Stochastic resonance in an asymmetric bistable system driven by coloured noises*

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The phenomenon of stochastic resonance is investigated in an asymmetric bistable system with coloured noises. The approximate Fokker–Planck equation is derived based on the Novikov theorem and the Fox approach. By applying the two-state theory, the expression of the signal-to-noise ratio is obtained in the adiabatic limit. The effects of the noise parameters on signal-to-ratio are discussed. It is found that the stochastic resonance phenomena appear in most cases and disappear in some special cases.

Keywords: stochastic resonance, asymmetric bistable system, coloured noises
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1. Introduction

In the last three decades, the phenomenon of stochastic resonance (SR) has been investigated extensively in bistable systems,[1−4] asymmetric bistable systems,[5,6] excited systems,[7−9] coupled oscillators,[10−12] even in the linear systems.[13,14] The output signal of the system may be amplified in the presence of an optimal level of noise and a weak periodic signal. The concept of noise-induce SR has been widely applied to many fields ranging from electronic circuits and lasers to chemical systems and the connection with some situations of biological interest.[15−21]

In the present paper, we study the SR in an asymmetric bistable system driven by multiplicative and additive coloured noises. In Section 2 the model with coloured noises are presented. In Section 3, the approximate Fokker–Planck equation, the analytic expressions of quasi-steady-state probability distribution and transition rates are derived in sequence. The signal-to-noise ratio (SNR) is obtained by applying the two-state theory in the adiabatic limit. In Section 4, according to the calculation results, we discuss the effects of the noise parameters, such as the noise intensity, the multiplicative or additive noise self-correlation time, on the SNR. A conclusion concludes the paper.

2. Model

We consider the model described by the following Langevin equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1-\theta x)x - [\beta + \xi(t)]\frac{x}{1+x} + \eta(t) + \varepsilon\cos(\omega t), \tag{1}$$

where θ and β are fixed parameters of the system. $\xi(t)$ and $\eta(t)$ are multiplicative noise and additive noise, respectively. The two noise terms are characterized by their mean and variance

$$\begin{aligned}
&\langle\xi(t)\rangle = \langle\eta(t)\rangle = 0,\\
&\langle\xi(t)\xi(t')\rangle = \frac{Q}{\tau_1}\exp\left[-\frac{|t-t'|}{\tau_1}\right],\\
&\langle\eta(t)\eta(t')\rangle = \frac{M}{\tau_2}\exp\left[-\frac{|t-t'|}{\tau_2}\right],\\
&\langle\xi(t)\eta(t')\rangle = \langle\eta(t)\xi(t')\rangle = 0,
\end{aligned} \tag{2}$$

where Q and M describe the intensities of the multiplicative and additive noises, respectively, τ_1 and τ_2 are the self-correlation times of the noises. $\varepsilon\cos(\omega t)$ is a weak low-frequency signal, i.e. $\varepsilon \ll 1$ and $\omega \ll 1$. Equation (1) can be considered as describing an overdamped motion of the state variable moving in a quasi-'free energy potential':

$$V(x) = \frac{\theta}{3}x^3 - \frac{x^2}{2} + \beta x - \beta\ln(x+1). \tag{3}$$

In the absence of external periodic force and noises terms, the fixed points of Eq. (1) strongly depend on

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θ and β: (i) For $\theta > 1$ and $0 < \beta < 1$, a stable fixed point $x_1 = 0$ and an unstable fixed point

$$x_2 = \frac{1-\theta+\sqrt{(1+\theta)^2-4\beta\theta}}{2\theta};$$

(ii) For $0 < \theta < 1$ and $0 < \beta < (1+\theta)^2/(4\theta)$, two stable fixed points $x_1 = 0$,

$$x_2 = \frac{1-\theta+\sqrt{(1+\theta)^2-4\beta\theta}}{2\theta}$$

and an unstable fixed point

$$x_3 = \frac{1-\theta-\sqrt{(1+\theta)^2-4\beta\theta}}{2\theta};$$

(iii) For $0 < \theta < 1$ and $\beta = (1+\theta)^2/(4\theta)$, a stable fixed point $x_1 = 0$ and an unstable fixed point $x_2 = (1-\theta)/(2\theta)$. In the forthcoming sections, we will consider only the parameters in a region of $0 < \theta < 1$ and $0 < \beta < (1+\theta)^2/(4\theta)$ in which case the model possesses two stable states.

3. Output SNR

By applying the Novikov theorem[22] and Fox's approach,[23] the approximate Fokker–Planck equation (FPE) corresponding to Eq. (1) can be written as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x)P(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[\sigma^2(x)P(x,t)\right]. \tag{4}$$

The drift coefficient $\mu(x)$ and the diffusion coefficient $\sigma^2(x)$ are

$$\begin{aligned}\mu(x) &= f(x) + \sigma(x)\frac{d\sigma(x)}{\mathrm{d}x},\\ \sigma^2(x) &= \frac{Q}{1-\tau_1 f'(x_s)}g^2(x) + \frac{M}{1-\tau_2 f'(x_s)},\end{aligned} \tag{5}$$

with

$$\begin{aligned}f(x) &= (1-\theta x)x - \beta\frac{x}{1+x} + \varepsilon\cos(\omega t),\\ g(x) &= -\frac{x}{1+x}, \quad x_s = \begin{cases} x_1 & x \le x_3 \\ x_2 & x \ge x_3 \end{cases}.\end{aligned} \tag{6}$$

The FPE is valid for[24]

$$1 - \tau_i f'(x_s) > 0, \quad (i = 1, 2). \tag{7}$$

The diffusion coefficient $\sigma^2(x)$ is a piecewise function, so the quasi-steady-state solution to Eq. (4) is also a piecewise function, which is given by

$$P_{st}(x) = \begin{cases} P_1(x), & x \le x_3, \\ P_2(x), & x \ge x_3. \end{cases} \tag{8}$$

For $P_{\mathrm{st}}(x)$, the continuity property $P_1(x_3) = P_2(x_3)$ and the normalization condition $\int_{-\infty}^{+\infty} P_{\mathrm{st}}(x)\mathrm{d}x = 1$ must be satisfied a priori. Since the frequency of the signal is very small, there is enough time for the system to reach the local equilibrium during the period of $1/\omega$. Then the quasi-steady-state probability distribution function corresponding to Eq. (4) can be obtained in the adiabatic limit

$$P_i(x) = \frac{N_i}{\sigma(x)}\exp\left[\int_x \frac{f(x')}{\sigma^2(x')}\mathrm{d}x'\right] = \frac{N_i}{\sigma(x)}\exp\left[U_i(x)\right], \quad i = 1, 2, \tag{9}$$

where N_i is the integral constant and $U_i(x)$ is the modified potential function, which is described as follows:

$$\begin{aligned}U_i(x) = {} & \frac{A_1}{B_1}x^3 + \frac{A_2}{B_1^2}x^2 + \frac{A_3}{B_1^3}x + \frac{A_4}{B_1^4}f_1(x) + \frac{1}{B_1^3 B_2}(A_5 + \frac{A_6}{B_1})f_2(x)\\ & + \left[\frac{E_1}{B_1}x + \frac{E_2}{B_1^2}f_1(x) + \frac{1}{B_1 B_2}\left(E_2 + \frac{E_3}{B_1}\right)f_2(x)\right]\varepsilon\cos(\omega t),\end{aligned} \tag{10}$$

where

$$\begin{aligned}&c_i = 1 - \tau_i f'(x_{\mathrm{s}}), \quad (i = 1, 2), \quad A_1 = -\frac{1}{3}c_1 c_2\theta, \quad A_2 = \frac{1}{2}c_1 c_2 B_1 - c_1 c_2^2\theta Q,\\ &A_3 = (2-\theta-\beta)c_1 c_2^3 Q^2 + (2-2\beta+3\theta)c_1^2 c_2^2 MQ - c_1^3 c_2\beta M^2,\\ &A_4 = \frac{1}{2}(1-\beta)c_1 c_2^4 Q^3 - \left(\frac{1}{2}\beta - 2\theta + 1\right)c_1^2 c_2^3 MQ^2 + \left(\frac{1}{2}\beta - 2\theta - \frac{3}{2}\right)c_1^3 c_2^2 M^2 Q + \frac{1}{2}c_1^4 c_2\beta M^3,\\ &A_5 = (-2+\theta+\beta)c_1^2 c_2^3 MQ^2 + (2\beta - 3\theta - 2)c_1^3 c_2^2 M^2 Q + c_1^4 c_2\beta M^3,\\ &A_6 = (\beta - 4\theta + 2)c_1^3 c_2^3 M^2 Q^2 + (4\theta - \beta + 3)M^3 Q + (\beta - 1)c_1^2 c_2^4 MQ^3 - c_1^5 c_2\beta M^4,\\ &B_1 = c_1 M + c_2 Q, \quad B_2 = \sqrt{c_1 c_2 MQ}, \quad E_1 = c_1 c_2, \quad E_2 = c_1 c_2^2 Q, \quad E_3 = -2c_1^2 c_2^2 QM,\\ &f_1(x) = \ln\left[c_1 M(x+1)^2 + c_2 Q x^2\right], \quad \text{and} \quad f_2(x) = \arctan\frac{B_1 x + c_1 M}{B_2}.\end{aligned} \tag{11}$$

In the absence of the periodic signal ($\varepsilon = 0$), the generalized potential function is

$$U(x) = \begin{cases} U_1(x), & x \le x_3, \\ U_2(x) + d, & x \ge x_3, \end{cases} \tag{12}$$

where $d = U_1(x_3) - U_2(x_3)$. Between the two states x_1 and x_2, the transition rates W_1 (the traverse direction is $x_1 \to x_2$) and W_2 (the traverse direction is $x_2 \to x_1$) can be obtained as

$$W_i = \frac{1}{2\pi}\sqrt{|V''(x_i)V''(x_3)|}\exp\left[U(x_3) - U(x_i)\right], \quad (i = 1, 2). \tag{13}$$

The expression of the SNR, in the asymmetric bistable system can be derived from the two-state approach with two stable states (x_1 and x_2) and given by[1,3,25]

$$\mathrm{SNR}_i = \frac{\pi W_3^2 \varepsilon^2}{4W_i}\left[1 - \frac{W_3^2\varepsilon^2}{2(W_i^2 + \omega^2)}\right]^{-1}, \quad (i = 1, 2), \tag{14}$$

where

$$W_3 = E_4 W_i, \quad (i = 1, 2),$$
$$E_4 = \frac{E_1}{B_1}(x_3 - x_i) + \frac{E_2}{B_1^2}\left[f_1(x_3) - f_1(x_i)\right] + \frac{1}{B_1 B_2}\left(E_2 + \frac{E_3}{B_1}\right)\left[f_2(x_3) - f_2(x_i)\right]. \tag{15}$$

4. Discussion

In conformity to the expression of SNR, Eq. (14), the effects of the additive and multiplicative noise intensities, the self-correlation times of the noises on the SNR can be discussed by numerical calculations.

The SNR as a function of the multiplicative noise intensity Q is plotted in Fig. 1 for different values of the multiplicative noise self-correlation time τ_1. If the additive noise intensity M=0.08, just like in Figs. 1(a) and 1(b), with the increase of Q, the function of SNR is non-monotonic. The curves show conventional SR phenomena. The peak value keeps invariant with the increase of τ_1, but its position moves to right. The function of SNR changes into a monotonic one for $M = 0.4$, which can be seen in Figs. 1(c) and 1(d). The SNR decreases with the value of Q increasing, and it increases with the value of τ_1 increasing. If we change the value of M from $M = 0.08$ to $M = 0.4$ and fix other parameters' values, the characteristic of the SNR produces tremendous change.

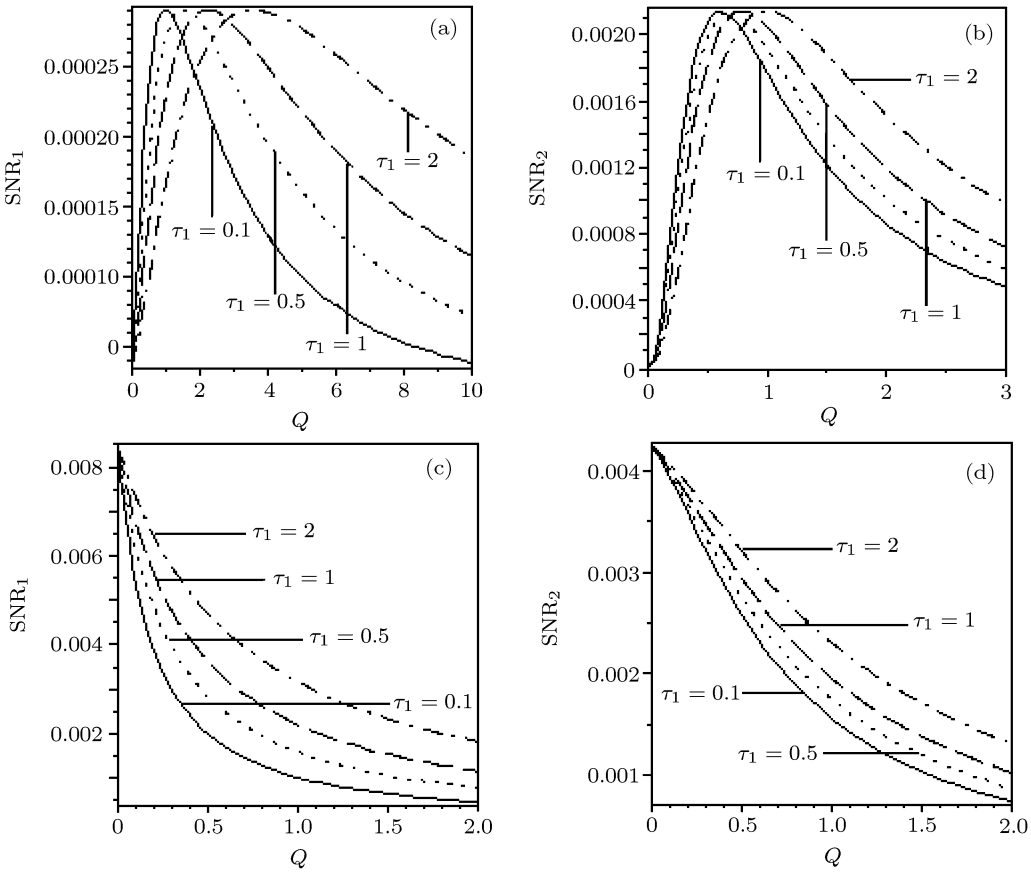


Fig. 1. Curves for SNR versus multiplicative noise intensity Q for different values of τ_1 at $\theta = 0.1$, $\beta = 2.6$, $\varepsilon = 0.1$, $\omega = 0.01$, $\tau_2 = 0.2$ and $M = 0.08$ ((a) and (b)), 0.4 ((c) and (d)), with the initial conditions being $x(0) = x_1$ ((a) and (c)), x_2 ((b) and (d)).

In Fig. 2, the SNRs are shown as a function of the multiplicative noise intensity Q for different values of the additive noise self-correlation time τ_2. For small additive noise intensity, which is presented in Figs. 2(a) and 2(b), the SNR plot has a single peak and it means that there appears SR. The peak becomes lower with the increase of τ_2, and its position moves to right at the same time. If $M = 0.4$, just like Figs. 2(c) and 2(d), the SR phenomenon disappears or weakens. When the initial condition is $x(0) = x_2$ as shown in Fig. 2(d), the SR phenomenon can be strengthened with the increase of τ_2.

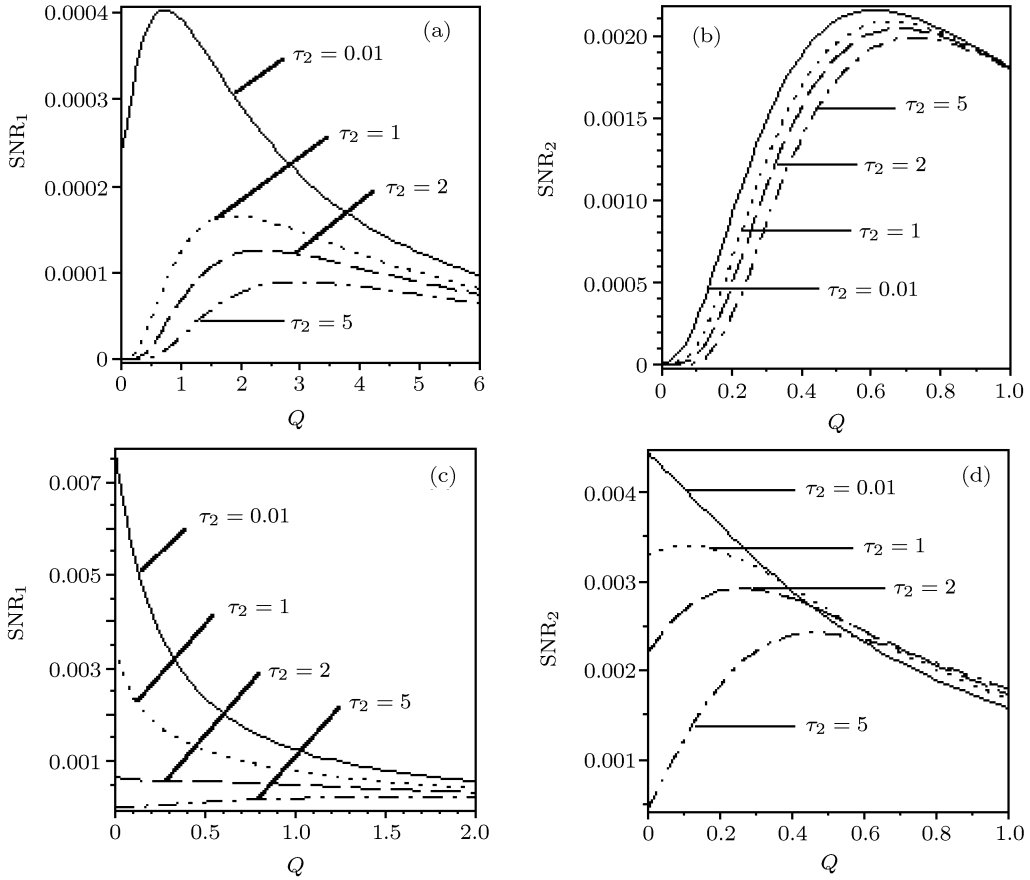


Fig. 2. Curves for SNR versus multiplicative noise intensity Q for different values of τ_2 at $\theta = 0.1$, $\beta = 2.6$, $\varepsilon = 0.1$, $\omega = 0.01$, $\tau_1 = 0.2$ and $M = 0.08$ ((a) and (b)), 0.4 ((c) and (d)), with the initial conditions being $x(0) = x_1$ ((a) and (c)), x_2 ((b) and (d)).

Figure 3 presents the curves for SNR versus additive noise intensity M with different values of τ_1. The SNRs decrease with the increase of τ_1 initially and then increase when the value of M is bigger, which is obvious in Fig. 3(b). With the increase of τ_1, the SR phenomenon is strengthened.

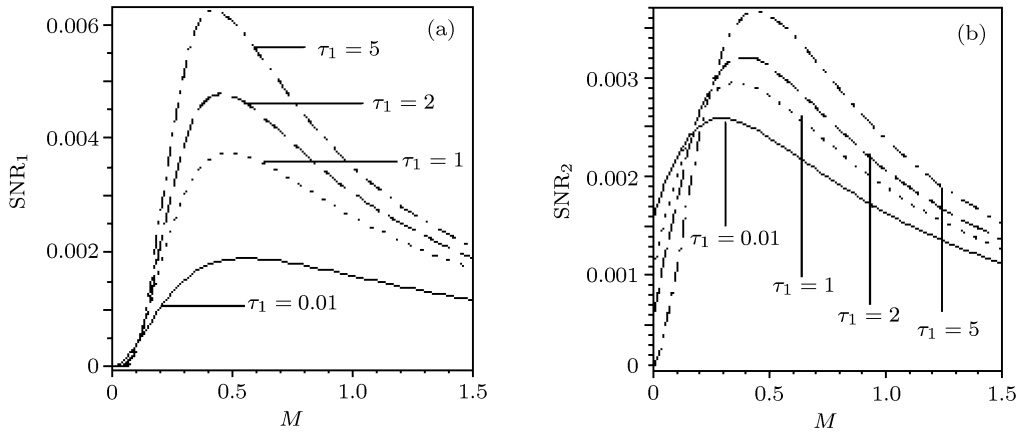


Fig. 3. Curves for SNR versus additive noise intensity M for different values of τ_1 at $\theta = 0.1$, $\beta = 2.6$, $\varepsilon = 0.1$, $\omega = 0.01$, $Q = 0.5$ and $\tau_2 = 0.2$, with the initial conditions being $x(0) = x_1$ (a), x_2 (b).

In Fig. 4, we analyse the influence of the additive noise self-correlation time τ_2 on SNR when the SNR is shown to be a function of additive noise intensity. Comparing this figure with Figs. 1(a) and 1(b) shows that these plots are similar, and the SR is observed in the two figures. The peak value keeps invariant with the increase of τ_2, but its position shifts to right. From Figs. 3 and 4, it is observed that the curve does not change much if we change the value of the multiplicative noise intensity Q. It is different from the effect of the additive noise intensity on the SNR.

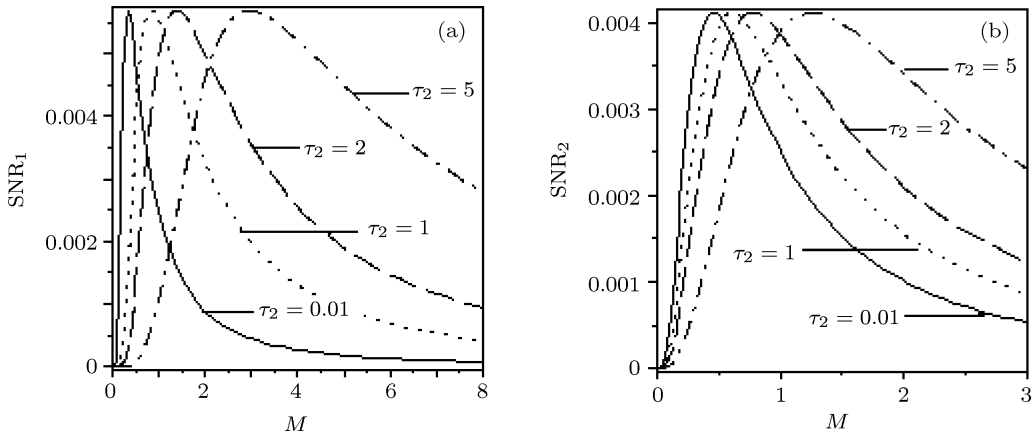


Fig. 4. Curves for SNR versus additive noise intensity M for different values of τ_2 at $\theta = 0.1$, $\beta = 2.6$, $\varepsilon = 0.1$, $\omega = 0.01$, $Q = 0.1$ and $\tau_1 = 0.2$ with the initial conditions being $x(0) = x_1$ (a), x_2 (b).

The curves for the SNR versus multiplicative noise self-correlation time τ_1 with different additive noise self-correlation times τ_2 are shown in Fig.5. For the initial condition $x(0) = x_1$ as shown in Fig. 5(a), there is one peak in the SNR for small values of τ_2. If the value of τ_2 is bigger, the SNR is monotonic with increase of τ_1. For the initial condition $x(0) = x_2$ as indicated in Fig. 5(b), there appears a resonance peak appearing on the curve.

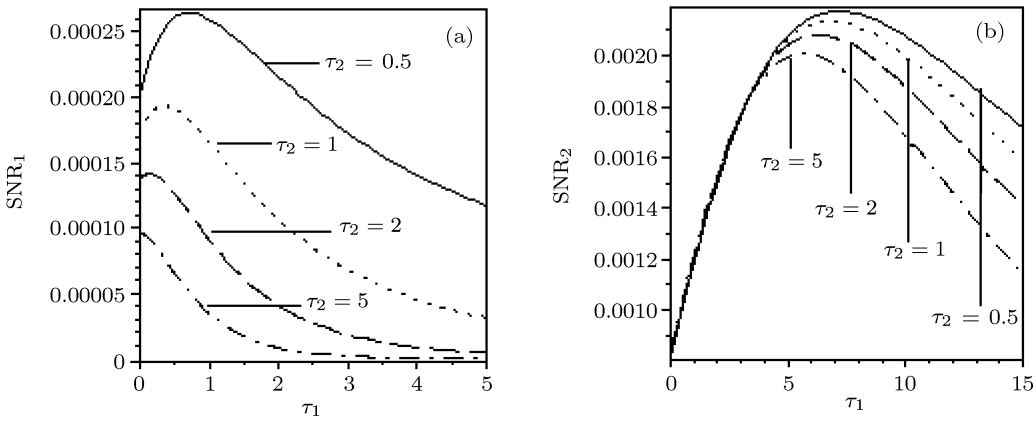


Fig. 5. Curves for SNR versus multiplicative noise self-correlation time τ_1 for different values of τ_2 at for $\theta = 0.1$, $\beta = 2.6$, $\varepsilon = 0.1$, $\omega = 0.01$, $Q = 2$ and $M = 0.1$, with the initial conditions being $x(0) = x_1$ (a), x_2 (b).

The curves for SNR versus additive noise self-correlation time τ_2 with different values of the multiplicative noise self-correlation time τ_1 are shown in Fig. 6, showing that SNR is non-monotonic with the increase of τ_2, namely the plots show SR.

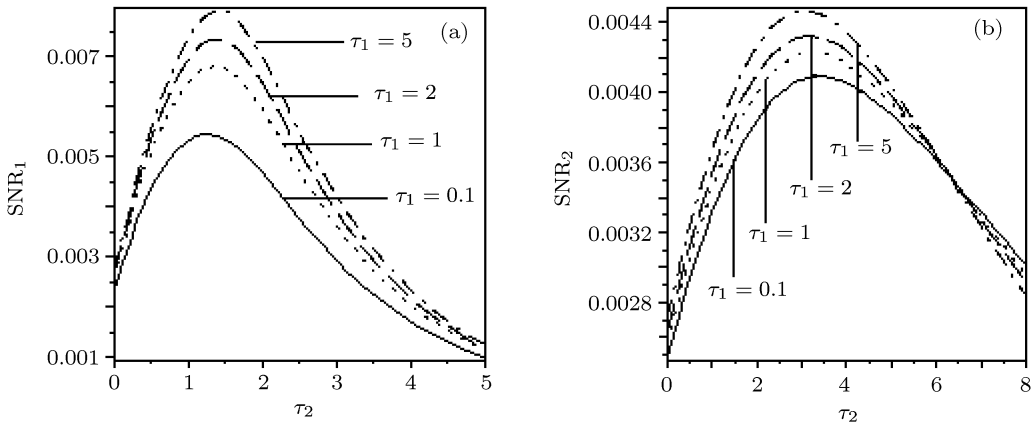


Fig. 6. Curves for SNR versus additive noise self-correlation time τ_2 for different values of τ_1 at $\theta = 0.1$, $\beta = 2.6$, $\varepsilon = 0.1$, $\omega = 0.01$, $Q = 0.1$ and $M = 1$, with the initial conditions being $x(0) = x_1$ (a), x_2 (b).

5. Conclusion

In this paper, we study the SR phenomenon in an asymmetric bistable system subjected to both coloured noises and periodic signal by using the theory of SNR proposed by McNamara and Wiesenfeld.[1] The SNR is dependent on the initial condition since the transition rate depends on initial condition. By virtue of the expression of SNR and through numerical computation, the following conclusions can be drawn from the present study. The effects of the multiplicative noise intensity and the additive noise intensity on SNR are different: the effect of the multiplicative noise intensity Q on SNR strongly depends on the value of the additive noise intensity. The plot for SNR-Q shows SR when the additive noise intensity is small, and the SR phenomenon disappears with the value of M turning big. However, the SNR is a non-monotonic function of additive noise intensity, and the plot does not change much if we change the value of the multiplicative noise intensity. The multiplicative (additive) noise self-correlation time changes only the position of the peak in the SNR-Q (SNR-M) plot, but not the value of the peak. In the SNR-Q (SNR-M) plot, the value and the position of the peak are both shifted by changing the additive (multiplicative) noise self-correlation time. The SNRs corresponding to the two kinds of the self-correlation times exhibit SR except the $\mathrm{SNR}_1 - \tau_1$ plot when the additive noise self-correlation time is big.

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