# Unifying the theory of integration within normal-, Weyl- and antinormal-ordering of operators and the *s*-ordered operator expansion formula of density operators*

Fan Hong-Yi(范洪义)†

*Department of Physics, Shanghai Jiao Tong University, Shanghai* 200030, *China*

(Received 18 September 2009; revised manuscript received 10 October 2009)

By introducing the *s*-parameterized generalized Wigner operator into phase-space quantum mechanics we invent the technique of integration within *s*-ordered product of operators (which considers normally ordered, antinormally ordered and Weyl ordered product of operators as its special cases). The *s*-ordered operator expansion (denoted by $\circledS\cdots\circledS$) formula of density operators is derived, which is

$$\rho=\frac{2}{1-s}\int\frac{\mathrm{d}^2\beta}{\pi}\left\langle-\beta\right|\rho\left|\beta\right\rangle\circledS\exp\left\{\frac{2}{s-1}\left(s|\beta|^2-\beta^*a+\beta a^\dagger-a^\dagger a\right)\right\}\circledS.$$

The *s*-parameterized quantization scheme is thus completely established.

**Keywords:** *s*-parameterized generalized Wigner operator, technique of integration within *s*-ordered product of operators, *s*-ordered operator expansion formula, *s*-parameterized quantization scheme

**PACC:** 0365, 0530, 4250

## 1. Introduction

The subject about operators and their classical correspondence has been a hot topic since the birth of quantum mechanics (QM) and now becomes a field named QM in phase space. Because Heisenberg's uncertainty principle prohibits the notion of a system being described by a point in phase space, only domains of minimum area $2\pi\hbar$ in phase space is allowed. Wigner[1] introduced a function whose marginal distribution gives the probability of a particle in coordinate space or in momentum space. The Wigner distribution is related to operators' Weyl ordering (or Weyl quantization scheme).[2] We notice that each phase space distribution is associated with a definite operator ordering for quantizing classical functions. For example, P-representation (as a density operator $\rho$'s classical correspondence) is actually $\rho$'s antinormally ordered expansion in terms of the completeness of coherent state $|z\rangle=\exp[-|z|^2/2+za^\dagger]\,|0\rangle$,[3,4]

$$\rho=\int\frac{\mathrm{d}^2z}{\pi}P\left(z\right)\left|z\right\rangle\left\langle z\right|, \tag{1}$$

because the coherent states compose a complete set $\int(\mathrm{d}^2z)/\pi\,|z\rangle\,\langle z|=1$.[5] The Wigner distribution function $W\left(p,x\right)$ of $\rho$, defined as $\mathrm{Tr}\left[\rho\Delta\left(p,x\right)\right]$, is proportional to the classical Weyl correspondence $h\left(p,x\right)$ of $\rho$ ($\rho$'s Weyl ordered expansion), i.e.,

$$\rho=\iint\mathrm{d}p\mathrm{d}x\Delta\left(p,x\right)h\left(p,x\right) \tag{2}$$

and

$$\mathrm{Tr}\left[\rho\Delta\left(p,x\right)\right]=\left(2\pi\right)^{-1}h\left(p,x\right)=W\left(p,x\right), \tag{3}$$

since the Wigner operator $\Delta\left(p,x\right)$ is complete too, $\iint\mathrm{d}p\mathrm{d}x\Delta\left(p,x\right)=1$. The original form of $\Delta\left(p,x\right)$ defined in the coordinate representation is[6]

$$\Delta\left(x,p\right)=\int\frac{\mathrm{d}u}{2\pi}\mathrm{e}^{\mathrm{i}up}\left|x+\frac{u}{2}\right\rangle\left\langle x-\frac{u}{2}\right| \tag{4}$$

for the Wigner operator in the entangled state representation refer to Ref. [7]. When $\rho=(1/2)^m\sum\limits_{l=0}^{m}\binom{m}{l}X^{m-l}P^nX^l$, $[X,P]=i$, $\hbar=1$, according to Eqs. (3) and (4), its classical correspondence

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10775097 and 10874174).

†Corresponding author. E-mail: fhym@sjtu.edu.cn

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is

$$\begin{aligned}
&2\pi \operatorname{Tr}\left[\left(\frac{1}{2}\right)^m \sum_{l=0}^{m}\binom{m}{l} X^{m-l}P^n X^l \Delta(x,p)\right] \\
&= \int \mathrm{d}u \mathrm{e}^{\mathrm{i}\,pu}\left\langle x-\frac{u}{2}\right|\left(\frac{1}{2}\right)^m \\
&\quad \times \sum_{l=0}^{m}\frac{m!}{l!\,(m-l)!} X^{m-l}P^n X^l \left|x+\frac{u}{2}\right\rangle \\
&= x^m \int \mathrm{d}u \mathrm{e}^{\mathrm{i}\,pu}\left\langle x-\frac{u}{2}\right| P^n \left|x+\frac{u}{2}\right\rangle \\
&= x^m \int \mathrm{d}u \mathrm{e}^{\mathrm{i}\,pu}\int \mathrm{d}p' \mathrm{e}^{-\mathrm{i}\,p'u} p'^n \\
&= x^m \int \mathrm{d}p' \delta(p-p')\, p'^n = x^m p^n, \qquad (5)
\end{aligned}$$

which is the original definition of Weyl quantization scheme (quantizing classical coordinate and momentum quantity $x^m p^n$ as the corresponding operators) expressed as[2]

$$x^m p^n \to \left(\frac{1}{2}\right)^m \sum_{l=0}^{m}\binom{m}{l} X^{m-l}P^n X^l, \qquad (6)$$

its right-hand side is in Weyl ordering, so we introduce the symbol $\vdots\ \ \vdots$ to characterize it,[8] i.e.,

$$\begin{aligned}
&\left(\frac{1}{2}\right)^m \sum_{l=0}^{m}\binom{m}{l} X^{m-l}P^n X^l \\
&= \vdots \left(\frac{1}{2}\right)^m \sum_{l=0}^{m}\binom{m}{l} X^{m-l}P^n X^l \vdots. \qquad (7)
\end{aligned}$$

It is worth emphasizing that the order of operators $X$ and $P$ is permuted within the Weyl ordering symbol,[8] a useful property which has been overlooked for a long time. Based on this fact a useful method called integration within Weyl ordered product of operators has been invented.[8]

Therefore, from Eqs. (6) and (7)

$$\begin{aligned}
x^m p^n &\to \vdots \left(\frac{1}{2}\right)^m \sum_{l=0}^{m}\binom{m}{l} X^{m-l}P^n X^l \vdots \\
&= \vdots X^m P^n \vdots. \qquad (8)
\end{aligned}$$

Following Eq. (2) we have

$$\vdots X^m P^n \vdots = \iint \mathrm{d}p\mathrm{d}x \Delta(x,p)\, x^m p^n, \qquad (9)$$

which implies $\Delta(x,p) = \vdots \delta(x-X)\,\delta(p-P) \vdots$, or $\Delta(\alpha) = \frac{1}{2}\vdots \delta\left(\alpha^* - a^\dagger\right)\delta(\alpha - a) \vdots$, $\alpha = (x+\mathrm{i}\,p)/\sqrt{2}$, a delta operator-function form in Weyl ordering.

Having realized that each phase space distribution accompanies a definite operator ordering for quantizing classical functions, we may think that each complete set of operators corresponds to an operator-ordering rule. In this work we shall introduce a complete set of operators characteristic of an $s$-parameter (the generalized Wigner operator) and then introduce a generalized quantization scheme with the $s$-parameter operator ordering. Historically, Cahill and Glauber[9] have introduced the $s$-parameterized quasiprobability distribution according to which the coherent state expectation of $\rho$, the Wigner function of $\rho$, and the $P$-representation of $\rho$ respectively corresponds to three distinct values of $s$, i.e., $s = 1, 0, -1$. However, the $s$-parameterized quantization scheme associated with the $s$-parameterized quasiprobability distribution has not been completely established, as the fundamental problem of what is $\rho$'s $s$-ordered operator expansion has not been touched yet. In other words, the problem of how to arrange any given operator in its $s$-ordered form has been unsolved, say for instance, no references has ever reported what is the $s$-ordered operator expansion of $\exp\left(\lambda a^\dagger a\right)$? $\left(\left[a, a^\dagger\right] = 1\right)$. In this work we shall solve this important problem by introducing the technique of integration within $s$-ordering of operators, which in the cases of $s = 1, 0, -1$, respectively goes to the technique of integration within normal-ordering, Weyl ordering and antinormal ordering of operators. In this way we can tackle these three techniques in a unified way. The paper is arranged as follows: In Section 2 we introduce the explicit $s$-parameterized Wigner operator $\Delta_s(\alpha)$ and then in Section 3 we establish one-to-one mapping between operators and their $s$-parameterized classical correspondence after proving the relation $2\pi \operatorname{Tr}\left[\Delta_{-s}(\alpha'^*, \alpha')\,\Delta_s(\alpha^*, \alpha)\right] = \delta(x'-x)(p'-p)$, where $\alpha = (x+\mathrm{i}\,p)/\sqrt{2}$. In Section 4 we introduce the symbol $\circledS \cdots \circledS$ denoting $s$-ordering of operators and the technique of integration within $s$-ordered product of operators. In Sections 5 and 6 we derive density operator's expansion formula in terms of $s$-ordered quantization scheme, such that the $s$-ordered expansion of $\exp\left(\lambda a^\dagger a\right)$ is obtained. In this way we develop and enrich the theory of phase space quantum mechanics.

## 2. The $s$-parameterized Wigner operator and quantization scheme

Our aim is to construct the $s$-parameterized quantization scheme, in other words, we want to construct a one-to-one correspondence between an operator and its classical correspondence in the sense of $s$-parameterized quasiprobability distribution. For this purpose we should introduce a generalised Wigner operator for the $s$-parameterized phase space theory. By analogy with the usual Wigner operator[6] we introduce a generalised Wigner operator for $s$-parameterized distributions,

$$\Delta_s(\alpha)=\int\frac{\mathrm{d}^2\beta}{2\pi^2}\exp\left(\frac{s|\beta|^2}{2}+\beta a^\dagger-\beta^* a-\beta\alpha^*+\beta^*\alpha\right). \tag{10}$$

Using the Baker–Hausdorff formula to put the exponential in normally ordered form, and using the technique of integration within normal product of operators,[10,11] for $s<1$, we obtain

$$\begin{aligned}\Delta_s(\alpha)&=\int\frac{\mathrm{d}^2\beta}{2\pi^2}{:}\exp\left[\frac{-(1-s)|\beta|^2}{2}+\beta a^\dagger-\beta^* a-\beta\alpha^*+\beta^*\alpha\right]{:}\\&=\frac{1}{(1-s)\pi}{:}\exp\left[\frac{-2}{1-s}\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]{:},\end{aligned} \tag{11}$$

which is named $s$-parameterized Wigner operator. In particular, when $s=0$, Eq. (11) reduces to the usual normally ordered Wigner operator[12]

$$\begin{aligned}\Delta_s(\alpha)\to\Delta(\alpha)&=\frac{1}{\pi}{:}\exp\left[-2\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]{:}\\&=\frac{1}{\pi}{:}\exp\left[-(x-X)^2-(p-P)^2\right]{:},\end{aligned} \tag{12}$$

where $X=\dfrac{a^\dagger+a}{\sqrt{2}}$, $P=\dfrac{\mathrm{i}\left(a^\dagger-a\right)}{\sqrt{2}}$. On the other hand, by putting the exponential in Eq. (10) within antinormal ordering symbol $\vdots\,\vdots$, we have, for $s<-1$,

$$\begin{aligned}\Delta_s(\alpha)&=\int\frac{\mathrm{d}^2\beta}{2\pi^2}\vdots\exp\left[\frac{-(-1-s)|\beta|^2}{2}+\beta a^\dagger-\beta^* a-\beta\alpha^*+\beta^*\alpha\right]\vdots\\&=\frac{1}{(-1-s)\pi}\vdots\exp\left[\frac{2}{1+s}\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]\vdots.\end{aligned} \tag{13}$$

In reference to the asymptotic expression of Delta function $\delta(x)=\lim_{\epsilon\to0}\dfrac{1}{\sqrt{\pi\epsilon}}\mathrm{e}^{-x^2/\epsilon}$, we have, for $s=-1$,

$$\begin{aligned}\Delta_{s=-1}(\alpha)&=\vdots\delta\left(a^\dagger-\alpha^*\right)\delta(a-\alpha)\vdots=\delta(a-\alpha)\,\delta\left(a^\dagger-\alpha^*\right)\\&=|\alpha\rangle\langle\alpha|,\end{aligned} \tag{14}$$

which is the pure coherent state, $|\alpha\rangle=\exp\left(-\frac{1}{2}|\alpha|^2+\alpha a^\dagger\right)|0\rangle$. Using

$${:}\exp\left[\left(\mathrm{e}^\lambda-1\right)a^\dagger a\right]{:}=\mathrm{e}^{\lambda a^\dagger a}, \tag{15}$$

we can convert Eq. (11) to the form

$$\begin{aligned}\Delta_s(\alpha)&=\frac{1}{(1-s)\pi}\mathrm{e}^{\frac{2}{1-s}\alpha a^\dagger}{:}\exp\left[\left(\frac{s+1}{s-1}-1\right)a^\dagger a\right]{:}\,\mathrm{e}^{\frac{2}{1-s}\alpha^* a-\frac{2}{1-s}|\alpha|^2}\\&=\frac{1}{(1-s)\pi}\mathrm{e}^{\frac{2}{1-s}\alpha a^\dagger}\mathrm{e}^{a^\dagger a\ln\frac{s+1}{s-1}}\mathrm{e}^{\frac{2}{1-s}\alpha^* a-\frac{2}{1-s}|\alpha|^2}.\end{aligned} \tag{16}$$

## 3. The $s$-parameterized quantization scheme

It follows from Eq. (11) that

$$2\int\mathrm{d}^2\alpha\Delta_s(\alpha)=\frac{2}{(1-s)\pi}\int\mathrm{d}^2\alpha{:}\exp\left[\frac{-2}{1-s}\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]{:}=1, \tag{17}$$

so $\Delta_s(\alpha)$ is complete and $\rho$ can be expanded as

$$\rho = 2\int \mathrm{d}^2\alpha \Delta_s(\alpha)\mathfrak{P}(\alpha), \tag{18}$$

which is a new classic-quantum mechanical correspondence between $\mathfrak{P}(\alpha)$ and $\rho$; when $s=0$, Eq. (18) yields the Weyl correspondence. By noting the form of $\Delta_{-s}(\alpha')$ and using $\int \frac{\mathrm{d}^2 z}{\pi}|z\rangle\langle z| = 1$, we calculate

$$\begin{aligned}
\mathrm{Tr}\left[\Delta_{-s}(\alpha')\Delta_s(\alpha)\right] &= G\int \frac{\mathrm{d}^2 z}{\pi}\langle z|\, \mathrm{e}^{\frac{2}{1+s}\alpha' a^\dagger}\mathrm{e}^{a^\dagger a \ln\frac{s-1}{s+1}}\mathrm{e}^{\frac{2}{1+s}\alpha'^* a}\mathrm{e}^{\frac{2}{1-s}\alpha a^\dagger}\mathrm{e}^{a^\dagger a\ln\frac{s+1}{s-1}}\mathrm{e}^{\frac{2}{1-s}\alpha^* a}|z\rangle \\
&= G\int \frac{\mathrm{d}^2 z}{\pi}\langle z|\, \mathrm{e}^{\frac{2}{1+s}\alpha' a^\dagger}\mathrm{e}^{\frac{-2}{1-s}\alpha'^* a}\mathrm{e}^{\frac{-2}{1+s}\alpha a^\dagger}\mathrm{e}^{\frac{2}{1-s}\alpha^* a}|z\rangle \\
&= G\mathrm{e}^{\frac{4\alpha'^*\alpha}{(1+s)(1-s)}}\int \frac{\mathrm{d}^2 z}{\pi}\exp\left[\frac{2z^*}{1+s}(\alpha'-\alpha) - \frac{2z}{1-s}(\alpha'^*-\alpha^*)\right] \\
&= \frac{1}{4\pi}\delta(\alpha'-\alpha)(\alpha'^*-\alpha^*)\,\mathrm{e}^{-\left(\frac{2}{1-s}+\frac{2}{1+s}\right)|\alpha|^2 - \frac{4\alpha'^*\alpha}{(1+s)(s-1)}} \\
&= \frac{1}{4\pi}\delta(\alpha'-\alpha)(\alpha'^*-\alpha^*) \\
&= \frac{1}{2\pi}\delta(q'-q)(p'-p),
\end{aligned} \tag{19}$$

where $G \equiv \dfrac{\mathrm{e}^{-\frac{2}{1-s}|\alpha|^2 - \frac{2}{1+s}|\alpha'|^2}}{(1+s)(1-s)\pi^2}$. Therefore, the classical function corresponding to $\rho$ (in the context of the $s$-parameterized quantization scheme) is given by

$$\begin{aligned}
2\pi\,\mathrm{Tr}\left[\Delta_{-s}(\alpha)\rho\right] &= 4\pi\int \mathrm{d}^2\alpha'\,\mathrm{Tr}\left[\Delta_{-s}(\alpha)\Delta_s(\alpha')\right]\mathfrak{P}(\alpha', s) \\
&= \int \mathrm{d}^2\alpha'\delta(\alpha-\alpha')(\alpha^*-\alpha'^*)\mathfrak{P}(\alpha', s) \\
&= \mathfrak{P}(\alpha, s).
\end{aligned} \tag{20}$$

Equation (20) is the reciprocal relation of Eq. (18). Thus we have established one-to-one mapping between operators and their $s$-parameterized classical correspondence. The $s$-parameterized quantization scheme is completed, of which the Weyl quantization is its special case.

# 4. Expansion formula of $|z\rangle\langle z|$ in terms of $s$-parameterized quantization scheme

When $\rho = |z\rangle\langle z|$, using Eq. (20) we have

$$\begin{aligned}
2\pi\,\mathrm{Tr}\left[\Delta_{-s}(\alpha)|z\rangle\langle z|\right] &= \frac{2}{1+s}\langle z|:\exp\left[\frac{-2}{1+s}\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]:|z\rangle \\
&= \frac{2}{1+s}\exp\left[\frac{-2}{1+s}(z^*-\alpha^*)(z-\alpha)\right],
\end{aligned} \tag{21}$$

which is the $s$-parameterized classical correspondence of $|z\rangle\langle z|$ in phase space. Equation (21) represents a kind of phase space distribution, since the integration over the phase space leads to the completeness

$$\int \frac{\mathrm{d}^2 z}{\pi}|z\rangle\langle z| \to \frac{2}{1+s}\int \frac{\mathrm{d}^2 z}{\pi}\exp\left[\frac{-2}{1+s}(z^*-\alpha^*)(z-\alpha)\right] = 1. \tag{22}$$

For this $s$-parameterized distribution we can define the $s$-ordered form of $|z\rangle\langle z|$ through the following formula

$$|z\rangle\langle z| = \frac{2}{1+s}\text{Ⓢ}\exp\left[\frac{-2}{1+s}\left(z^*-a^\dagger\right)(z-a)\right]\text{Ⓢ}, \tag{23}$$

where Ⓢ$\cdots$Ⓢ is the $s$-ordering symbol. This definition is consistent with those well-known ordered formulas of $|z\rangle\langle z|$. Indeed, when in Eq. (23) $s=0$, Ⓢ$\cdots$Ⓢ converts to Weyl ordering $\vdots\cdots\vdots$, so Eq. (23) reduces to

$$|z\rangle\langle z| = 2\vdots \exp\left[-2\left(z^*-a^\dagger\right)(z-a)\right]\vdots, \tag{24}$$

as expected;[7] when in Eq. (23) $s=1$, Ⓢ$\cdots$Ⓢ becomes normal ordering,[10]

$$|z\rangle\langle z| =: \exp\left[-\left(z^*-a^\dagger\right)(z-a)\right]:, \tag{25}$$

which is as expected too; when $s=-1$, Ⓢ$\cdots$Ⓢ becomes antinormal ordering,

$$\begin{aligned}|z\rangle\langle z| &= \lim_{\epsilon\to 0}\frac{2}{\epsilon}\vdots\exp\left[\frac{2}{\epsilon}\left(z^*-a^\dagger\right)(z-a)\right]\vdots\\ &= \vdots\delta\left(z^*-a^\dagger\right)\delta(z-a)\vdots\end{aligned} \tag{26}$$

still is as expected.

## 5. The technique of integration within $s$-ordered product of operators

Let us introduce the technique of integration within $s$-ordered product of operators (IWSOP) by listing some properties of the $s$-ordered product of operators which are defined through Eq. (23):

1) The order of Boson operators $a$ and $a^\dagger$ within an $s$-ordered symbol can be permuted, even though $\left[a,a^\dagger\right]=1$.

2) $c$-numbers can be taken out of the symbol Ⓢ$\cdots$Ⓢ as one wishes.

3) An $s$-ordered product of operators can be integrated or differentiated with respect to a $c$-number provided that the integration is convergent.

4) The vacuum projection operator $|0\rangle\langle 0|$ has the $s$-ordered product form (see Eq. (23))

$$|0\rangle\langle 0| = \frac{2}{1+s}\text{Ⓢ}\exp\left(\frac{-2}{1+s}a^\dagger a\right)\text{Ⓢ}. \tag{27}$$

5) the symbol Ⓢ$\cdots$Ⓢ becomes : : for $s=1$, $\vdots\vdots$ for $s=0$, and $\vdots\vdots$ for $s=-1$.

## 6. Density operator's expansion formula in terms of $s$-ordered quantization scheme

Using Eqs. (1) and (23), we have the expansion within Ⓢ$\cdots$Ⓢ,

$$\rho = \int\frac{\mathrm{d}^2z}{\pi}P(z)|z\rangle\langle z| = \frac{2}{1+s}\int\frac{\mathrm{d}^2z}{\pi}P(z)\,\text{Ⓢ}\exp\left[\frac{-2}{1+s}\left(z^*-a^\dagger\right)(z-a)\right]\text{Ⓢ}. \tag{28}$$

Substituting Mehta's expression $P(z)$[13] in Eq. (28)

$$P(z) = \mathrm{e}^{|z|^2}\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|\rho|\beta\rangle\,\mathrm{e}^{|\beta|^2+\beta^*z-\beta z^*}, \tag{29}$$

where $|\beta\rangle$ is also a coherent state, $\langle-\beta|\,\beta\rangle=\mathrm{e}^{-2|\beta|^2}$, we have

$$\begin{aligned}\rho &= \frac{2}{1+s}\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|\rho|\beta\rangle\,\mathrm{e}^{|\beta|^2}\int\frac{\mathrm{d}^2z}{\pi}\text{Ⓢ}\exp\left[|z|^2+\beta^*z-\beta z^*-\frac{2}{1+s}\left(z^*-a^\dagger\right)(z-a)\right]\text{Ⓢ}\\ &= \frac{2}{1-s}\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|\rho|\beta\rangle\,\text{Ⓢ}\exp\left[\frac{2}{s-1}\left(s|\beta|^2-\beta^*a+\beta a^\dagger-a^\dagger a\right)\right]\text{Ⓢ},\end{aligned} \tag{30}$$

which is the density operator's expansion formula in terms of $s$-ordered quantization scheme. In particular, for $s=0$, Eq. (30) becomes

$$\rho = 2\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|\rho|\beta\rangle\vdots\exp\left[2\left(\beta^*a-\beta a^\dagger+a^\dagger a\right)\right]\vdots, \tag{31}$$

which is the formula for converting $\rho$ into its Weyl-ordered form;[7,8] while for $s=-1$, Eq. (30) becomes

$$\rho = 2\int \frac{\mathrm{d}^2\beta}{\pi}\left\langle -\beta\right|\rho\left|\beta\right\rangle \vdots \exp\left[-\left(|\beta|^2+\beta^* a-\beta a^\dagger + a^\dagger a\right)\right]\vdots, \tag{32}$$

which is the formula converting $\rho$ into its antinormally ordered form,[14] as expected.

## 7. Application

We now use Eq. (30) to derive the $s$-ordered expansion of $\mathrm{e}^{\lambda a^\dagger a}$. From Eq. (15) and the IWSOP technique we have

$$\begin{aligned}
\mathrm{e}^{\lambda a^\dagger a} &= \frac{2}{1-s}\int \frac{\mathrm{d}^2\beta}{\pi}\left\langle -\beta\right|\exp\left[\left(1-\mathrm{e}^{\lambda}\right)|\beta|^2\right]\left|\beta\right\rangle \circledS \exp\left\{\frac{2}{s-1}\left(s|\beta|^2-\beta^* a+\beta a^\dagger - a^\dagger a\right)\right\}\circledS \\
&= \frac{2}{1-s}\int \frac{\mathrm{d}^2\beta}{\pi}\circledS \exp\left[\left(-1-\mathrm{e}^{\lambda}\right)|\beta|^2+\frac{2}{s-1}\left(s|\beta|^2-\beta^* a+\beta a^\dagger - a^\dagger a\right)\right]\circledS \\
&= \frac{2}{1+s-s\mathrm{e}^{\lambda}+\mathrm{e}^{\lambda}}\circledS \exp\left[\frac{2\left(\mathrm{e}^{\lambda}-1\right)}{1+s-s\mathrm{e}^{\lambda}+\mathrm{e}^{\lambda}}a^\dagger a\right]\circledS,
\end{aligned} \tag{33}$$

which is a new formula. For $s=1$, $\circledS\cdots\circledS \to :\ :$, Eq. (33) reduces to Eq. (15) as expected; for $s=0$, $\circledS\cdots\circledS \to \vdots\,\vdots$, Eq. (33) becomes the Weyl ordering expansion,[7]

$$\mathrm{e}^{\lambda a^\dagger a} = \frac{2}{1+\mathrm{e}^{\lambda}}\vdots \exp\left[\frac{2\mathrm{e}^{\lambda}-2}{1+\mathrm{e}^{\lambda}}aa^\dagger\right]\vdots; \tag{34}$$

and for $s=-1$, $\circledS\cdots\circledS \to \vdots\,\vdots$, Eq. (33) becomes[10]

$$\mathrm{e}^{\lambda a^\dagger a} = \mathrm{e}^{-\lambda}\vdots \exp\left[\left(1-\mathrm{e}^{-\lambda}\right)aa^\dagger\right]\vdots, \tag{35}$$

which is also correct. Furthermore, we consider the $s$-ordered expansion of the generalized Wigner operator itself. Using Eqs. (10) and (30), we have

$$\begin{aligned}
\Delta_s\left(\alpha^*,\alpha\right) &= \frac{2}{\left(1-s\right)^2\pi}\int \frac{\mathrm{d}^2\beta}{\pi}\left\langle -\beta\right|:\exp\left[\frac{-2}{1-s}\left(a^\dagger-\alpha^*\right)\left(a-\alpha\right)\right]:\left|\beta\right\rangle \\
&\quad \times\circledS\exp\left[\frac{2}{s-1}\left(s|\beta|^2-\beta^* a+\beta a^\dagger - a^\dagger a\right)\right]\circledS \\
&= \frac{2}{\left(1-s\right)^2\pi}\int \frac{\mathrm{d}^2\beta}{\pi}\circledS \mathrm{e}^{-2|\beta|^2}\exp\left\{\frac{2}{s-1}\left[\left(-\beta^*-\alpha^*\right)\left(\beta-\alpha\right)\right.\right. \\
&\quad \left.\left.+s|\beta|^2-\beta^* a+\beta a^\dagger - a^\dagger a\right]\right\}\circledS \\
&= \frac{2}{\left(1-s\right)^2}\circledS\delta\left[\frac{2}{s-1}\left(a^\dagger-\alpha^*\right)\right]\delta\left[\frac{2}{s-1}\left(a-\alpha\right)\right]\circledS,
\end{aligned} \tag{36}$$

which, in the case of $s=0$, becomes the Weyl-ordered form of the usual Wigner operator

$$\Delta\left(\alpha\right) = \frac{1}{2}\vdots\delta\left(\alpha^*-a^\dagger\right)\delta\left(\alpha-a\right)\vdots.$$

In summary, by introducing the $s$-parameterized generalized Wigner operator into phase-space quantum mechanics, we have proposed the technique of integration within the $s$-ordered product of operators (which considers the normally ordered, antinormally ordered and Weyl-ordered product of operators as its special cases). The $s$-ordered operator expansion (denoted by $\circledS\cdots\circledS$) formula of density operators is derived. The theory of integration within normal-, Weyl- and antinormal-ordering of operators can now be tackled in a unified way. The $s$-parameterized quantization scheme is completely established, of which the Weyl quantization is its special case. For the mutual transformation between the Weyl-ordering and $X-P$ (or $P-X$) ordering of operators we refer to Ref. [15]. For other possible methods to perform integration over ket-bra operator we refer to Ref. [16].

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