Quantum phase transition and entanglement in Heisenberg XX spin chain with impurity*

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In this paper, we study the quantum phase transition and the effect of impurity on the thermal entanglement between any two lattices in three-qubit Heisenberg XX chain in a uniform magnetic field. We show that the quantum phase transition always appears when impurity parameter is an arbitrary constant and unequal to zero, the external magnetic field and impurity parameters have a great effect on it. Also, there exists a relation between the quantum phase transition and the entanglement. By modulating the temperature, magnetic field and the impurity parameters, the entanglement between any two lattices can exhibit platform-like behaviour, which can be used to realize entanglement switch.

Keywords: Heisenberg XX chain, thermal entanglement, impurity lattice, quantum phase transition
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1. Introduction

Entanglement is not only a key concept to distinguish between the classical world and quantum world but is also very important in the quantum information processing as an indispensable resource.[1−6] Entanglement can be realised in many physical contexts, a typical example is the spin chains that can describe the interaction of qubits in solid physical systems and in many other systems such as quantum dots,[7] electronic spins,[8] and optical lattices.[9] Therefore, there have been numerous studies on Ising model[10] and all kinds of Heisenberg XX models,[11] XY models,[12−14] XYZ models,[15] and pairwise entanglement with impurity.[16−21] It has provided a very advantageous theoretical framework for the quantum spin systems, in which Heisenberg spin chain model has been used for the calculation of quantum dot-based study.[22] Impurity plays an important role in the one-dimensional (1D) quantum system, even a small defect may destroy the whole properties of the quantum system. Therefore, it is meaningful to investigate the entanglement with impurity. Shan *et al.*[23] have demonstrated that the entanglement can be controlled by introducing impurities into the systems. Meanwhile, the entanglement shares many features with quantum phase transition (QPT) for a many-body system.[24,25] The QPT is the structural change in property of the ground state, which occurs at an absolute zero temperature and is purely driven by quantum fluctuation. Concerning the three-qubit Heisenberg XX chain, previous literatures are limited to the study of entanglement between any two lattices or entanglement between any two lattices in external magnetic field, where as little studies are available when it contains impurity in the external magnetic field. Reference [18] studied the effect of impurity on the thermal entanglement between any two lattices in three-qubit systems without external magnetic field.

Because the effects of magnetic field on the entanglement are very great, in this paper, we will study the QPT and the effect of impurity and external magnetic field on the thermal entanglement between any two lattices in periodic boundary conditions in three-qubit Heisenberg XX chain. We discuss two cases, the first case is the entanglement between normal lattices while the second is the entanglement between a normal lattice and an impurity lattice. The results show that thermal entanglement exhibits a platform-like behaviour when adjusting temperature, impurity and external magnetic field. Because of this property, the entanglement switch can be realised at a certain temperature.

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2. The model and its solution

We first recall the definition of concurrence[26,27] between a pair of qubits. Let ρ_{ij} be the density matrix of the pair which can be either pure or mixed. The concurrence corresponding to the density matrix is defined as

$$C = \max\{0, 2\max(\lambda_i) - \sum_{i=1}^{4} \lambda_i\}, \tag{1}$$

where the quantities λ_i $(i = 1 \sim 4)$ are the square roots in descending order of the eigenvalues of the operator

$$R_{ij} = \rho_{ij}(\sigma_i^y \otimes \sigma_j^y)\rho_{ij}^*(\sigma_i^y \otimes \sigma_j^y). \tag{2}$$

Here ρ_{ij}^* denotes the complex conjugate of ρ_{ij}, and σ_i^y and σ_j^y are the usual Pauli matrix. All the matrix elements in the density matrix can be calculated from the different spin–spin correlation functions. The eigenvalues of R_{ij} are real and non-negative even though R_{ij} is not necessarily Hermitian. The values of concurrence C range from zero to one. When $C = 0$, the system is in an unentangled state, when $C = 1$, the system is in a maximally entangled state.

The state of the system at thermal equilibrium is represented by the density operator

$$\begin{aligned}\rho(T) &= \frac{1}{Z}\exp\left(-\frac{H}{\kappa_\beta T}\right) \\ &= \frac{1}{Z}\sum_{i=0}^{7}\exp\left(\frac{E_i}{\kappa_\beta T}\right)|\Psi_i\rangle\langle\Psi_i|,\end{aligned} \tag{3}$$

where $Z = \mathrm{Tr}[\exp(-H/k_\beta T)]$ is the partition function, H is the system Hamiltonian, k_β is the Boltzmann constant and has been set to 1 hereafter, T is the temperature. As $\rho(T)$ represents a thermal state, the entanglement in the state is called thermal entanglement.[28] In this paper, we introduce impurity into the thermal state.

We consider a 1D physical Heisenberg XX model of three spin-1/2 particles with the nearest-neighbouring interactions. The first site is impurity and others are normal particles. Here we impose the periodic boundary condition ($\sigma_3^\pm = \sigma_1^\pm$, $\sigma_3^z = \sigma_1^z$), then the corresponding Hamiltonian can be written as

$$\begin{aligned}H_{XX} &= J_1(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+ + \sigma_1^+\sigma_3^- + \sigma_1^-\sigma_3^+) \\ &\quad + J(\sigma_2^+\sigma_3^- + \sigma_2^-\sigma_3^+) + B\sum_{n=1}^{3}\sigma_n^z,\end{aligned} \tag{4}$$

where J_1 denotes the coupling between the normal site and the impurity site, B is the strength of external magnetic field, $\sigma_n^\alpha(\alpha = x, y, z)$ are the Pauli matrices of the n-th qubit, $\sigma_n^\pm = \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y)$ are the raising and lowering operators, and J is the exchange interaction constant between the normal sites, positive (negative) J corresponds to the antiferromagnetic (ferromagnetic) case. In the standard basis $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$, the eigenvalues of the Hamiltonian (Eq. (4)) are analytically obtained as

$$\begin{aligned}&E_0 = -3B, \quad E_1 = -B - J, \quad E_2 = B - J, \\ &E_3 = \frac{1}{2}(-2B + J - \sqrt{J^2 + 8J_1^2}), \\ &E_4 = \frac{1}{2}(2B + J - \sqrt{J^2 + 8J_1^2}), \\ &E_5 = \frac{1}{2}(-2B + J + \sqrt{J^2 + 8J_1^2}), \\ &E_6 = \frac{1}{2}(2B + J + \sqrt{J^2 + 8J_1^2}), \quad E_7 = 3B\end{aligned} \tag{5}$$

with the corresponding eigenstates

$$\begin{aligned}&|\Psi_0\rangle = |000\rangle, \quad |\Psi_1\rangle = \frac{1}{N_1}(-|001\rangle + |010\rangle), \\ &|\Psi_2\rangle = \frac{1}{N_2}(-|101\rangle + |110\rangle), \\ &|\Psi_3\rangle = \frac{1}{N_3}(a_1|001\rangle + a_1|010\rangle + |100\rangle), \\ &|\Psi_4\rangle = \frac{1}{N_4}(a_2|011\rangle + |101\rangle + |110\rangle), \\ &|\Psi_5\rangle = \frac{1}{N_5}(a_3|001\rangle + a_3|010\rangle + |100\rangle), \\ &|\Psi_6\rangle = \frac{1}{N_6}(a_4|011\rangle + |101\rangle + |110\rangle), \\ &|\Psi_7\rangle = |111\rangle,\end{aligned} \tag{6}$$

where

$$\begin{aligned}a_1 &= -\frac{J^2 + 2J_1^2 - J\sqrt{J^2 + 8J_1^2}}{J_1(-3J + \sqrt{J^2 + 8J_1^2})}, \\ a_2 &= -\frac{J + \sqrt{J^2 + 8J_1^2}}{2J_1}, \\ a_3 &= -\frac{-J^2 - 2J_1^2 - J\sqrt{J^2 + 8J_1^2}}{J_1(3J + \sqrt{J^2 + 8J_1^2})}, \\ a_4 &= -\frac{J - \sqrt{J^2 + 8J_1^2}}{2J_1}, \\ N_3 &= \sqrt{2a_1^2 + 1}, \\ N_4 &= \sqrt{a_2^2 + 2}, \\ N_5 &= \sqrt{2a_3^2 + 1}, \\ N_6 &= \sqrt{a_4^2 + 2}.\end{aligned} \tag{7}$$

Then the partition function

$$
\begin{aligned}
Z = & \exp\left(\frac{3B}{T}\right) + \exp\left(\frac{B+J}{T}\right) + \exp\left(\frac{-B+J}{T}\right) + \exp\left(\frac{2B-J+\sqrt{J^2+8J_1^2}}{2T}\right) \\
& + \exp\left(\frac{-2B-J+\sqrt{J^2+8J_1^2}}{2T}\right) + \exp\left(\frac{2B-J-\sqrt{J^2+8J_1^2}}{2T}\right) \\
& + \exp\left(\frac{-2B-J-\sqrt{J^2+8J_1^2}}{2T}\right) + \exp\left(\frac{-3B}{T}\right).
\end{aligned} \tag{8}
$$

The explicit expression of the density matrix of the state

$$
\begin{aligned}
\rho(T) = & \frac{1}{Z}\Bigg[\exp\left(\frac{3B}{T}\right)|\Psi_0\rangle\langle\Psi_0| + \exp\left(\frac{B+J}{T}\right)|\Psi_1\rangle\langle\Psi_1| \\
& + \exp\left(\frac{-B+J}{T}\right)|\Psi_2\rangle\langle\Psi_2| + \exp\left(\frac{2B-J+\sqrt{J^2+8J_1^2}}{2T}\right)|\Psi_3\rangle\langle\Psi_3| \\
& + \exp\left(\frac{-2B-J+\sqrt{J^2+8J_1^2}}{2T}\right)|\Psi_4\rangle\langle\Psi_4| + \exp\left(\frac{2B-J-\sqrt{J^2+8J_1^2}}{2T}\right)|\Psi_5\rangle\langle\Psi_5| \\
& + \exp\left(\frac{-2B-J-\sqrt{J^2+8J_1^2}}{2T}\right)|\Psi_6\rangle\langle\Psi_6| + \exp\left(\frac{-3B}{T}\right)|\Psi_7\rangle\langle\Psi_7|\Bigg].
\end{aligned} \tag{9}
$$

We turn to the pairwise entanglement between the normal sites 2 and 3, so we need to gain the reduced density $\rho_{23}(T) = \mathrm{Tr}_1[\rho(T)]$. If $\rho_{23}^x = \mathrm{Tr}_1(|\Psi_x\rangle\langle\Psi_x|)$, in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, we have

$$
\begin{aligned}
\rho_{23}^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{23}^1 = \frac{1}{N_1^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\rho_{23}^2 &= \frac{1}{N_2^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{23}^3 = \frac{1}{N_3^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1^2 & a_1^2 & 0 \\ 0 & a_1^2 & a_1^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\rho_{23}^4 &= \frac{1}{N_4^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & a_2^2 \end{pmatrix}, \quad \rho_{23}^5 = \frac{1}{N_5^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_3^2 & a_3^2 & 0 \\ 0 & a_3^2 & a_3^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\rho_{23}^6 &= \frac{1}{N_6^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & a_4^2 \end{pmatrix}, \quad \rho_{23}^7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{10}
$$

then we have

$$
\begin{aligned}
\rho_{23}(T) = & \frac{1}{Z}\Bigg[\exp\left(-\frac{E_0}{T}\right)\rho_{23}^0 + \exp\left(-\frac{E_1}{T}\right)\rho_{23}^1 + \exp\left(-\frac{E_2}{T}\right)\rho_{23}^2 \\
& + \exp\left(-\frac{E_3}{T}\right)\rho_{23}^3 + \exp\left(-\frac{E_4}{T}\right)\rho_{23}^4 + \exp\left(-\frac{E_5}{T}\right)\rho_{23}^5 \exp\left(-\frac{E_6}{T}\right)\rho_{23}^6 + \exp\left(-\frac{E_7}{T}\right)\rho_{23}^7\Bigg] \\
= & \frac{1}{Z}\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & w & y & 0 \\ 0 & y & w & 0 \\ 0 & 0 & 0 & v \end{pmatrix},
\end{aligned} \tag{11}
$$

where the parameters u, v, y, w are

$$\begin{aligned}
u &= \exp\left(-\frac{E_0}{T}\right)+\frac{1}{N_3^2}\exp\left(-\frac{E_3}{T}\right)+\frac{1}{N_5^2}+\exp\left(-\frac{E_5}{T}\right),\\
v &= \frac{a_2^2}{N_4^2}\exp\left(-\frac{E_4}{T}\right)+\frac{a_4^2}{N_6^2}\exp\left(-\frac{E_6}{T}\right)+\exp\left(-\frac{E_7}{T}\right),\\
y &= -\frac{1}{N_1^2}\exp\left(-\frac{E_1}{T}\right)-\frac{1}{N_2^2}\exp\left(-\frac{E_2}{T}\right)+\frac{a_1^2}{N_3^2}\exp\left(-\frac{E_3}{T}\right)\\
&\quad+\frac{1}{N_4^2}\exp\left(-\frac{E_4}{T}\right)+\frac{a_3^2}{N_5^2}\exp\left(-\frac{E_5}{T}\right)+\frac{1}{N_6^2}\exp\left(-\frac{E_6}{T}\right),\\
w &= \frac{1}{N_1^2}\exp\left(-\frac{E_1}{T}\right)+\frac{1}{N_2^2}\exp\left(-\frac{E_2}{T}\right)+\frac{a_1^2}{N_3^2}\exp\left(-\frac{E_3}{T}\right)\\
&\quad+\frac{1}{N_4^2}\exp\left(-\frac{E_4}{T}\right)+\frac{a_3^2}{N_5^2}\exp\left(-\frac{E_5}{T}\right)+\frac{1}{N_6^2}\exp\left(-\frac{E_6}{T}\right).
\end{aligned} \tag{12}$$

From Eqs. (1), (2), and (3), we obtain the concurrence

$$C_{23}=\max\left\{\frac{2}{Z}(|y|-\sqrt{uv}),0\right\}. \tag{13}$$

Making use of completely similar methods we obtain the reduced density between the impurity site 1 and the normal site 3

$$\rho_{13}(T)=\frac{1}{Z}\begin{pmatrix} u' & 0 & 0 & 0\\ 0 & w' & y' & 0\\ 0 & y' & x' & 0\\ 0 & 0 & 0 & v' \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned}
u' &= \exp\left(-\frac{E_0}{T}\right)+\frac{1}{N_1^2}\exp\left(-\frac{E_1}{T}\right)+\frac{a_1^2}{N_3^2}\exp\left(-\frac{E_3}{T}\right)+\frac{a_3^2}{N_5^2}\exp\left(-\frac{E_5}{T}\right),\\
v' &= \frac{1}{N_2^2}\exp\left(-\frac{E_2}{T}\right)+\frac{1}{N_4^2}\exp\left(-\frac{E_4}{T}\right)+\frac{1}{N_6^2}\exp\left(-\frac{E_6}{T}\right)+\exp\left(-\frac{E_7}{T}\right),\\
y' &= \frac{a_1}{N_3^2}\exp\left(-\frac{E_3}{T}\right)+\frac{a_2}{N_4^2}\exp\left(-\frac{E_4}{T}\right)+\frac{a_3}{N_5^2}\exp\left(-\frac{E_5}{T}\right)+\frac{a_4}{N_6^2}\exp\left(-\frac{E_6}{T}\right),\\
w' &= \frac{1}{N_1^2}\exp\left(-\frac{E_1}{T}\right)+\frac{a_1^2}{N_3^2}\exp\left(-\frac{E_3}{T}\right)+\frac{a_2^2}{N_4^2}\exp\left(-\frac{E_4}{T}\right)+\frac{a_3^2}{N_5^2}\exp\left(-\frac{E_5}{T}\right)+\frac{a_4^2}{N_6^2}\exp\left(-\frac{E_6}{T}\right),\\
x' &= \frac{1}{N_2^2}\exp\left(-\frac{E_2}{T}\right)+\frac{1}{N_3^2}\exp\left(-\frac{E_3}{T}\right)+\frac{1}{N_4^2}\exp\left(-\frac{E_4}{T}\right)+\frac{1}{N_5^2}\exp\left(-\frac{E_5}{T}\right)+\frac{1}{N_6^2}\exp\left(-\frac{E_6}{T}\right),
\end{aligned} \tag{15}$$

then the concurrence between the impurity site 1 and the normal site 3 is

$$C_{13}=\max\left\{\frac{2}{Z}(|y'|-\sqrt{u'v'}),0\right\}. \tag{16}$$

3. Results and discussion

3.1. QPT

With $J_1=\beta J$ (β is an arbitrary constant, and the results are the same when $\beta=|a|$ and $\beta=-|a|$, and a is arbitrary constant and unequal to zero, here we only consider the conditions when $\beta>0$), we obtain the eigenvalues of 1D XX model as a function of the parameter B/J for $\beta=0.5$,0.8,1.0 and 2.0, as shown in Fig. 1. As can be seen from the figure, there exists the crossing pattern among ground-state eigenvalues when B/J increases, and the ground state jumps from a state to another at the crossing point. Obviously, with the changing of B/J, there is a QPT in this system. When $0<\beta<1$, the

crossing points are at -0.5, 0 and 0.5; the changes of ground state are: $|\Psi_7\rangle \rightarrow |\Psi_2\rangle \rightarrow |\Psi_1\rangle \rightarrow |\Psi_0\rangle$, $|\Psi_0\rangle$ and $|\Psi_7\rangle$ are disentangled states, $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are entangled states. When $\beta = 1$, there exist two degenerate states, which are $E_1 = E_3 = -(B + J)$ and $E_2 = E_4 = B - J$, and there accompanies a W state $|\Psi_3\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$. Moreover, it could be seen from Fig. 1(c) that the ground state jumps from a disentangled state $|\Psi_7\rangle$ to an entangled one ($|\Psi_2\rangle$ or $|\Psi_4\rangle$), then it jumps from an entangled one ($|\Psi_3\rangle$ or $|\Psi_1\rangle$) to a disentangled one $|\Psi_0\rangle$ at the transformation points respectively. When $\beta > 1$, the ground state jumps from a disentangled state $|\Psi_7\rangle$ to an entangled state $|\Psi_4\rangle$, then it jumps from an entangled state $|\Psi_3\rangle$ to a disentangled state $|\Psi_0\rangle$ at the transformation points. The results indicate that with the occurrence of QPT, the ground state changes between disentangled state and entangled state, so we can further understand the relation between the entanglement and the QPT. By means of the analysis of Fig. 1, we find that the phase transition points are at $B/J = -0.5, 0, 0.5$ when $0 < \beta \leq 1$, but when $\beta > 1$, except for the point $B/J = 0$, the other two points shift to the direction of the higher parameters $|B/J|$. Therefore, by controlling of the parameter B/J we can change the phase transition point.

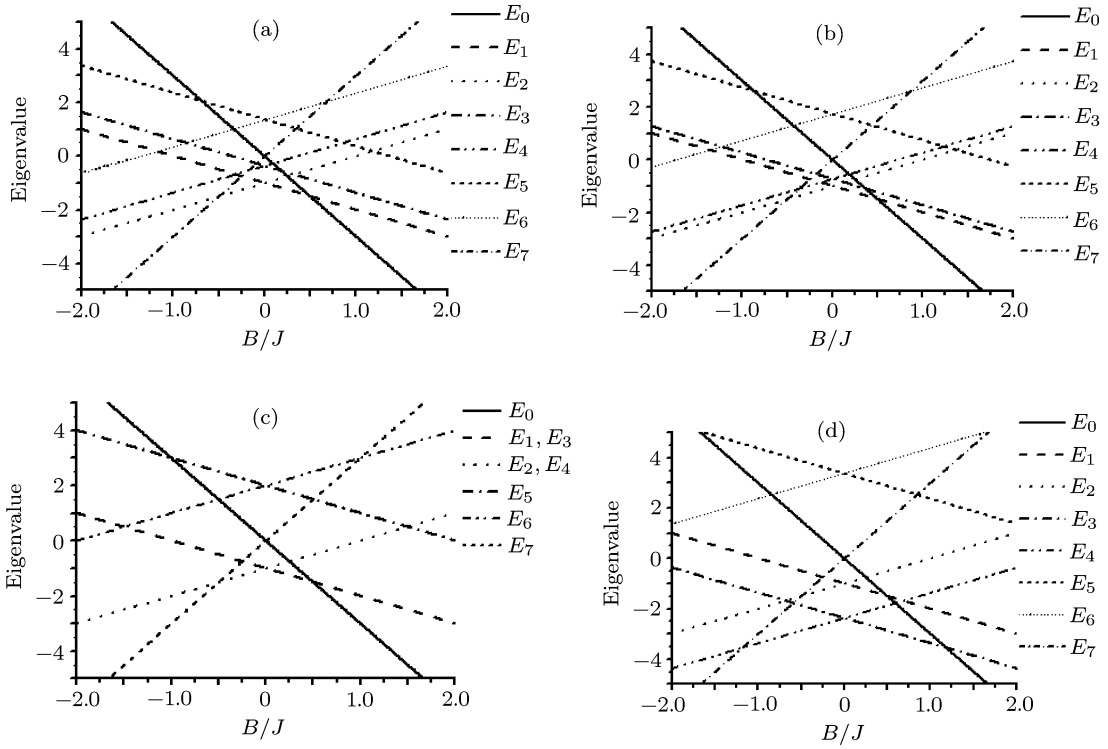


Fig. 1. The eigenvalues as a function of the parameter B/J at different values of parameter β for different distributions. (a) $\beta = 0.5$, (b) $\beta = 0.8$, (c) $\beta = 1$, (d) $\beta = 2$.

3.2. Entanglement between normal lattices

3.2.1. The ferromagnetic ($J < 0$) case

We consider the numerical results in the case $J = -1$ for simplicity. Figure 2 shows the nearest-neighbouring concurrence between the normal lattices as a function of B and J_1 at certain temperature. The results indicate that in the region $|J_1| > 1$, when the temperature and the impurity parameters are fixed, the concurrence will increase rapidly with the increase of magnetic field. It ranges from 0 (or very close to 0) to a fixed value and exhibits a platform-like region, then it quickly decreases to 0. Taking advantage of this nature, we can control the concurrence by adjusting the magnetic field, and make entanglement switch. Figures 3(a) and 3(b) give the width of entanglement switch at a certain temperature for different impurity parameters. As can be seen from Fig. 3, regardless of high or low temperature, the greater the values of the impurity parameters are, the wider the width of the entanglement switches are.

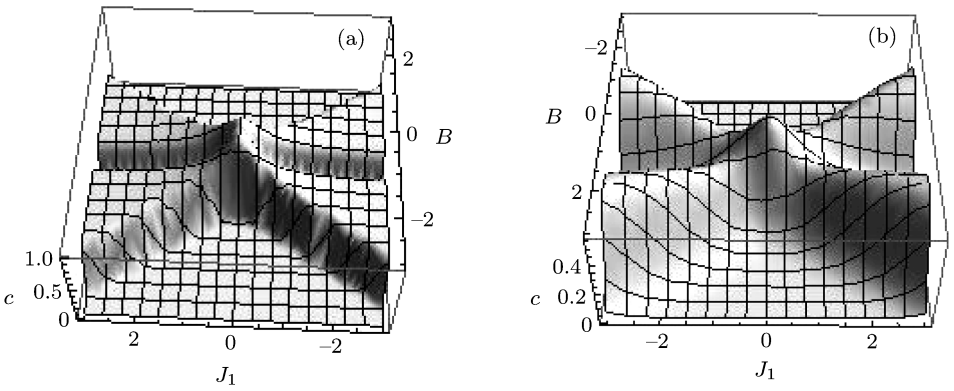


Fig. 2. The concurrence between normal lattices as functions of B and J_1 at certain temperature when $J = -1$ for simplicity. (a) $T = 0.1$, (b) $T = 0.5$.

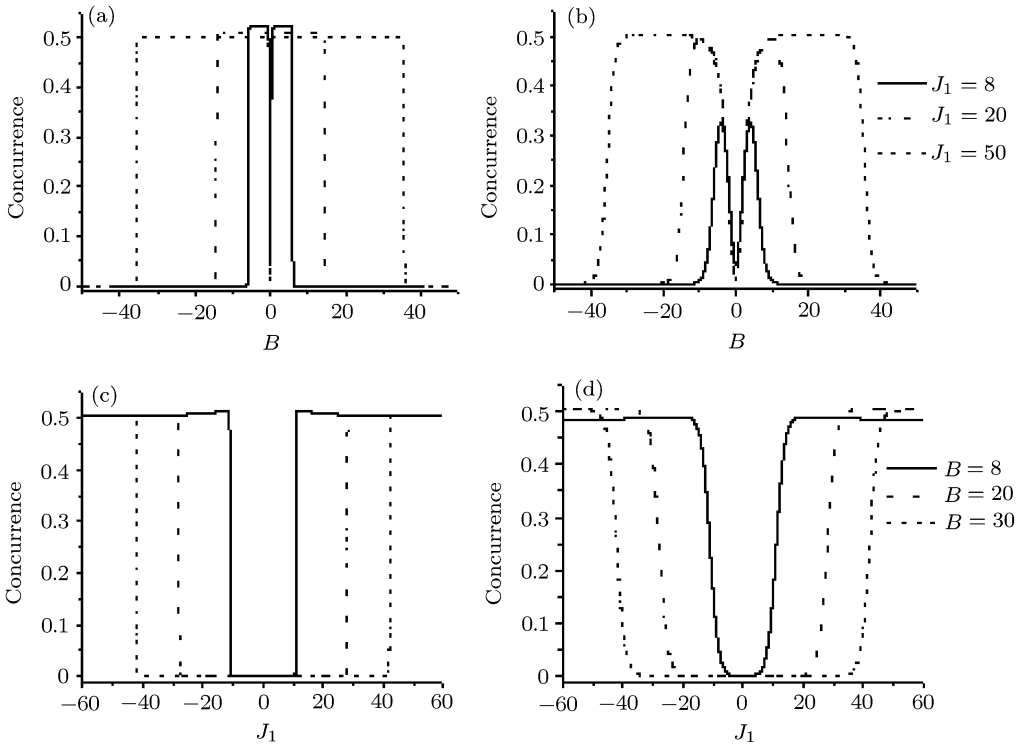


Fig. 3. (a) and (b) show concurrences between normal lattices as a function of B for different values of impurity parameters J_1, when $J = -1$ for simplicity. (c) and (d) show concurrences between normal lattices as a function of constant J_1 for different values of external magnetic fields, when $J = -1$ for simplicity. (a), (c) $T = 0.1$, (b), (d) $T = 2$.

Moreover, when the temperature is high, we can obtain an ideal switch only in the case that the parameter is relatively larger, as shown in Fig. 3(b). In addition, from Figs. 2 and 3(a) and 3(b), it can be seen that the value of the concurrence decreases monotonically with increasing T.

Figures 3(c) and 3(d) show the concurrence as a function of J_1 at a certain temperature when external magnetic fields are fixed and $J = -1$ for simplicity. From Fig. 2, we also find that the entanglement switch can be realised by modulating impurity parameters J_1, as shown in Figs. 3(c) and 3(d). A comparison among curves in Figs. 3(c) and 3(d) shows that the entanglement switch realised by modulating impurity parameters J_1 is different from that realised by modulating external magnetic fields. The former realises conversion from zero to a fixed value, the latter realises conversion from zero to a fixed value then to zero. Because the realization of the continuous change of impurity parameters is more difficult, the entanglement switch realised by modulating magnetic field is more practical.

3.2.2. The antiferromagnetic ($J > 0$) case

By considering the numerical results when J is chosen to be 1 for simplicity, the results are similar to those in the ferromagnetic case, as shown in Fig. 4. It shows that a similar analysis can be carried out about the entanglement switch which can also be realised by modulating magnetic field or impurity parameters, and the figures about the entanglement switch are very similar to Fig. 3.

It is interesting to find that the entanglement peak is higher at $T = 0.1$ in Fig. 4(a) than that at $T = 2$ in Fig. 4(b) for the antiferromagnetic ($J > 0$) case. A similar analysis can be carried out for the ferromagnetic ($J < 0$) case in Fig. 2. In addition, when $T \to 0$, $J_1 \to 0$ and $B \to 0$, the values of the peak of the maximal entanglement in Fig. 4(a) and Fig. 2(a) all reach 1. This is similar to the result in Ref. [18]. The concurrence tends to approach a fixed value when B and J_1 increase, unlike the result in Ref. [18] in which the concurrence tends to zero without the effect of external magnetic field.

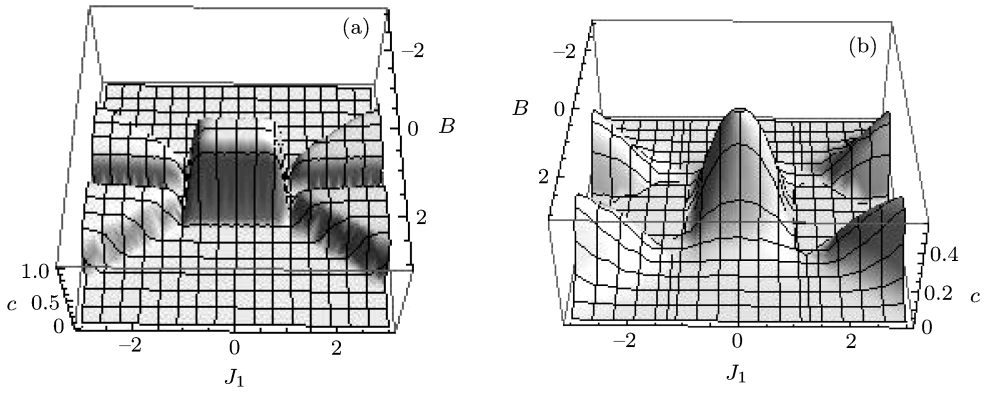


Fig. 4. The concurrence between normal lattices as a function of B and J_1 at certain temperature when J is chosen to be 1 for simplicity. (a) $T = 0.1$, (b) $T = 0.5$.

3.3. Entanglement between the normal lattice and the impurity lattice

3.3.1. The ferromagnetic ($J < 0$) case

In this case, J is chosen to be -1 for simplicity. Figure 5 shows the nearest-neighbouring concurrence between the normal lattice 3 and the impurity lattice 1 as a function of B and J_1 at fixed temperatures. A similar behaviour can be seen in Fig. 2, only the entanglement peak does not appear when $T \to 0$, $J_1 \to 0$.

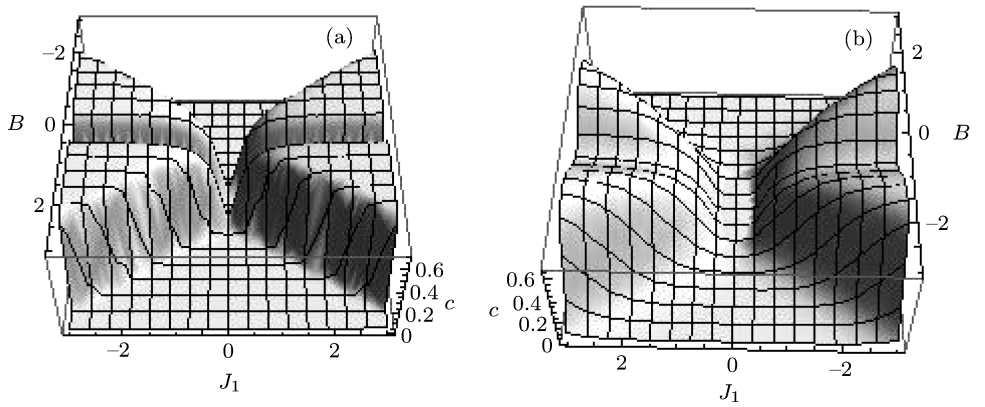


Fig. 5. The concurrence between the normal lattice and the impurity lattice as a function of B and J_1 at certain temperatures when J is chosen to be -1 for simplicity. (a) $T = 0.1$, (b) $T = 0.5$.

From Eqs. (14), (15) and (16), we obtain the concurrence between the normal lattice 3 and the impurity lattice 1, and its first derivative is given by

$$\begin{aligned} f'(B) = {} & \mathrm{e}^{(1+B)/T}(-1+\mathrm{e}^{2B/T})\{\mathrm{e}^{2B/T}[3\mathrm{e}^{1/T}+\mathrm{e}^{2B/T}+(3+a+b)\mathrm{e}^{(1+2B)/T}+3\mathrm{e}^{(1+4B)/T}][c-4\mathrm{e}^{-B/T}\,|g|] \\ & -\frac{4d}{cD}\mathrm{e}^{-1/T}(1+\mathrm{e}^{2B/T})[(1+\mathrm{e}^{2B/T})(a-b+aD+bD)\mathrm{e}^{1/T} \\ & +2D(1+\mathrm{e}^{2B/T})+(a-b)cD\mathrm{e}^{(1+B)/T}|-g\mathrm{e}^{-B/T}|]\}/2d^2(1+\mathrm{e}^{2B/T})^2T, \end{aligned} \tag{17}$$

where

$$a=\mathrm{e}^{(1+D)/2T}, \quad b=\mathrm{e}^{(1-D)/2T},$$

$$c=\frac{1}{D}\sqrt{\mathrm{e}^{(-2(B+1))/T}[(-b+bD+a+aD)\mathrm{e}^{(1+2B)/T}+2D(2\mathrm{e}^{1/T}+\mathrm{e}^{2B/T})][(a-b+aD+bD)\mathrm{e}^{1/T}+2D+4D\mathrm{e}^{(1+2B)/T}]},$$

$$d=\mathrm{e}^{1/T}+\mathrm{e}^{2B/T}+(-1+a+b)\mathrm{e}^{(1+2B)/T}+\mathrm{e}^{(1+4B)/T}, \quad g=\frac{(a-b)(1+\mathrm{e}^{2B/T})J_1}{D}, \quad D=\sqrt{8J_1^2+1}.$$

From Eq. (17), when $B=0$, the value of the first derivative of concurrence is zero $f'(B)=0$, this indicates that there is a critical point at which there exhibits a sudden change, as shown in Figs. 6(a) and 6(b). This means that the external magnetic field is not used as entanglement switch. Figures 6(c) and 6(d) shows the concurrence between the normal lattice and the impurity lattice as a function of constant J_1 at different values of external magnetic fields for different distributions, when temperature is fixed and J is chosen to be -1 for simplicity. It shows the entanglement switch can also be realised by modulating impurity parameters J_1.

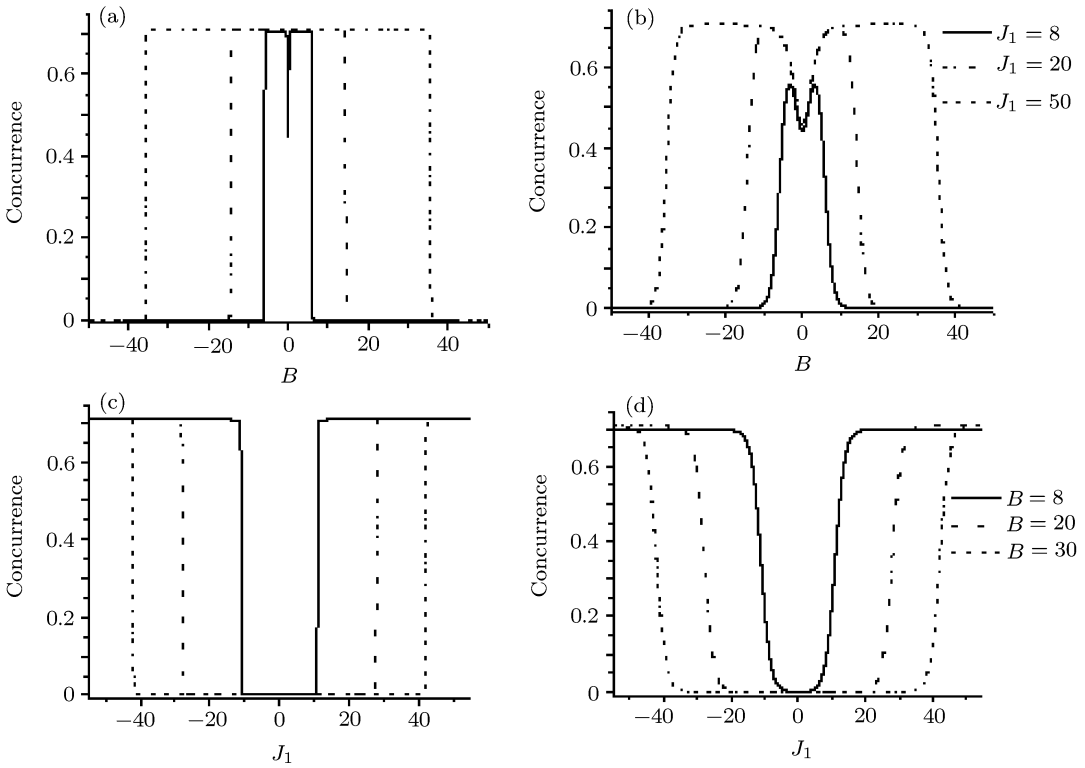


Fig. 6. The concurrence between the normal lattice and the impurity lattice as a function of B (or J_1) for various values of J_1 (or B), when $J=-1$ for simplicity. (a) $T=0.1$, (b) $T=2$, (c) $T=0.1$, (d) $T=2$.

3.3.2. The antiferromagnetic ($J>0$) case

The behaviour in this case is similar to that in the ferromagnetic ($J<0$) case, the values of entanglement in platform region is approximately 0.7 because of the effect of impurity, as shown in Figs. 5–7; it is higher than that between the normal lattice in platform region, which is approximately 0.5 (Figs. 2–4).

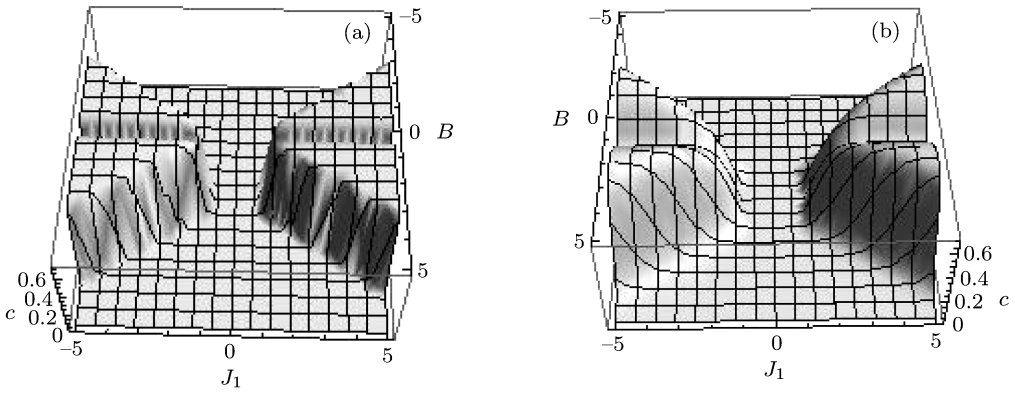


Fig. 7. The concurrence between the normal lattice and the impurity lattice as a function of B and J_1 at certain temperatures when J is chosen to be 1 for simplicity. (a) $T = 0.1$, (b) $T = 0.5$.

4. Conclusion

In this paper, we obtain the analytical expression of the QPT and concurrence in the three-qubit Heisenberg XX chain with impurity in the presence of the external magnetic field. From the above discussion, we can see that there are quantum phase transitions in the three-qubit Heisenberg XX chain, with corresponding relations between the entanglement and the QPT. In the condition of $J_1 = \beta J$, the phase transition occurs at the points $B/J = -0.5, 0, 0.5$ when $0 < \beta \leq 1$, but when $\beta > 1$, except the point $B/J = 0$, the other two points shift to the direction of the higher value of $|B/J|$. Therefore, we can modulate the phase transition by controlling the parameter β. No matter what is the concurrence between the normal lattices or between the normal lattice and the impurity lattice, the higher the temperature is, the smaller the value of the concurrence is. Moreover, due to the effect of impurity and external magnetic field, the values of concurrence is approximately fixed with increasing B and J_1. For the normal lattices, the entanglement switch can be realised by modulating external magnetic field and impurity parameter, but for the impurity lattices, the external magnetic field is not used as entanglement switch, for the reason that the concurrence has a sudden change when $B = 0$. Moreover, the lower the temperature is, the more quickly the sudden change varies.

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