Control and synchronization of second Julia sets*

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A visualization of Julia sets of the complex Henon map system with two complex variables is introduced in this paper. With this method, the optimal control function method is introduced to this system and the control and synchronization of its Julia sets are achieved. Control and synchronization of generalized Julia sets are also achieved with this optimal control method. The simulations illustrate the efficacy of this method.

Keywords: Henon map, Julia set, optimal control, synchronization

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1. Introduction

In 1976, the French astronomer Michel Henon studied the following famous two dimensional iterative map[1]

$$x_{k+1} = 1 + y_k - ax_k^2,\ a = 1.4,$$
$$y_{k+1} = bx_k,\ b = 0.3.$$

At present, results about the real Henon map system are extensive and relate to nonlinear qualitative analysis, chaos and control, applications in physics and so on.[2−4] In general, for the system in the complex domain, more attentions are given to fractals and properties of complex dynamics, such as the Julia set, Fatou set, Mandelbrot set and so on. These theories have extensive applications in many fields, such as the application of Julia set in physics. Specially, the strange attractor and the boundary of the attractive region of the recursive function on the complex plane C are associated with chaotic dynamic systems, which is the Julia set broadly emerging in engineering applications, such as the basic comprehension of the notorious small denominator problem plaguing Hamiltonian mechanics[5] and the scaling behaviour of circle maps modeling successfully the onset of chaos in many real physical situations.[6] The direct applications of Julia set include the analysis of the spectral and localization properties of quantum states in hierarchical tight-binding models.[7] The study of the complex dynamic properties for the two dimensional Henon map in the complex domain C^2 is also abundant, such as the Julia set.[8−11]

In addition, chaos control and synchronization have been studied extensively and many methods are introduced to control chaotic systems.[12,13] And sometimes we need to control the behaviours described by Julia set. Recently, the authors[14,15] have studied the control and synchronization of Julia sets in the complex plane. But the fixed point of the system needs to be known. In this paper, we give a visualization of Julia sets of Henon map with two complex variables in Section 2. Then we take the optimal control and synchronize methods on the Julia sets of two dimensional complex Henon map in Sections 3 and 4 respectively. For this control method, we do not need to know the fixed point of the system. In Section 5, with the optimal control method, control and synchronization of generalized Julia sets are also achieved. The simulations illustrate the efficacy of this methods.

2. Julia set of the complex Henon map

The general form of the complex variable Henon map in the complex domain C^2 is

$$z_{n+1} = 1 - az_n^2 + bw_n,$$
$$w_{n+1} = z_n, \tag{1}$$

where $a,\ b \in C$. In fact, take the first equation of (1) to the second one, we can get the following second

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order equation:

$$z_{n+1} = 1 - az_n^2 + bz_{n-1}.$$

Let $H : C^2 \to C^2$ be the complex Henon map (1), that is $H(z, w) = (1 - az^2 + bw, z)$. Let

$$K^+ = \{(z, w) \in C^2 | \{H^n(z, w)\}_{n\geq 1} \text{ is bounded in } C\},$$

$$K^- = \{(z, w) \in C^2 | \{H^n(z, w)\}_{n\leq 1} \text{ is bounded in } C\}$$

denote the sets of points whose forward/backward orbit is bounded. And let

$$J = \partial K^+ \cap \partial K^-,$$

which is called the Julia set of the complex Henon map, where ∂K^+, ∂K^- denote the boundary of K^+, K^- respectively.

Since z, w are complex numbers, we suppose $(z, w) = (x, y, s, t)$, where x, y, s, t are real numbers. Therefore J is a four dimensional set from the definition. In order to depict Julia set with simulation, one of the simplest methods is to fix one of the coordinates in J such as w. That is to say that if $(z, w) = (x, y, s, t)$, we can consider the set with the form $J_{w_0} = J \cap \{w = w_0\}$. Then we can depict the Julia set in the $z = (x, y)$ plane. The other method is to fix one of the four real coordinates and we can depict Julia set by three dimensional simulations. For example, we can fix $t = t_0$ and consider the set of points with the form (x, y, s, t_0) and depict them in three dimension (x, y, s). In this paper, we take the second method to depict Julia set of the complex Henon map and take $t = 0$.

For the complex polynomial f, its Julia set has the following properties.

(i) $J(f)$ is nonempty and bounded;

(ii) $J(f)$ is fully invariant, that is $J(f) = f(J(f)) = f^{-1}(J(f))$;

(iii) $J(f) = J(f^p)$, for any positive integer p;

(iv) If w is an attractive fixed point of f, then $\partial A(w) = J(f)$, where $A(w)$ is the attractive domain of the attractive fixed point w, that is, $A(w) = \{z \in C : f^k(z) \to w, (k \to \infty)\}$, which is also true when $w = \infty$.

3. Optimal control of Julia set of the complex Henon map

From the definition of the Julia set of the complex Henon map, it is easy to see that the boundedness of the iteration of the system is important to the structure of its Julia set. Therefore, the Julia set control can be achieved by controlling the trajectories of the Henon map. For example, we can consider the stability of the fixed point of the complex Henon map. Thus, we can achieve the control of the attractive domain of the fixed point and then the control of the Julia set will be achieved.

The fixed point of system (1) is $[z^*, w^*] = (-c + \sqrt{c^2 + a})/a$, where $c = (1 - b)/2$. We will consider the stability of the fixed point. For example, we will find a control signal u_n such that the fixed point $[z^*, w^*]$ of the controlled system

$$\begin{aligned} z_{n+1} &= 1 - az_n^2 + bw_n + u_n, \\ w_{n+1} &= z_n \end{aligned} \tag{2}$$

is stable and $[z_{n+1}, w_{n+1}] \to [z^*, w^*]$, $n \to \infty$.

According to the optimal control function method,[16] we take the control signal $u_n = -k[1 - az_n^2 + bw_n - z_n]$, then the controlled system (2) becomes

$$\begin{aligned} z_{n+1} &= 1 - az_n^2 + bw_n - k[1 - az_n^2 + bw_n - z_n], \\ w_{n+1} &= z_n. \end{aligned} \tag{3}$$

Obviously, $[z^*, w^*]$ is also a fixed point for the controlled system (3), which is generated by the map

$$\begin{aligned} \widetilde{H}(z, w) &= ((1 - k)f(z, w) + kz, z) \\ &= ((1 - k)(1 - az^2 + bw) + kz, z), \end{aligned} \tag{4}$$

where $f(z, w) = 1 - az^2 + bw$. A block diagram of the control scheme is shown in Fig. 1.

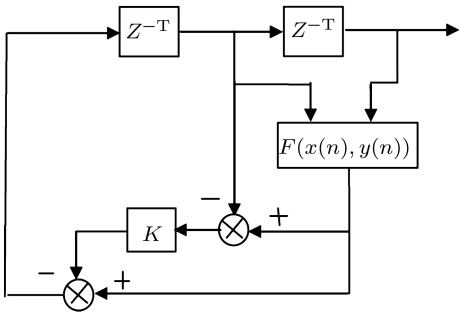


Fig. 1. Block diagram of the control method.

Case 1 $b = 0$

In this case the controlled system (3) becomes

$$\begin{aligned} z_{n+1} &= (1 - k)(1 - az_n^2) + kz_n \\ &= -a(1 - k)\left(z_n - \frac{k}{2a(1 - k)}\right)^2 + (1 - k) \end{aligned}$$

$$+\frac{k^2}{4a(1-k)},$$
$$w_{n+1} = z_n. \tag{5}$$

Obviously, the iteration of z_n is independent of w_n in this case. Let $g(z) = (1-k)(1-az^2) + kz$. It is well known that $|g'(z^*)| < 1$ is one of the conditions to guarantee the stability of the fixed point z^*, that is, $|-2a(1-k)z^* + k| < 1$. Then we have

$$\left|\frac{1-\sqrt{1+4a}}{\sqrt{1+4a}} + k\right| < 1. \tag{6}$$

For a given parameter a, the scope of the control parameter k can be obtained from the above inequality (6). In fact, the controlled system in this case is the iteration of the polynomial. From the property (iv) of Julia set of the complex polynomial, the attractive domain can be changed by adjusting the control parameter k and then the objective of Julia set control is achieved.

In general, there are two basic methods to calculate Julia set: the boundary scanning method and the inverse iteration method. The second method is to calculate the previous term by the backward iteration of the map. And the previous terms are dense in its Julia set. Thus, we can analyse the changing of Julia set with the control parameter k by using the inverse image of $g(z) = (1-k)(1-az^2) + kz$.

For the convenience to discuss the changing of Julia set, take a linear transformation $t_n = z_n - k/(2a(1-k))$ on the controlled system (5), then

$$t_{n+1} = -a(1-k)t_n^2 + \frac{k^2 - 2k + 4a(1-k)^2}{4a(1-k)}. \tag{7}$$

Since the linear transformation makes only the properties of the two systems change rigidly but does not change their essential properties, we convert to discuss the Julia set control of system (7). Let $h(t) = -a(1-k)t^2 + (k^2 - 2k + 4a(1-k)^2)/(4a(1-k))$. Consider the following equation

$$|-a(1-k)|t^2 - t - \left|\frac{k^2 - 2k + 4a(1-k)^2}{4a(1-k)}\right| = 0,$$

which has a positive root

$$T = \frac{1 + \sqrt{1 + |k^2 - 2k + 4a(1-k)^2|}}{2|a(1-k)|}.$$

When $|t| > T$,

$$\begin{aligned}
&|h(t)| \\
&\geq \left||-a(1-k)t^2| - \left|\frac{k^2 - 2k + 4a(1-k)^2}{4a(1-k)}\right|\right| \\
&= \left|\left|\frac{-a(1-k)t^2}{T^2}\right| T^2 - \left|\frac{k^2 - 2k + 4a(1-k)^2}{4a(1-k)}\right|\right| \\
&= \left|\left|\frac{-a(1-k)t^2}{T^2}\right| \frac{T + \left|\dfrac{k^2 - 2k + 4a(1-k)^2}{4a(1-k)}\right|}{|-a(1-k)|}\right. \\
&\quad \left. - \left|\frac{k^2 - 2k + 4a(1-k)^2}{4a(1-k)}\right|\right| \\
&= \left(\frac{|t|}{T}\right)^2 T + \left|\frac{k^2 - 2k + 4a(1-k)^2}{4a(1-k)}\right| \left(\left(\frac{|t|}{T}\right)^2 - 1\right) \\
&> \left(\frac{|t|}{T}\right)^2 T.
\end{aligned}$$

Hence $|h^n(z)| > (|z|/T)^{2^n} T \to \infty\ (n \to \infty)$. Therefore the Julia set of the controlled system (7) is contained in the set $\{t : |t| \leq T\}$.

With the property (iv) $J(f) = \partial A(z^*) = \partial A(\infty)$, we can discuss the structure of Julia set of the controlled system (7) with the method in Refs. [14] and [15].

Since

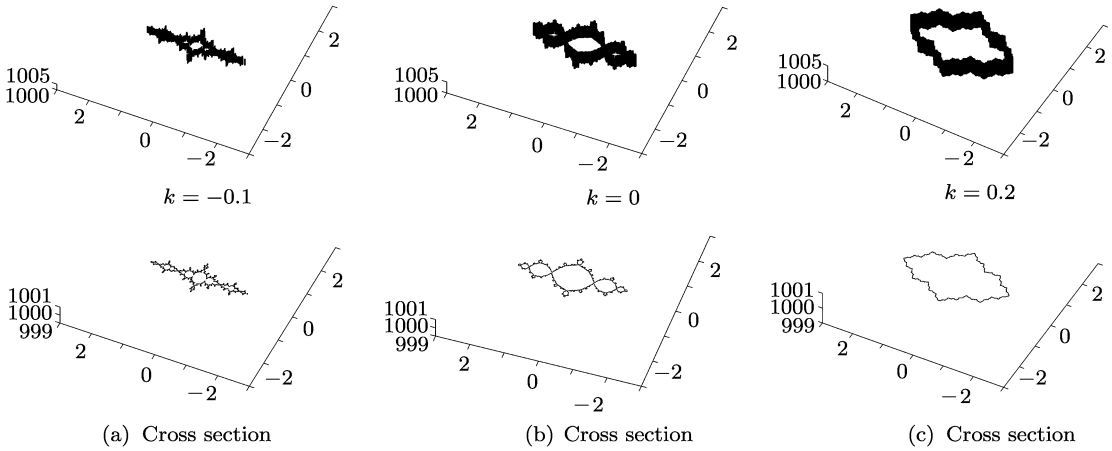
$$(h^{-1}(t))^2 = \frac{1+4a}{4a^2} - \frac{1}{4a^2(1-k)^2} - \frac{t}{a(1-k)},$$

we have

$$\begin{aligned}
&|h^{-1}(t)|^2 \\
&= \left|\frac{1+4a}{4a^2} - \frac{1}{4a^2(1-k)^2} - \frac{t}{a(1-k)}\right| \\
&\geq \left|\frac{1}{4a^2(1-k)^2}\right| + \left|\frac{t}{a(1-k)}\right| - \left|\frac{1+4a}{4a^2}\right|.
\end{aligned}$$

From above equation, it is easy to see that the value of $|h^{-1}(t)|$ increases with the value of k increasing when $k < 1$, which indicates that its inverse image extends outwards compared to the original point. Therefore, its Julia set extends outwards compared to the original point. The following simulations illustrate the analysis and the efficacy of the method, where we take $a = 1, b = 0$. From inequality (6) and $k < 1$, we know $-0.4472 < k < 1$. We have rotated the simulations for having a better observation.

The bottom simulations in Fig. 2 are the cross sections of the top respectively. In fact, they are also the Julia set control of the complex system $z_{n+1} = z_n^2 + c$ with this control method, where c is a complex number. From the above simulations, it is easy to see that the changing of Julia sets is corresponding to our conclusions. Specially, Fig. 2(b) is the original Julia set of the system without control.



(a) Cross section (b) Cross section (c) Cross section

Fig. 2. Julia set control with parameter k when $a = 1, b = 0$.

Case 2 $b \neq 0$

We will continue to consider the stability of the fixed point of the controlled system (3) with $b \neq 0$.

First, the Jacobi matrix of the map $\widetilde{H}$ is

$$\boldsymbol{D\widetilde{H}} = \begin{bmatrix} -2a(1-k)z + k & b(1-k) \\ 1 & 0 \end{bmatrix}.$$

The fixed points are attractive when the module of the eigenvalues of the Jacobi matrix at the fixed points is less than 1. That is

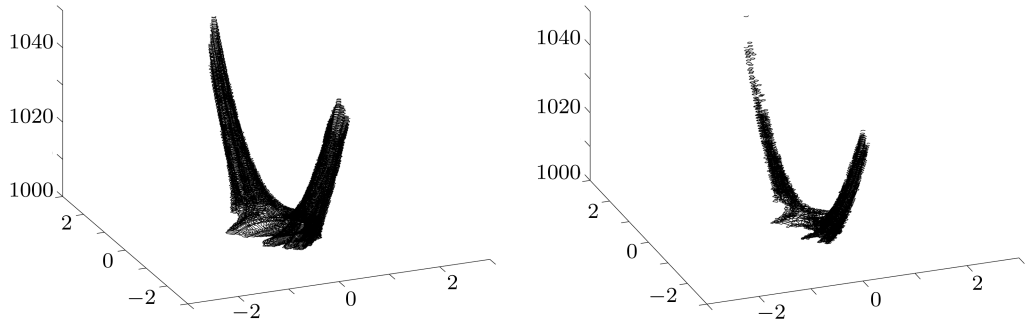
$$\left| \frac{-(2a(1-k)z^* - k)}{2} \pm \frac{\sqrt{(2a(1-k)z^* - k)^2 + 4b(1-k)}}{2} \right| < 1,$$

then we can get a simple form

$$|b(1-k) - 1| < |2a(1-k)z^* - k|.$$

Thus we can immediately obtain the scope of k and so we can control the trajectory of Henon map by choosing an appropriate k. For example, taking $a = 1.4$, $b = 0.3$, we have $k < 0.3481$. The following simulations illustrate the effectivity of the method.

From Fig. 3, we can see the changing of Julia set of the controlled system. When k decreases, the corresponding Julia set shrinks according to the original point.



(a) $k = -0.3$ (b) $k = 0.15$

Fig. 3. Julia set control with parameter k when $a = 1.4, b = 0.3$.

4. Synchronization of Julia sets of Henon map

In Ref. [15], the authors give the following definition of synchronization of Julia sets. Consider two systems in the complex domain C

$$x_{n+1} = f(x_n), \tag{8}$$

$$y_{n+1} = g(y_n). \tag{9}$$

Their Julia sets are determinate when the parameters of the systems are given. Without loss of generality, we let them be J and J^* respectively. In order to make the Julia sets of systems (8) and (9) associated with each other, we will couple systems (8) and (9). Add two coupled items into systems (8) and (9)

$$x_{n+1} = f(x_n) + p(x_n, y_n; k), \tag{10}$$

$$y_{n+1} = g(y_n) + q(x_n, y_n; k), \tag{11}$$

where p and q are the coupled items about x_n and y_n, and k is the coupling strength. It is obvious that one Julia set is corresponding with a value of k and denote them J_k^1 and J_k^2 about systems (10) and (11) respectively. Julia sets synchronization of systems (8) and (9) was taken to occur if

$$\lim_{k\to k_0}(J_k^1 \cup J_k^2 - J_k^1 \cap J_k^2) = \emptyset \tag{12}$$

for some k_0 (k_0 may be ∞). Specifically, when $q(x_n, y_n; k) = 0$, Eq. (12) becomes

$$\lim_{k\to k_0}(J_k^2 \cup J^* - J_k^2 \cap J^*) = \emptyset.$$

Then the Julia set of system (10) will synchronize with J^*, that is, the Julia set of system (10) will turn to be the Julia set of system (9) with the changing of the control parameter k. Thus, the synchronization of the original Julia sets of systems (8) and (9) is achieved.

In the following, we will consider the synchronization of Julia sets of the complex Henon map with a nonlinear coupling term from the optimal function control, where we take $q(x_n, y_n; k) = 0$. Substitute $g(y_n, y_{n-1}) = 1 - a_2 y_n^2 + b_2 y_{n-1}$ for z_n in u_n in Section 3, that is, the coupled term $p = -k(f(x_n, x_{n-1}) - g(y_n, y_{n-1})) = -k((1 - a_1 x_n^2 + b_1 x_{n-1}) - (1 - a_2 y_n^2 + b_2 y_{n-1}))$. Then,

$$x_{n+1} = f(x_n, x_{n-1}) - k(f(x_n, x_{n-1}) - g(y_n, y_{n-1})),$$

which with system (11) derives

$$\begin{aligned} & y_{n+1} - x_{n+1} \\ = {} & (1-k)(f(x_n, x_{n-1}) - g(y_n, y_{n-1})) \\ = {} & (1-k)(a_1 x_n^2 - a_2 y_n^2 + b_2 y_{n-1} - b_1 x_{n-1}), \end{aligned}$$

Since the Julia set of Henon map is a closed set,[10] we can obtain Julia set by calculating the iteration of points in a bounded region D. Thus only the iteration of points in D is needed to consider. In order to get the Julia set, we consider only the points whose trajectories are also in D since $z \notin J$ if there is an n_0 such that $f^{n_0}(z) \notin D$ from the property (ii). Since D is a bounded region, there is an $M > 0$ such that $|z| < M$ for all $z \in D$.

Let $a = \max\{|a_1|, |a_2|\}$, $b = \max\{b_1, |b_2|\}$ and $\alpha = 2|1-k|aM$, $\beta = |1-k|b$, then

$$\begin{aligned} & |y_{n+1} - x_{n+1}| \\ \le {} & |1-k|(a|x_n^2 - y_n^2| + b|y_{n-1} - x_{n-1}|) \\ \le {} & |1-k|(a2M|y_n - x_n| + b|y_{n-1} - x_{n-1}|) \\ = {} & \alpha|y_n - x_n| + \beta|y_{n-1} - x_{n-1}| \\ = {} & a_1|y_n - x_n| + b_1|y_{n-1} - x_{n-1}| \\ \le {} & a_1(\alpha|y_{n-1} - x_{n-1}| + \beta|y_{n-2} - x_{n-2}|) \\ & + b_1|y_{n-1} - x_{n-1}| \\ = {} & (a_1\alpha + \beta)|y_{n-1} - x_{n-1}| + b_1\alpha|y_{n-2} - x_{n-2}| \\ = {} & a_2|y_{n-1} - x_{n-1}| + b_2|y_{n-2} - x_{n-2}| \\ \le {} & \cdots \\ \le {} & a_{n-1}|y_2 - x_2| + b_{n-1}|y_1 - x_1| \\ = {} & a_{n-1}|(1-k)[(a_1 - a_2)x_1^2 + (b_2 - b_1)x_0]|, \end{aligned} \tag{13}$$

since we take the same iterative original values $x_0 = y_0, x_1 = y_1$, where

$$\begin{aligned} a_{n+1} &= a_n\alpha + b_n, \\ b_{n+1} &= a_n\beta. \end{aligned} \tag{14}$$

It is easy to see that $|(1-k)[(a_1 - a_2)x_1^2 + (b_2 - b_1)x_0]|$ is bounded from the above discussion and we hope $a_n \to 0\,(n \to \infty)$ in order to synchronize their trajectories. Obviously, $(0, 0)$ is a fixed point of Eq. (14) and we will discuss the stability of this fixed point. Consider the Jacobi matrix of Eq. (14)

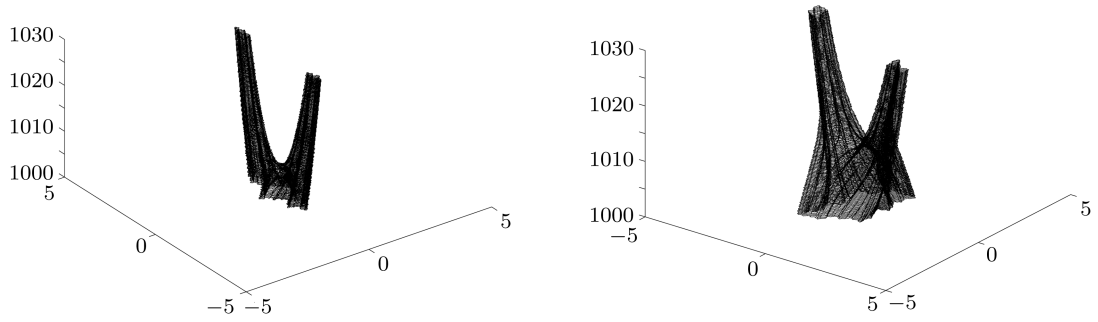
$$\boldsymbol{D} = \begin{vmatrix} \alpha & 1 \\ \beta & 0 \end{vmatrix}$$

and require that all its eigenvalues are less than 1 in absolute value at the fixed point, i.e.

$$\left|\frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}\right| < 1,$$

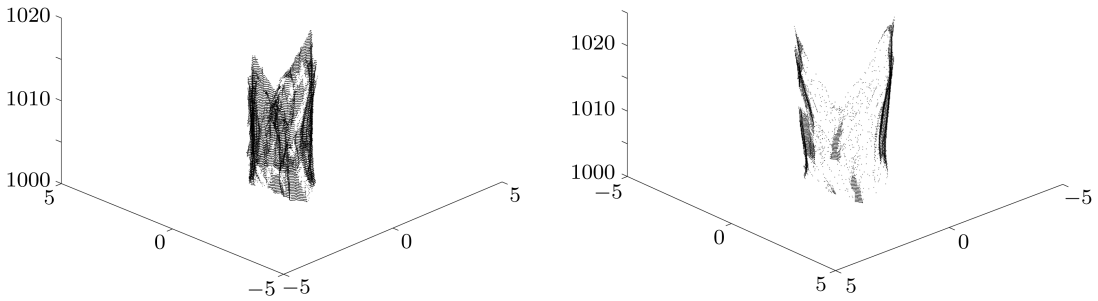
then we obtain $|\beta - 1| < |\alpha|$. From this, we can get the scope of the coupling parameter k. The following simulations illustrate the synchronization status of Julia set with the coupling parameter k.

Figure 4 is the original Julia sets of systems (8) and (9) and Fig. 5 is the procession of synchronization of Julia sets of the coupled systems (10) and (11). From Fig. 5, it is easy to see that the Julia set of system (10) is changing to the one of system (9) under the coupled control method with the increasing of coupled parameter k, which approves our conclusion. In order to have a clear description, we give a cross section to illustrate the synchronization of Julia sets in the following simulations.



(a) $a_1 = 1$, $b_1 = 0.1$ (b) $a_2 = 0.5 - 0.5\mathrm{i}$, $b_2 = -0.2$

Fig. 4. Original Julia sets of systems (8) and (9).



(a) $k = -0.35$ (b) $k = 0.55$

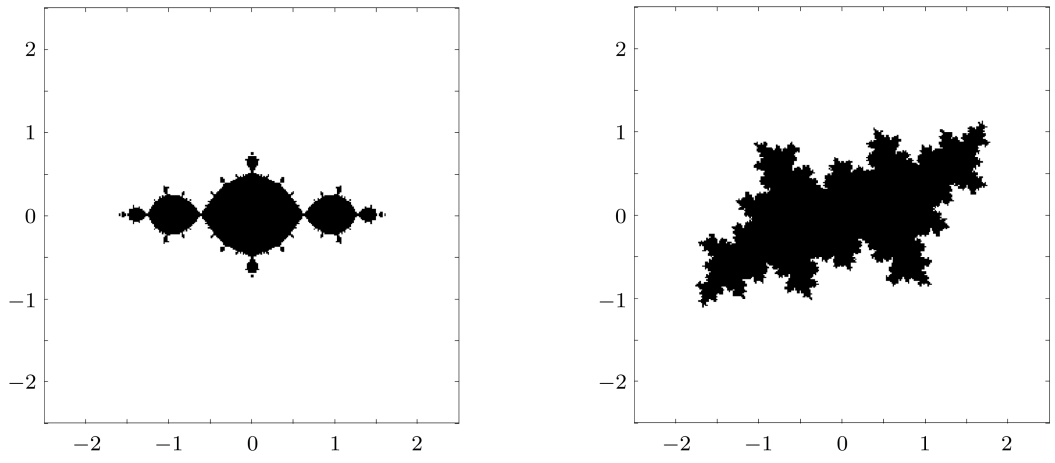
Fig. 5. Julia sets synchronization with parameter k.

In fact, $b_i = 0\,(i = 1, 2)$ in this case, then we have

$$\begin{aligned} x_{n+1} &= 1 - a_1 x_n^2 - k[(1 - a_1 x_n^2) - (1 - a_2 y_n^2)], \\ y_{n+1} &= 1 - a_2 y_n^2. \end{aligned} \tag{15}$$

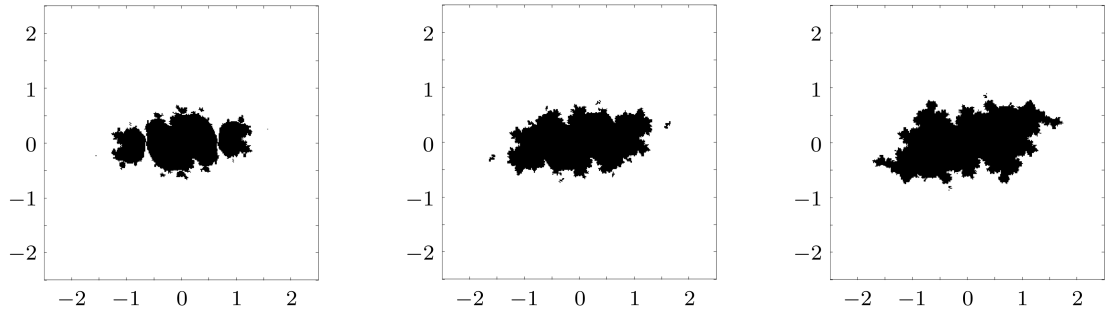
The following simulations illustrate the synchronization changing of the Julia sets and the boundary of the shadow is the corresponding Julia set.

It is obvious that the original Julia set Fig. 6(a) is changing with the coupling parameter k and it is approximate with the Julia set Fig. 6(b) with k increasing. And Fig. 7 depicts the changing, which approves our conclusion about the Julia sets synchronization.



(a) $a_1 = 1$ (b) $a_2 = 0.5 - 0.5\mathrm{i}$

Fig. 6. Original Julia set of systems (8) and (9) when $b_i = 0\,(i = 1, 2)$.



(a) $k = 0.1$ (b) $k = 0.25$ (c) $k = 0.35$

Fig. 7. Julia set synchronization of system (15) with parameter k.

5. Optimal control and synchronization of generalized Julia sets

In Section 3, we analyse the Julia set control of the special case $b = 0$ in system (1), which is also a special form of the following equation on the complex plane C

$$z_{n+1} = z_n^r + c, \tag{16}$$

where c is a complex number and $r > 1$. The Julia set of system (16) has many applications in physics. For example, in Ref. [17], the authors give the applications of generalized Mandelbrot and Julia sets generated by Eq. (16) in physics. In Ref. [18], the authors take the feedback control method to achieve the control of the generalized Julia set, but the fixed point is needed to compute. Introduce the optimal control item to Eq. (16), then

$$\begin{aligned} z_{n+1} &= z_n^r + c - k(z_n^r + c - z_n) \\ &= (1-k)z_n^r + kz_n + (1-k)c. \end{aligned} \tag{17}$$

Obviously, the fixed point will not be changed in this method. However, from the controlled system (17), it is easy to see that we need not to know the fixed point.

Let $s(z) = (1-k)z^r + kz + (1-k)c$ and consider the following equation

$$|1-k|t^r - (|k|+1) - |(1-k)c| = 0,$$

which has a positive root R from the graphics of the two functions $x_1(z) = z^r$ and $x_2(z) = (1+|k|)z/(|1-k|) + (|(1-k)c|)/(|1-k|)$. When $|z| > R$,

$$\begin{aligned} |s(z)| &= |(1-k)z^r + kz + (1-k)c| \\ &\geq |1-k||z^r| - |k||z| - |(1-k)c| \\ &= |1-k|\frac{|z|^r}{R^r}\frac{(|k|+1)R + |(1-k)c|}{|1-k|} - |k||z| - |(1-k)c| \\ &= \left(\frac{|z|}{R}\right)^r R + |k||z|\left(\left(\frac{|z|}{R}\right)^{r-1} - 1\right) + |(1-k)c|\left(\left(\frac{|z|}{R}\right)^r - 1\right) \\ &> \left(\frac{|z|}{R}\right)^r R > R, \end{aligned}$$

then

$$|s^m(z)| > \left(\frac{|z|}{R}\right)^{r^m} R \to \infty \text{ as } k \to \infty.$$

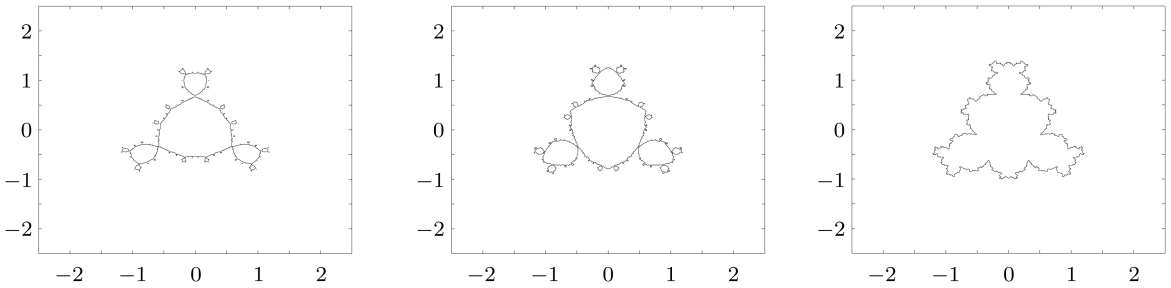
Therefore, the Julia set of the controlled system (17) is contained in the set $\{z : |z| < R\}$.

The discussion of the structure and changing of generalized Julia set of $s(z)$ is similar to the discussion in Section 3, which can be seen from the following simulations, where we take $r = 3, c = i$ in system (16).

Figure 8(a) is the special case $k = 0$ and it is also the original Julia set of system (16). Figures 8(b) and

8(c) are the changing of the Julia set of the controlled system (17). Obviously, the Julia set is extending according to the original point with the control parameter k increasing.

By the similar method to that in Section 4, we can discuss the synchronization of Julia sets of two different systems with different parameters in system (16).



(a) $k = 0$ (b) $k = 0.1$ (c) $k = 0.2$

Fig. 8. Julia set control of the controlled system (17) with parameter k.

6. Conclusion

It is difficult to depict the Julia set of the two dimensional complex Henon map in simulation figures since it needs four variables. In this paper, we fix one of the four variables (we take the fourth variable $t = 0$) to depict the Julia set of the two dimensional complex Henon map. Based on this, we achieve the control and synchronization of the Julia sets of the complex Henon map by applying the optimal control function method and also we achieve the control and synchronization of generalized Julia sets. The simulations illustrate the effectivity of this control method.

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