Filtering noisy chaotic signal via sparse representation based on random frame dictionary*

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The denoising problem of impure chaotic signals is addressed in this paper. A method based on sparse representation is proposed, in which the random frame dictionary is generated by a chaotic random search algorithm. The numerical simulation shows the proposed algorithm outperforms those recently reported alternative denoising methods.

Keywords: chaotic signal, denoising, sparse representation, random frame dictionary

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1. Introduction

In past decades, the study of nonlinear dynamical systems, especially chaotic systems, has drawn much attention. Some typical applications of chaotic signals are chaotic communication[1,2] and array signal processing.[3,4] They all depend on the synchronization property of chaotic system. However, if the level of additive noise in the transmitting channel exceeds 10% of the driving signal, synchronization will fail,[5−7] and coherent detection is unavailable in this case. Therefore, the development of denoising methods for noisy chaotic signals is very important.

Many methods for cleaning noisy chaotic signal have been proposed.[8−13] The objective of them is to remove additive whiten Gaussian noise (AWGN) $\boldsymbol{w}$, where $\boldsymbol{w} = [w(1), \ldots, w(t)]$ (standard deviation is σ), from the noisy chaotic signal $\boldsymbol{y} = [y(1), \ldots, y(t)]$, and then $\tilde{\boldsymbol{s}}$, the estimate of original signal $\boldsymbol{s} = [s(1), \ldots, s(t)]$ is obtained. These methods can be divided into three categories: one is to exploit the cost function,[8,9] one depends on using manifold decomposition methods,[10,11] and the other relies on using projection filtering method.[12,13] The sparse representation (SR)[14,15] is a recently reported method, which belongs to the first category. The principle of it is to transform the denoising issue to a searching for optimisation problem, which is to seek the sparsest representation over overcomplete dictionaries, i.e.,

$$\min_{\boldsymbol{D},\boldsymbol{x}}\{\|\boldsymbol{y} - \boldsymbol{D}\boldsymbol{x}\|_2 + \lambda\|\boldsymbol{x}\|_2\}, \tag{1}$$

where $\|\cdot\|_2$ is l^2 norm, and λ is the Lagrange multiplier; the $m \times k$ overcomplete dictionary $\boldsymbol{D}$ contains k atoms $\boldsymbol{d}_j$ as its columns, $\boldsymbol{D} = [\boldsymbol{d}_1, \ldots, \boldsymbol{d}_k]$, $(m < k)$;[14] $\boldsymbol{x}$ denotes the sparse coefficients. It is assumed that the observed signal $\boldsymbol{y}$ can, as sparse as possible, be represented as a linear combination of these atoms. Since the $\boldsymbol{D}$ contains the structure information of the original signal $\boldsymbol{s}$, the AWGN cannot be sparsely represented by this dictionary. When Eq. (1) reaches its minimum, we can then separate the ideal signal and the AWGN.

2. 'Free-type' random frame dictionary

Expression (1) can be rewritten as

$$\min_{\boldsymbol{D},\boldsymbol{x}}\{\|\boldsymbol{y} - \boldsymbol{D}\boldsymbol{x}\|_2 + \lambda\|\boldsymbol{x}\|_0\},\ N_{\rm in} \le N_{\rm thr}, \tag{2}$$

where $\|\cdot\|_0$ is the l^0 norm, counting the nonzero entries of a vector, $N_{\rm in}$ denotes the number of nonzero entries of each vector $\boldsymbol{x}$, and $N_{\rm thr}$ denotes the threshold of number of nonzero entries of each vector $\boldsymbol{x}$. Without loss of generality, we restrict $N_{\rm in}$ to be much less than the threshold $N_{\rm thr}$, so the problem can then be simplified as to find

$$\min_{\boldsymbol{D},\boldsymbol{x}}\{\|\boldsymbol{y} - \boldsymbol{D}\boldsymbol{x}\|_2\}. \tag{3}$$

The Moore–Pearson inverse $\boldsymbol{D}^+$ is defined as[16]

$$\boldsymbol{D}^+ = \boldsymbol{D}^{\rm T}(\boldsymbol{D}\boldsymbol{D}^{\rm T})^{-1},\ m < k. \tag{4}$$

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Transform expression (3) to a cost function, in which only dictionary $\boldsymbol{D}$ is variable, i.e.,

$$J(\boldsymbol{D}) = \arg\min_{\boldsymbol{D}}\{\|\boldsymbol{y} - \boldsymbol{D}\text{select}(\boldsymbol{D}^{+}\boldsymbol{y})\|_2\}, \quad (5)$$

where select$(\cdot)$ denotes the selection operation, in which it makes N_{in} nonzero entries of each vector keep constant.

The generation of $\boldsymbol{D}$ is a key issue. In Ref. [17], the random Gaussian frames are used, and their performances are compared with the numerical lower bound. This work enlightens us. We propose a 'free-type' random frame dictionary $\boldsymbol{F}$, which is generated by using chaotic random search (CRS) algorithm, and Eq. (5) can be rewritten as

$$J(\boldsymbol{F}) = \arg\min_{\boldsymbol{F}}\{\|\boldsymbol{y} - \boldsymbol{F}\text{select}(\boldsymbol{F}^{+}\boldsymbol{y})\|_2\}. \quad (6)$$

The CRS algorithm is outlined as follows:

a. **Task** Generate a random frame dictionary $\boldsymbol{F}$ until the cost function $J(\boldsymbol{F})$ reaches its minimum.

b. **Initialisation** Choose initial element values, minimum and maximum 'search variance', $\delta^2_{\max}$, $\delta^2_{\min}$, respectively. Set $\delta^2(1) = \delta^2_{\max}$, where $\delta^2(1)$ denotes the value of search variance when iterated one time. Choose a contract factor κ ($\kappa \geq 1$) and maximal iteration times $z_{\max}$. Let $\boldsymbol{H} = [h_{\text{qp}}]$, $(q = 1, 2, \ldots, m,\ p = 1, 2, \ldots, k)$, which is a default dictionary.

c. **Iteration to convergence** The chaotic carrier matrix is defined as $\boldsymbol{C}(z) = [c_{\text{qp}}(z)]$, $(q = 1, 2, \ldots, m;\ p = 1, 2, \ldots, k)$, z is the number of iterations. The single time operation of each chaotic carrier $c_{\text{qp}}(z)$ is assigned

$$c_{\text{qp}}(z+1) = \text{mod}\bigg(\sin\frac{\mu}{c_{\text{qp}}(z)} + \theta c_{\text{qp}}(z) \times (1 - c_{\text{qp}}(z)), 1\bigg), \quad (7)$$

where μ and θ denote the default parameters, which are usually set to $\mu = 2$, $1 \leq \theta \leq 4$. Then, we have

$$\tilde{\boldsymbol{H}} = \boldsymbol{H}(z-1) + \boldsymbol{C}(z)\delta^2(z), \quad (8)$$

where $\tilde{\boldsymbol{H}}$ is an auxiliary dictionary. If

$$J(\tilde{\boldsymbol{H}}) < J(\boldsymbol{H}(z-1)), \quad (9)$$

update below we have

$$\begin{aligned} \boldsymbol{H}(z) &= \tilde{\boldsymbol{H}}, \\ J(\boldsymbol{H}(z)) &= J(\tilde{\boldsymbol{H}}), \\ \delta^2(z+1) &= \delta^2_{\max}. \end{aligned} \quad (10)$$

Else if $J(\tilde{\boldsymbol{H}}) \geq J(\boldsymbol{H}(z-1))$, the old values of the matrix are reserved

$$\begin{aligned} \boldsymbol{H}(z) &= \boldsymbol{H}(z-1), \\ J(\boldsymbol{H}(z)) &= J(\boldsymbol{H}(z-1)), \\ \delta^2(z+1) &= \delta^2(z)\kappa^{-1}; \end{aligned} \quad (11)$$

if $\delta^2(z+1) < \delta^2_{\min}$, then reset $\delta^2(z+1) = \delta^2_{\max}$. This process will be stopped after $z_{\max}$ or when $J(\boldsymbol{H}(z)) \leq \varepsilon$, for some prescribed $\varepsilon \geq 0$. When the cost function $J(\boldsymbol{F})$ ($\boldsymbol{F}$=$\boldsymbol{H}$) reaches its minimum, this means that the AWGN is reduced by a *Maximum a posteriori* (MAP) estimation way.

3. Simulations

Two chaotic signals are used to test the performance of the above method by computer simulation. The two signals $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generated by the logistic map and the Hénon mapping, respectively. i.e.,

$$s_1(t+1) = c_1 s_1(t)(1 - s_1(t)), \quad (12)$$

$$\begin{aligned} s_2(t+1) &= 1 - c_2(s_2(t))^2 + s_3(t), \\ s_3(t+1) &= c_3 s_2(t), \end{aligned} \quad (13)$$

where $t = 1, 2, 3, \ldots, n$, (n is set to 4800 in simulation), and the initialization states are $s_1(0) = 0.1$ and $s_2(0) = 0.1$. The systematic parameters of the maps are set to $c_1 = 4$, $c_2 = 1.4$, and $c_3 = 0.3$. The vectors of the observed signals are denoted by $\boldsymbol{y}_1 = \boldsymbol{s}_1 + \boldsymbol{w}_1$ and $\boldsymbol{y}_2 = \boldsymbol{s}_2 + \boldsymbol{w}_2$, where $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are AWGNs, with zero mean and standard deviations δ_1 and δ_2, respectively.

In the simulation, we combine the generated random frame dictionary with the basis pursuit algorithm,[18] which is a fast algorithm for solving $\boldsymbol{x}$ after the dictionary is achieved, to form basis pursuit denoising algorithm based on overcomplete random frame dictionary. The results of this method will be given compared with the algorithm based on overcomplete Haar dictionary and the algorithm based on overcomplete discrete cosine transform (DCT) dictionary. The performance measurement is the signal-to-noise ratios (SNR), which is defined as

$$\text{SNR}_p \triangleq 10\log_{10}\bigg(\sum_{\text{t}=1}^{n}(\tilde{s}_p(t))^2 \Big/ \sum_{t=1}^{n}(s_p(t) - \tilde{s}_p(t))^2\bigg), \quad p = 1, 2, \quad (14)$$

where $\tilde{\boldsymbol{s}}_p$ the estimated signal and $\boldsymbol{s}_p$ denotes the original signal. Experimental results for these two observed signals under the different noise level are shown in Figs. 1 and 2. It can seen from Figs. 1 and 2,

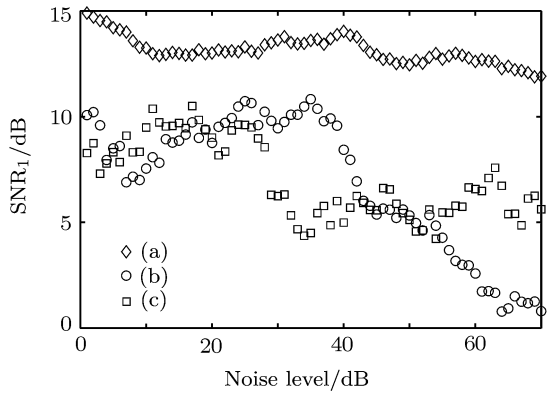


Fig. 1. Performance comparison on $\boldsymbol{y}_1$ denoising: (a) overcomplete random frame dictionary, (b) overcomplete Haar dictionary, and (c) overcomplete DCT dictionary.

our method outperforms the basis pursuit denoising algorithm based on overcomplete Haar dictionary and that based on overcomplete DCT dictionary. The SNR curves for both basis pursuit denoising algorithms in Figs. 1 and 2 show tendencies of continuously decreasing with increasing noise level.

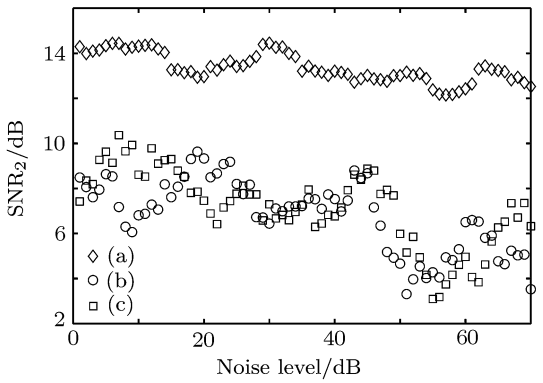


Fig. 2. Performance comparison on $\boldsymbol{y}_2$ denoising: (a) overcomplete random frame dictionary, (b) overcomplete Haar dictionary, and (c) overcomplete DCT dictionary.

From experimental results above, we can see the superiority of our method. This results from the following: first, the nonlinear signal denoising problem is actually a sparsity constrained approximation problem;[17] second, random frame dictionaries are asymptotically good in the worst case, when using l^0 regularization or its approximation l^1 regularization.[19] To confirm that these statements are still effective in this study, we evaluate the performance of the random frame dictionary generated by using our method under the average distortion criteria.[17] The distortion is defined as follows:

$$\text{Distortion}_p \triangleq 10\log_{10}\left(\sum_{t=1}^{n}(s_p(t)-\tilde{s}_p(t))^2 \Big/ \sum_{t=1}^{n}(\tilde{s}_p(t))^2\right), \quad p=1,2. \tag{15}$$

For $m = 6$, 8, and 10, and $N = m$, we generate $k = 32$ random frame dictionaries from $\boldsymbol{s}_1$, respectively. For each m value, 1000 Monte Carlo runs are executed. The average distortion is then

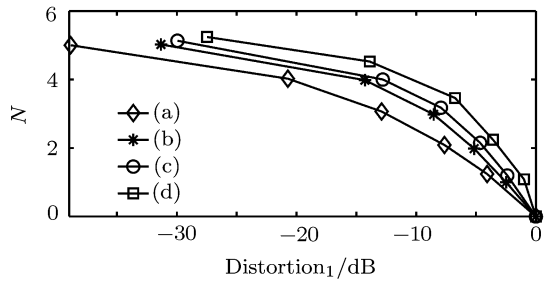


Fig. 3. Comparisons of the numerical lower bound with (a) optimal solutions, (b) overcomplete random frame dictionary, (c) overcomplete Haar dictionary, and (d) overcomplete DCT dictionary, when $N = 6$.

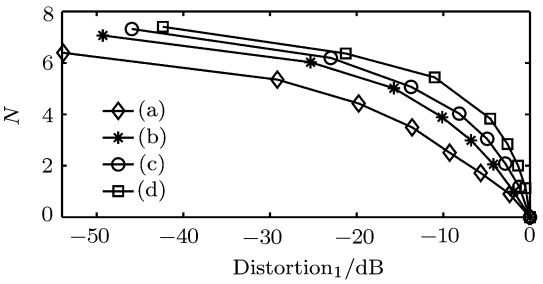


Fig. 4. Comparisons of the numerical lower bound with (a) optimal solutions, (b) overcomplete random frame dictionary, (c) overcomplete Haar dictionary, and (d) overcomplete DCT dictionary, when $N = 8$.

calculated by using optimal sparse representation algorithm.[17] These results are compared with the numerical bound of Theorem 2.1 in Ref. [17]. Figures 3–5 show the comparison results.

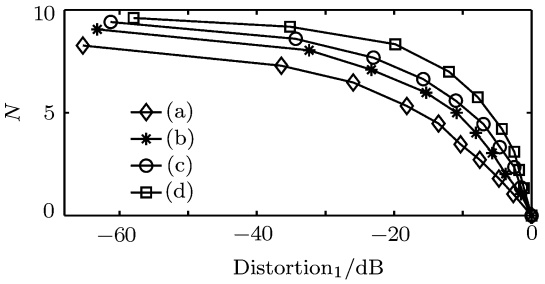


Fig. 5. Comparisons of the numerical lower bound with (a) optimal solutions, (b) overcomplete random frame dictionary, (c) overcomplete Haar dictionary, and (d) overcomplete DCT dictionary, when $N = 10$.

These results indicate that $m = 6$, $m = 8$, and $m = 10$ random frame dictionaries ($k = 32$) perform, respectively, about 5.03, 5.21, and 6.18 dB from the numerical lower bound of Theorem 2.1 in Ref. [17] for a sparsity of $NN_{\text{in}}/N_{\text{thr}} = 0.5N$. It is also seen that the performance of random frame dictionary hardens about their average as N grows. All these results confirm that those statements are still effective here.

4. Conclusions

An SR denoising algorithm based on random frame dictionary for treating noisy chaotic signals has been proposed in this paper. Its properties are discussed and its performance is compared with other two algorithms, i.e., the basis pursuit denoising algorithm based on overcomplete Haar dictionary and the basis pursuit denoising algorithm based on overcomplete DCT dictionary. The numerical simulation indicated that our method has better denoising effect, and confirmed that random frame dictionaries are asymptotically good in the worst case, when using l^0 regularization.

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