guarantee taht the multi-agent system in leaderless case achieves consensus under any given initial condition $r(0)$ and $p(0)$, i.e.,

$$\begin{cases} \lim\limits_{t\to\infty}(r_i(t) - r_j(t)) = 0,\\ \lim\limits_{t\to\infty}(p_i(t) - p_j(t)) = 0, \end{cases} \quad i,j = 1,...,n. \quad (29)$$

Proof Note that system (27) is in the same form of (12) of the leader-following case, though its state is $(n-1)$-dimensional. Also, $\bar{\boldsymbol{H}}_k$ plays the same role as $\boldsymbol{H}_k$ does in the leader-following case. Therefore, similar to the proof of Theorem 1, we have

$$\lim_{t\to\infty}\begin{pmatrix}\xi_1(t)\\ \psi_1(t)\end{pmatrix} = 0.$$

From system (26), we have

$$\begin{aligned}\psi_0(t) &= \psi_0(0)\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} p_i(0),\\ \xi_0(t) &= \xi_0(0) + \psi_0(0)t\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} r_i(0) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} p_i(0)t. \quad (30)\end{aligned}$$

Noticing that $\boldsymbol{U}^{\mathrm{T}}1 = (\sqrt{n}, 0, \ldots, 0)^{\mathrm{T}}$, we have

$$\begin{aligned}r(t) - \frac{1}{\sqrt{n}}\xi_0(t)\mathbf{1} &= \boldsymbol{U}\left(\bar{s} - \frac{1}{\sqrt{n}}\xi_0(t)\boldsymbol{U}^{\mathrm{T}}\mathbf{1}\right)\\ &= \boldsymbol{U}\begin{pmatrix}0\\ \xi_1(t)\end{pmatrix} \to 0, \quad t\to\infty,\end{aligned}$$

and

$$\begin{aligned}p(t) - \frac{1}{\sqrt{n}}\psi_0(t)\mathbf{1} &= \boldsymbol{U}\left(\bar{q} - \frac{1}{\sqrt{n}}\psi_0(t)\boldsymbol{U}^{\mathrm{T}}\mathbf{1}\right)\\ &= \boldsymbol{U}\begin{pmatrix}0 & \psi_1(t)\end{pmatrix} \to 0, \quad t\to\infty,\end{aligned}$$

which implies that Eq. (29) is satisfied.

Remark 2 We define the centre of the multi-agent system as $\bar{r} = \sum_{i=1}^{n} r_i/n$ and the velocity of the center as $\bar{p} = \sum_{i=1}^{n} p_i/n$. The time derivative of center is given by

$$\dot{\bar{r}} = \frac{1}{n}\sum_{i=1}^{n}\dot{r_i} = \frac{1}{n}\sum_{i=1}^{n} p_i = \bar{p}, \quad (31)$$

and

$$\begin{aligned}&\dot{\bar{p}}\\ &= \frac{1}{n}\sum_{i=1}^{n}\dot{p_i}\\ &= \frac{1}{n}\sum_{i=1}^{n} u_i\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(-\kappa\sum_{j\in\mathcal{N}_i} a_{ij}(r_i - r_j) - \kappa\sum_{j\in\mathcal{N}_i} a_{ij}(p_i - p_j)\Big)\\ &= 0. \quad (32)\end{aligned}$$

Then, we can get that

$$\begin{aligned}\bar{p}(t) = \bar{p}(0) &= \frac{1}{n}\sum_{i=1}^{n} p_i(0) = \frac{1}{\sqrt{n}}\psi_0(t),\\ \bar{r}(t) &= \bar{r}(0) + \bar{p}(0)t = \frac{1}{\sqrt{n}}\xi_0(t). \quad (33)\end{aligned}$$

The centre $\bar{r}$ of the closed loop system described by (11) will run at a constant speed $\sum_{i=1}^{n} p_i(0)/n$, which is the average initial velocity of all agents. The position and velocity of every agent will tend to the position and velocity of centre respectively, which means that the multi-agent network achieve consensus.

5. Simulation results

In this section, we illustrate theoretical results derived in the above section and provide two numerical simulations. Without loss of generality, we take $m = 3$ in numerical simulations. Take $\kappa = 5$ and $\gamma = 1$. In leader-follower case, consider a multi-agent system with one leader and six followers. The interconnection topology is arbitrarily switched with switching period 1 among four graphs $\bar{\mathcal{G}}_i (i = 1,2,3,4)$. The Laplacian matrices $\boldsymbol{L}_i (i = 1,2,3,4)$ for the four subgraphs $\mathcal{G}_i (i = 1,2,3,4)$ are

$$\boldsymbol{L}_1 = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ -1 & -1 & 3 & 0 & -1 & 0\\ -1 & 0 & 0 & 3 & -1 & -1\\ 0 & 0 & -1 & -1 & 3 & -1\\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix},$$

$$\boldsymbol{L}_2 = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$\boldsymbol{L}_3 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$\boldsymbol{L}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

and the diagonal matrices for the interconnection relationship between the leader and the followers are

$$\boldsymbol{B}_1 = \mathrm{diag}(1,0,0,1,0,0),$$

$$\boldsymbol{B}_2 = \mathrm{diag}(1,0,1,1,0,0),$$

$$\boldsymbol{B}_3 = \mathrm{diag}(1,0,1,1,1,0),$$

$$\boldsymbol{B}_4 = \mathrm{diag}(1,1,1,1,1,1).$$

The initial positions and velocities of the all agents are randomly produced. The position and velocity errors in Fig. 1 are defined as $\|r_i - r_0\|$ and $\|p_i - p_0\|$ respectively. Figure 1 shows that the follower-agents can track the leader.

In leaderless case, consider a multi-agent system with six agents. The interconnection topology is also arbitrarily switched with switching period 1 among four graphs $\mathcal{G}_i (i = 1, 2, 3, 4)$ described as follows. The Laplacian matrices $\boldsymbol{L}_i (i = 1, 2, 3, 4)$ are also defined as above. The position and velocity errors in Fig. 2 are defined as $\|r_i - \bar{r}\|$ and $\|p_i - \bar{p}\|$ respectively. Figure 2 shows the multi-agent system in leaderless case can achieve consensus.

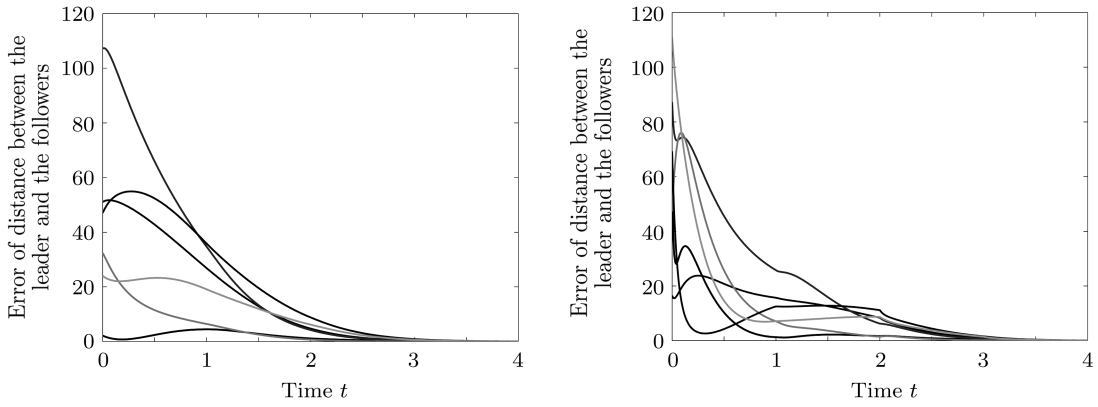


Fig. 1. Position and velocity tracking errors of six followers.

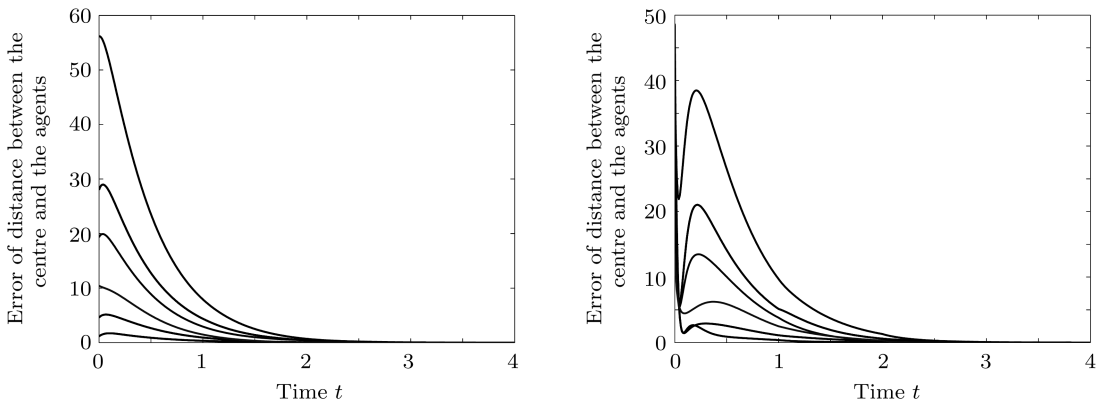


Fig. 2. Position and velocity tracking errors of six agents.

6. Conclusions

We have investigated the control of multi-agent systems with varying interconnection topologies using graph theory and stability theory. The proposed consensus strategies are neighbour-based law, i.e. each agent is regulating its position and velocity based on its neighbour agents. There are two different cases considered in this study. One is the leader-following case and the other is the leaderless case. For the first one, a common Lyapunov function (CLF) was constructed to analyse the stability of the closed system. While for the second case, we use a Schur orthogonal transformation which decomposes the group tracking dynamics into two subsystems: one represses the leader and the other represses a leader-following subsystem, for which the obtained result of the first case can be applied directly. Some sufficient conditions were given to achieve consensus of these mobile agents with the proposed local control strategies in both leader-following and leaderless cases respectively. Moreover, several numerical simulations were shown to verify the theoretical analysis.

References

[1] Vicsek T, Czirok A, Jacob E B, Cohen I and Schochet O 1995 *Phys. Rev. Lett.* **75** 1226
[2] Amritkar R and Jalan S 2003 *Physica* A **321** 220
[3] Reynolds C W 1987 *ACM SIGGRAPH Conference Proceedings* **21** 25
[4] Okubo A 1986 *Advances in Biophysics* **22** 1
[5] Jadbabaie A, Lin J and Morse A S 2003 *IEEE Trans. Automatic Control* **48** 988
[6] Murray R M 2007 *ASME J. Dynamic Systems, Measurement, and Control* **129** 571
[7] Mu S, Chu T and Wang L 2005 *Physica* A **351** 211
[8] Gao L and Cheng D 2005 *IEEE Trans. Automatic Control* **50** 1913
[9] Olfati-Saber R and Murray R M 2004 *IEEE Trans. Automatic Control* **49** 1520
[10] Hong Y, Hu J and Gao L 2006 *Automatica* **42** 1177
[11] Hu J and Hong Y 2007 *Physica* A **374** 853
[12] Gao L, Cheng D and Hong Y 2008 *J. Control Theory and It's Applications* **6** 357
[13] Sun Y Z and Ruan J 2008 *Chin. Phys.* B **17** 4137
[14] Hong Y, Chen G and Bushnell L 2008 *Automatica* **44** 846
[15] Peng K and Yang Y 2009 *Physica* A **388** 193
[16] Li Y M and Guan X P 2009 *Chin. Phys.* B **18** 3355
[17] Hong Y, Gao L, Cheng D and Hu J 2007 *IEEE Trans. Automatic Control* **52** 943
[18] Li H, Lin P and Zhang C X 2008 *Chin. Phys.* B **17** 4458
[19] Godsil C and Royle G 2001 *Algebraic Graph Theory* (New York: Springer-Verlag)
[20] Horn R and Johnson C 1985 *Matrix Analysis* (New York: Cambbridge Univ. Press)