Form invariance and new conserved quantity of generalised Birkhoffian system*

Mei Feng-Xiang(梅凤翔) and Wu Hui-Bin(吴惠彬)†

Faculty of Science, Beijing Institute of Technology, Beijing 100081, *China*

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A form invariance and a conserved quantity of the generalised Birkhoffian system are studied. First, a definition and a criterion of the form invariance are given. Secondly, through the form invariance, a new conserved quantity can be deduced. Finally, an example is given to illustrate the application of the result.

Keywords: generalised Birkhoffian system, form invariance, conserved quantity

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1. Introduction

In 1983, Santilli pointed out: "Birkhoffian Mechanics is the most general possible mechanics that can be constructed from Hamiltonian Mechanics via the transformation theory."[1] Up to now, great progress has been achieved on its research.[1−15] Reference [4], when discussing the generalised quasi-symmetry transformation of Birkhoffian system, added a term to the Birkhoff equations, and called it generalised Birkhoff equations. A mechanical system whose motion is described by the generalised Birkhoff equations or a physical system whose state is described by the generalised Birkhoff equations is called a generalised Birkhoffian system. Reference [16] presented the fundamental theoretical framework for the dynamics of generalised Birkhoffian system. Some results have been obtained on the research of generalised Birkhoffian system.[16−20] In this study, for generalised Birkhoffian system, the form invariance and the new conserved quantity deduced by it directly are discussed.

2. Form invariance of generalised Birkhoffian system

The generalised Birkhoff equations have the form[4,16]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\Lambda_\mu=0,\quad(\mu,\nu=1,2,\ldots,2n),\tag{1}$$

where $B=B(t,\boldsymbol{a})$ is the Birkhoffian, $R_\mu=R_\mu(t,\boldsymbol{a})$ are Birkhoff's functions, and $\Lambda_\mu=\Lambda_\mu(t,\boldsymbol{a})$ are the added terms. When $\Lambda_\mu=0\quad(\mu=1,2,\ldots,2n)$, equations (1) become Birkhoff equations. So, in theory, the generalised Birkhoff equations are more general than Birkhoff equations. Also, when transforming differential equations into a Birkhoffian system, it is easier to use Eq. (1) than Birkhoff equations.

Choosing the infinitesimal transformation of time t and variables a^μ as

$$\begin{aligned}&t^*=t+\varepsilon\xi_0(t,\boldsymbol{a}),\\&a^{\mu*}(t^*)=a^\mu(t)+\varepsilon\xi_\mu(t,\boldsymbol{a}),\end{aligned}\tag{2}$$

and supposing that dynamical functions B, R_μ and Λ_μ, undergoing the infinitesimal transformation (2), become B^*, R^*_μ and Λ^*_μ respectively, we have

$$\begin{aligned}B^*&=B(t^*,\boldsymbol{a}^*)\\&=B(t,\boldsymbol{a})+\varepsilon X^{(0)}(B)+O(\varepsilon^2),\\R^*_\mu&=R_\mu(t^*,\boldsymbol{a}^*)\\&=R_\mu(t,\boldsymbol{a})+\varepsilon X^{(0)}(R_\mu)+O(\varepsilon^2),\\\Lambda^*_\mu&=\Lambda_\mu(t^*,\boldsymbol{a}^*)\\&=\Lambda_\mu(t,\boldsymbol{a})+\varepsilon X^{(0)}(\Lambda_\mu)+O(\varepsilon^2),\end{aligned}\tag{3}$$

where

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_\mu\frac{\partial}{\partial a^\mu}.\tag{4}$$

Definition If the form of Eq. (1) keeps invariant when dynamical functions B, R_μ and Λ_μ are replaced

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†Corresponding author. E-mail: huibinwu@bit.edu.cn

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by B^*, R^*_μ and Λ^*_μ respectively, then the corresponding invariance is called form invariance.

According to the above definition, we have

$$\left(\frac{\partial R^*_\nu}{\partial a^\mu}-\frac{\partial R^*_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B^*}{\partial a^\mu}-\frac{\partial R^*_\mu}{\partial t}+\Lambda^*_\mu=0. \quad (5)$$

Substituting expression (3) into Eq. (5) and neglecting the terms of ε^2 and the higher order terms, we obtain

$$\begin{aligned}&\left[\frac{\partial}{\partial a^\mu}X^{(0)}(R_\nu)-\frac{\partial}{\partial a^\nu}X^{(0)}(R_\mu)\right]\dot{a}^\nu\\&-\frac{\partial}{\partial a^\mu}X^{(0)}(B)-\frac{\partial}{\partial t}X^{(0)}(R_\mu)\\&+X^{(0)}(\Lambda_\mu)=0.\end{aligned} \quad (6)$$

Criterion If infinitesimal generators ξ_0, ξ_μ satisfy Eq. (6), the corresponding invariance is a form invariance of generalised Birkhoffian system.

Equation (6) is called the criterion equation of form invariance for generalised Birkhoffian system.

3. New conserved quantity deduced by form invariance

Under certain condition, the form invariance of generalised Birkhoffian system can deduce a new conserved quantity. We can obtain the following result.

Proposition If ξ_0, ξ_μ, the generators of form invariance for generalised Birkhoffian system, and gauge function $G_F = G_F(t, \boldsymbol{a})$ satisfy the following structure equation

$$\begin{aligned}&X^{(0)}\left[X^{(0)}(R_\mu)\right]\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right)\\&-X^{(0)}\left[X^{(0)}(B)\right]-X^{(0)}(B)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\\&+X^{(0)}(R_\mu)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu\\&+X^{(0)}(\Lambda_\mu)\left[\xi_\mu-\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right)\xi_0\right]\\&+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_F=0,\end{aligned} \quad (7)$$

where

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right)\frac{\partial}{\partial a^\mu},\\&\Omega^{\mu\nu}\Omega_{\nu\rho}=\delta^\mu_\rho,\quad \Omega_{\nu\rho}=\left(\frac{\partial R_\rho}{\partial a^\nu}-\frac{\partial R_\nu}{\partial a^\rho}\right),\end{aligned} \quad (8)$$

then the form invariance of generalised Birkhoffian system leads to the following new conserved quantity

$$I_F=X^{(0)}(R_\mu)\xi_\mu-X^{(0)}(B)\xi_0+G_F=\text{const}. \quad (9)$$

Proof

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_F=&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}[X^{(0)}(R_\mu)]\xi_\mu+X^{(0)}(R_\mu)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}[X^{(0)}(B)]\xi_0\\&-X^{(0)}(B)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0-X^{(0)}[X^{(0)}(R_\mu)]\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right)\\&+X^{(0)}[X^{(0)}(B)]+X^{(0)}(B)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0-X^{(0)}(R_\mu)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu-X^{(0)}(\Lambda_\mu)\left[\xi_\mu-\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right)\xi_0\right]\\=&\left\{\left[\frac{\partial}{\partial a^\mu}X^{(0)}(R_\nu)-\frac{\partial}{\partial a^\nu}X^{(0)}(R_\mu)\right]\Omega^{\nu\rho}\left(\frac{\partial B}{\partial a^\rho}+\frac{\partial R_\rho}{\partial t}-\Lambda_\rho\right)-\frac{\partial}{\partial a^\mu}X^{(0)}(B)\right.\\&\left.-\frac{\partial}{\partial t}X^{(0)}(R_\mu)+X^{(0)}(\Lambda_\mu)\right\}\left[\Omega^{\mu\tau}\left(\frac{\partial B}{\partial a^\tau}+\frac{\partial R_\tau}{\partial t}-\Lambda_\tau\right)\xi_0-\xi_\mu\right]\\=&0.\end{aligned}$$

This completes the proof.

The Birkhoffian system is a special case of generalised Birkhoffian system, namely the case when $\Lambda_\mu = 0$ ($\mu = 1, 2, \ldots, 2n$). So, we have

Corollary For a Birkhoffian system, if the infinitesimal generators of form invariance ξ_0, ξ_μ and the gauge function $G_F = G_F(t, \boldsymbol{a})$ satisfy the following structure equation

$$X^{(0)}\left[X^{(0)}(R_\mu)\right]\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\right)-X^{(0)}\left[X^{(0)}(B)\right]-X^{(0)}(B)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0+X^{(0)}(R_\mu)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_F=0, \quad (10)$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\right)\frac{\partial}{\partial a^\mu}, \quad (11)$$

then the form invariance of Birkhoffian system leads to the new conserved quantity (9).

The above result has been given in Ref. [21].

4. Illustrative example

In order to illustrate the above new result, we give an example.

Suppose that a generalised Birkhoffian system of fourth order is as follows:

$$
\begin{aligned}
&R_1 = 0,\\
&R_2 = a^1 - a^4 - (a^2 + a^3)t + (a^2 \sin t + a^4 \cos t)\cos t,\\
&R_3 = a^1 - a^4 - (a^2 + a^3)t,\\
&R^4 = -(a^2 \sin t + a^4 \cos t)\sin t,\\
&B = 0, \quad \Lambda_1 = \Lambda_3 = 0,\\
&\Lambda_2 = -2(a^2 \sin t + a^4 \cos t)\sin t,\\
&\Lambda_4 = -2(a^2 \sin t + a^4 \cos t)\cos t. \qquad (12)
\end{aligned}
$$

Deduce the new conserved quantity by using form invariance of the system.

In order to find the generators of form invariance, through some calculation, we get

$$
\begin{aligned}
X^{(0)}(B) &= 0, \quad X^{(0)}(R_1) = 0, \quad X^{(0)}(\Lambda_1) = 0, \quad X^{(0)}(\Lambda_3) = 0;\\
X^{(0)}(R_2) &= \xi_1 - \xi_4 - (\xi_2 + \xi_3)t - (a^2 + a^3)\xi_0 + (\xi_2 \sin t + \xi_4 \cos t + a^2\xi_0 \cos t - a^4\xi_0 \sin t)\cos t\\
&\quad -(a^2 \sin t + a^4 \cos t)\xi_0 \sin t;\\
X^{(0)}(R_3) &= \xi_1 - \xi_4 - (\xi_2 + \xi_3)t - (a^2 + a^3)\xi_0;\\
X^{(0)}(R_4) &= -(\xi_2 \sin t + \xi_4 \cos t + a^2\xi_0 \cos t - a^4\xi_0 \sin t)\sin t - (a^2 \sin t + a^4 \cos t)\xi_0 \cos t;\\
X^{(0)}(\Lambda_2) &= -2(\xi_2 \sin t + \xi_4 \cos t + a^2\xi_0 \cos t - a^4\xi_0 \sin t)\sin t - 2(a^2 \sin t + a^4 \cos t)\xi_0 \cos t;\\
X^{(0)}(\Lambda_4) &= -2(\xi_2 \sin t + \xi_4 \cos t + a^2\xi_0 \cos t - a^4\xi_0 \sin t)\cos t + 2(a^2 \sin t + a^4 \cos t)\xi_0 \sin t.
\end{aligned}
$$

Choosing the generators as

$$
\xi_0 = \xi_2 = \xi_4 = 0, \quad \xi_1 = (a^2 + a^3)t + 1, \quad \xi_3 = a^2 + a^3, \qquad (13)
$$

we have

$$
\begin{aligned}
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 = 0,\\
&X^{(0)}(B) = X^{(0)}(R_1) = X^{(0)}(R_4) = X^{(0)}(\Lambda_1) = X^{(0)}(\Lambda_2) = X^{(0)}(\Lambda_3) = X^{(0)}(\Lambda_4) = 0;\\
&X^{(0)}(R_2) = X^{(0)}(R_3) = 1.
\end{aligned}
$$

Then from structure equation (7) we get

$$
G_F = 0.
$$

Also, from the expression of conserved quantity (9), we obtain

$$
I_F = X^{(0)}(R_3)\xi_3 = \xi_3 = a^2 + a^3 = \text{const.} \qquad (14)
$$

5. Conclusions

The generalised Birkhoffian system is the 5th stage in the development of classical mechanics. The research on the system possesses important significance. In this paper, we discuss the form invariance and the new conserved quantity deduced by it for the system. The obtained result presents a way for the problem of integration of the generalised Birkhoffian system.

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