Some discussions about the variable separating method for solving nonlinear models*

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Through analysing the exact solution of some nonlinear models, the role of the variable separating method in solving nonlinear equations is discussed. We find that rich solution structures of some special fields of these equations come from the nonzero seed solution. However, these nonzero seed solutions is likely to result in the divergent phenomena for the other field component of the same equation. The convergence and the signification of all field components should be discussed when someone solves the nonlinear equation using the variable separating method.

Keywords: variable separating method, nonzero seed solution, nonlinear equation

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1. Introduction

The recent development of the nonlinear wave theory clarifies the role of the 'soliton' in various systems. The soliton structures and their properties of (1+1)-dimensional integrable models have been very well understood. In (2+1)-dimensional cases, although some significant models such as the Kadomtsev–Petviashvili equation, Davey–Stewartson equation, Nizhnik–Novikov–Veselov (NNV) equation, sine-Gordon equation, etc. have also been established in nonlinear physics, the dynamics of nonlinear waves is not understood well enough in comparison with that in (1+1)-dimensional cases. In order to reveal the dynamics of nonlinear waves in higher dimensions we must reveal more solution structures. To find the exact solution of the nonlinear models is an important and complex work. There are some wealth of methods for finding special solutions of a nonlinear partial differential equation. Some of the most important methods are the inverse scattering transformation, bilinear method, symmetry reductions, Bäcklund and Darboux transformations, and so on.[1−3] Recently, the variable separating procedure has been established and used widely to solve (2+1)-dimensional nonlinear models with multi-component.[4−8]

The main thought of the variable separating method is: through using the arbitrariness of the seed solution in the Bäcklund transformation and supposing the form of variable separation of the solution, one can obtain the special exact solutions with some arbitrary functions.[6−10]

The suggestion of the variable separating method reveals many new nonlinear phenomena. The rich structures of exact solution resulting from the method of 'variable separation' hint us to consider the reason why these nonlinear phenomena have not been found for many years and these new phenomena can be or cannot be realised. We know that in the variable separating method, due to the role of the nonzero seed solution, some arbitrary functions are included in a certain field component. Therefore, the rich solution structures can be obtained in some special field components. It should be stated that the nonzero seed solution has been added to the other physical field or potential of the same equation in procedure of variable separation. One can see that physical field or potential with the added nonzero part is likely to be divergent. For generation, every field component has some special physical significance, and it is necessary to discuss the seed solution influence on different field components. Many typical nonlinear models such as, higher-order Bröer–Kaup equation (HBKE), Nizhnik–Novikov–Vesselev equation (NNVE), the asymmetric NNV equation (ANNVE), dispersive long wave equation (DLWE), Broer–Kaup–Kupershmidt equation (BKKE), belong to (2+1)-dimensional nonlinear multi-component equation, through Bäcklund transformation with nonzero seed solution these equations can be transformed to bilinear or trilinear equations

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possessing the nonzero seed solution about a function f.

Taking

$$f(x,y,t)=p_1(x,t)+p_2(x,t)q(y,t), \tag{1}$$

the original equation can be separated into two (1+1) dimensional equations in which one is an equation about $p_1(x,t)$ and $p_2(x,t)$, and the other is an equation about $q(y,t)$. Owing to the arbitrariness of the nonzero seed solution, the functions $p_1(x,t)$ and $p_2(x,t)$ can be arbitrarily selected. In some special cases, the function $q(y,t)$ also possesses some arbitrariness.

The solution form of a certain field component for most of (2+1)-dimensional multi-component nonlinear equations can be written as

$$\begin{aligned} U &= a(\log(f))_{xy} \\ &= a\left(-\frac{p_2q_y(p_{1x}+p_{2x}q)}{(p_1+p_2q)^2}+\frac{p_{2x}q_y}{p_1+p_2q}\right), \end{aligned} \tag{2}$$

where p_1 and p_2 can be selected arbitrarily. Since the seed solution is determined by p_1 and p_2, the arbitrariness of the seed solution is taken place from the functions p_1 and p_2, and the solution form of the other field component is related to the nonzero seed solution. We know that the rich phenomena of the field U come from arbitrariness of the functions p_1 and p_2, which is likely to result in the divergence of the other field component in the same equation. Taking NNVE, ANNVE, BKKE and DLWE as four examples, we discuss the role of the seed solution in different field components in this paper.

The paper is organised as follows. In Section 2, the solution structure of the four models are presented with the help of the Bäcklund transformation and variable separation procedure. The properties of some special solutions are discussed in Section 3. While a summary and a discussion are in Section 4.

2. Exact solutions of the four (2+1)-dimensional nonlinear models

In this section, we list some known models which can be solved by means of the special variable separation approach such that they possess a common quantity expressed as Eq. (2).

(A) NNV model. For concreteness and completeness to see the detailed variable separation approach, we review the procedure for the NNV model[11]

$$u_t+u_{xxx}+u_{yyy}=3(uv)_x+3(uw)_y, \tag{3}$$

$$v_y=u_x, \tag{4}$$

$$w_x=u_y. \tag{5}$$

Introducing the Bäcklund transformation

$$\begin{aligned} u&=-2(\ln f)_{xy}+u_0,\\ v&=-2(\ln f)_{xx}+v_0,\\ w&=-2(\ln f)_{yy}+w_0, \end{aligned} \tag{6}$$

in Eqs. (3)–(5), which become a trilinear equation about f

$$\begin{aligned} &-f(f_{xxxx}f_y+f_xf_{yt}-2f_{xy}f_{yyy}+f_tf_{xy}+f_xf_{yyyy}\\ &+4f_{xxxy}f_x)+2f_{xxx}(ff_{xy}+f_xf_y)+6f_x^2f_{xxy}\\ &-6f_{xy}f_xf_{xx}+6f_y^2f_{yyx}-f_y(4ff_{yyyx}+ff_{xt}\\ &-2f_xf_t-2f_xf_{yyy}+6f_{xy}f_{yy})\\ &+f^2(f_{xxxxy}+f_{yyyyx}+f_{xyt})\\ &+3f(f_xf_y-ff_{xy})(w_{0y}+v_{0x})\\ &+v_0(6ff_xf_{xy}+3ff_yf_{xx}-3f^2f_{xxy}-6f_x^2f_y)\\ &+w_0(6ff_yf_{xy}-3f^2f_{yyx}-6f_xf_y^2+3f_xf_{yy}f)=0. \end{aligned} \tag{7}$$

The $\{u_0,\ v_0,\ w_0\}$ in Eq. (6) is an arbitrary seed solution and has been taken as

$$u_0=0,\quad v_0=v_0(x,t),\quad w_0=w_0(y,t), \tag{8}$$

in Eq. (7) for simplicity. Taking the form of variable separation

$$f=p_1(x,t)+p_2(x,t)q(y,t), \tag{9}$$

we can obtain

$$\begin{aligned} &(2f_x-f\partial_x)(p_{1xxx}p_2-p_{2xxx}p_1\\ &+3(p_{2xx}p_{1x}-p_{1xx}p_{2x})+p_2p_{1t}-p_1p_{2t}\\ &+3v_0(p_1p_{2x}-p_2p_{1x}))+(p_1p_{2x}-p_2p_{1x})\\ &\times(2p_2-q_y^{-1}f\partial_y)(-q_{3y}-q_t+3q_yw_0)=0. \end{aligned} \tag{10}$$

Because $p_1(x,t)$ and $p_2(x,t)$ are x independent and q is y independent, Eq. (10) can be separated into two equations

$$\begin{aligned} &p_{2t}p_1-p_2p_{1t}\\ &=3v_0(p_1p_{2x}-p_2p_{1x})+p_{1xxx}p_2-p_{2xxx}p_1\\ &\quad+3(p_{2xx}p_{1x}-p_{1xx}p_{2x}), \end{aligned} \tag{11}$$

$$q_t=3q_yw_0-q_{3y}. \tag{12}$$

Considering the arbitrariness of the functions v_0 and w_0, p_1, p_2 and q can be selected arbitrarily. Substituting Eq. (9) into Eq. (6), we can obtain the form of solution for the NNV equation in the special case

$$\begin{aligned} U &= -2(\ln f)_{xy} \\ &= -\frac{2p_{2x}q_y}{p_1+p_2q} + \frac{2(p_{1x}+p_{2x}q)p_2q_y}{(p_1+p_2q)^2}, \qquad (13) \\ v &= -2(\ln f)_{xx} + v_0 \\ &= -2\frac{p_{1xx}+p_{2xx}q}{p_1+p_2q} + \frac{2(p_{1x}+p_{2x}q)^2}{(p_1+p_2q)^2} + v_0, \qquad (14) \\ w &= -2(\ln f)_{yy} + w_0 \\ &= -\frac{2p_2q_{yy}}{p_1+p_2q} + \frac{2p_2^2q_y^2}{(p_1+p_2q)^2} + w_0, \qquad (15) \end{aligned}$$

where v_0 and w_0 are related to p_1, p_2, and q by

$$\begin{aligned} v_0 &= \frac{1}{3(p_1p_{2x}-p_2p_{1x})}[p_{2t}p_1 - p_2p_{1t} + p_{2xxx}p_1 \\ &\quad -p_{1xxx}p_2 - 3(p_{2xx}p_{1x} - p_{1xx}p_{2x})], \qquad (16) \\ w_0 &= \frac{1}{3q_y}(q_t + q_{3y}). \qquad (17) \end{aligned}$$

Because $p_1(x,t)$ and $p_2(x,t)$ are two arbitrary functions of (x,t), and $q(y,t)$ is an arbitrary function of (y,t) in Eqs. (13)–(17), the physical field U possesses much rich solution structures. However, one can see that the selection of the p_1 and p_2 determines the form of the functions v_0 and w_0, i.e. the form of the other physical fields or potentials v and w is related to the selection of the p_1 and p_2.

(B) ANNV system. The similar formula is valid for the ANNV system[12]

$$u_t + u_{xxx} + 3[uv]_x = 0, \quad u_x = v_y. \qquad (18)$$

A special variable separation solution reads

$$\begin{aligned} u &= U, \qquad (19) \\ v &= -2\frac{p_{1xx}+p_{2xx}q}{p_1+p_2q} + \frac{2(p_{1x}+p_{2x}q)^2}{(p_1+p_2q)^2} + v_0, \qquad (20) \end{aligned}$$

where $p_1(x,t)$, $p_2(x,t)$, and $q(y,t)$ are arbitrary functions of $\{x,t\}$ and $\{y\}$, respectively; while v_0 is related to $p_1(x,t)$ and $p_2(x,t)$ by

$$\begin{aligned} v_0 &= \frac{1}{3(p_2p_{1x}-p_1p_{2x})}[p_{1t}p_2 - p_1p_{2t} + p_{1xxx}p_2 \\ &\quad -p_{2xxx}p_1 - 3(p_{1xx}p_{2x} - p_{2xx}p_{1x})]. \qquad (21) \end{aligned}$$

(C) BKK equation system. The (2+1)-dimensional BKK equation is as follows:

$$v_{ty} - v_{xxy} + 2(vv_x)_y + 2u_{xx} = 0, \qquad (22)$$

$$u_t + u_{xx} + 2(vu)_x = 0,$$

which may be derived from the inner parameter-dependent symmetry constraints of the Kadomtsev-Petviashvili (KP) equation.[13] When we take $y = x$, the BKK system can be used to describe the propagation of long waves in shallow water.[14] For the same reason as given above we can obtain the solution of variable separation

$$\begin{aligned} u &= U, \qquad (23) \\ v &= \frac{p_{1x}+p_{2x}q}{p_1+p_2q} + v_0, \qquad (24) \\ v_0 &= -\frac{p_1p_{2xx} - p_2p_{1xx} + p_1p_{2t} - p_2p_{1t}}{p_1p_{2x} - p_2p_{1x}}, \qquad (25) \end{aligned}$$

where $p_1(x,t)$, $p_2(x,t)$, and $q(y)$ are the arbitrary functions of $\{x,t\}$ and $\{y\}$, respectively.

(D) DLW equation system.

$$\begin{aligned} v_{yt} + u_{xx} + v_xv_y + vv_{xy} &= 0, \qquad (26) \\ u_t + v_x + uv_x + vu_x + v_{xxy} &= 0 \qquad (27) \end{aligned}$$

was first obtained by Boiti *et al.*[15] as a compatibility condition for a 'weak' Lax pair. For the (2+1)-dimensional DLW system, there is a special variable separation solution in the form

$$\begin{aligned} u &= U - 1, \qquad (28) \\ v &= 2\frac{p_{1x}+p_{2x}q}{p_1+p_2q} + v_0, \qquad (29) \\ v_0 &= -\frac{1}{p_1p_{2x} - p_2p_{1x}} \\ &\quad \times (p_1p_{2t} - p_2p_{1t} + p_1p_{2xx} - p_2p_{1xx}). \qquad (30) \end{aligned}$$

In Eqs. (28)–(30), $p_1(x,t)$ and $p_2(x,t)$ are the arbitrary functions of $\{x,t\}$ and $q(y)$ is the arbitrary function function of $\{y\}$. Obviously, the forms of the physical field U are the same in the four nonlinear models, and thus there are the same solution structures in these models for the physical field U. The solution form about the other field component may be different because of the different relationships in $p_1(x,t)$ and $p_2(x,t)$.

3. The analysis of some special exact solution for the four models

From Section 2, we see that the forms of solution for the special field component U described by

Eq. (2) are the same for the four models. The discussion about U is applicable to the four equations. Taking

$$\begin{aligned} p_1 &= 1+\exp(\eta_1)+\exp(\eta_2)+a_{13}\exp(\eta_1+\eta_3),\\ p_2 &= (1+a_{12}\exp(\eta_1)+a_{13}\exp(\eta_3)\\ &\quad +m\exp(\eta_1+\eta_3))\times\exp(-\omega_2 t),\\ q &= \exp(l_2 y), \quad \eta_1 = k_1 x-\omega_1 t,\\ \eta_3 &= k_3 x-\omega_3 t \end{aligned} \tag{31}$$

in Eq. (2), one can obtain the solution which describes the interaction between two 'dromions'. When the seed solution is determined by Eq. (16), or (21), or (25) or (30), the coupling constants, a_{ij}, ω_i, k_i, and m can be arbitrarily selected in Eq. (31) and there are no restraints among ω_i, k_i, and l_i, the function $f = p_1 + p_2 q$ has the form of a three 'plane' solitons solution presented by Hirota,[16]

$$\begin{aligned} f &= 1+\exp(\eta_1)+\exp(\eta_2)+\exp(\eta_3)\\ &\quad +a_{12}\exp(\eta_1+\eta_2)+a_{13}\exp(\eta_1+\eta_3)\\ &\quad +a_{23}\exp(\eta_2+\eta_3)+m\exp(\eta_1+\eta_2+\eta_3), \end{aligned} \tag{32}$$

where η_1 and η_3 are the same as those in Eq. (31), and

$$\eta_2 = l_2 y-\omega_2 t. \tag{33}$$

Take $m = a_{12}a_{13}a_{23}$, $a_{12}\neq 0$, $a_{13}\neq 0$, $a_{23}\neq 0$. U is an elastic interacting solution between two dromions as described in Fig. 1, and the other field components v, $v' = v - v_0$ are shown in Fig. 2. In Fig. 1 the three line solitons are determined by

$$\eta_1 = x-t, \quad \eta_2 = \frac{3}{2}y-\frac{1}{3}t, \quad \eta_3 = 2x-4t.$$

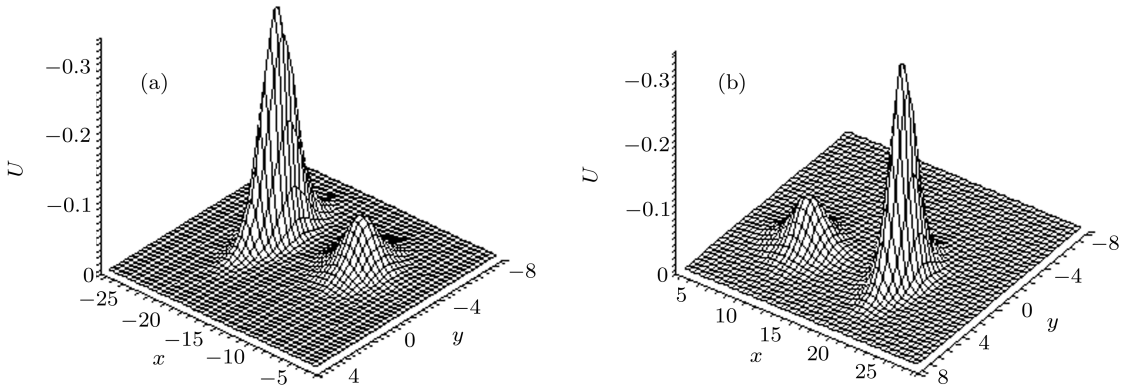
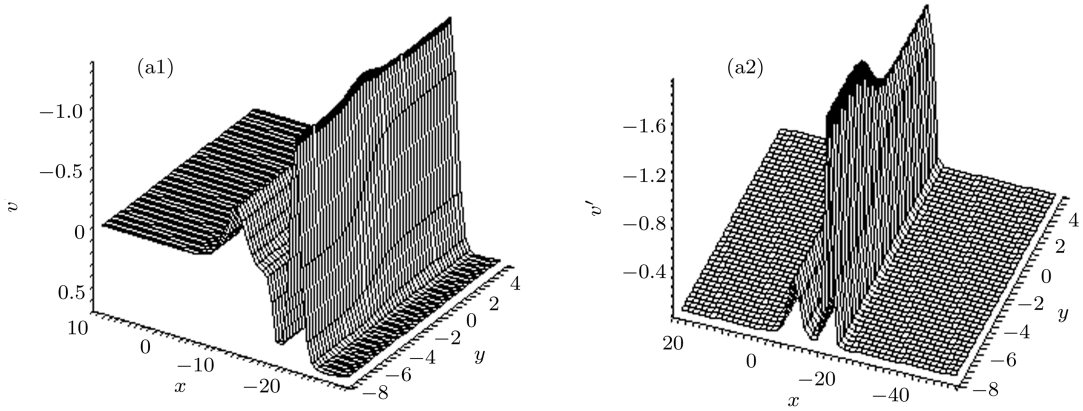


Fig. 1. The interacting plot of two dromions about U, determined by using Eqs. (31), (32), and (33), where $(a_{12},\ a_{13},\ a_{23},\ m) = (3/2, 2, 5/2, 15/2)$. The time for the figure is (a) $t = -10.0$, (b) $t = 10.0$.

From Fig. 1, we see that the shapes of two dromions are kept unchanged after interaction. Figure 2 shows the plots of the corresponding other field component v. Figures 2(a1) and 2(a2) are the plots of the function v and $v' = v - v_0$ for the NNVE and ANNVE systems; Figures 2(b1) and 2(b2) are the field v plots for BKKE and dispersive long wave equation system (DLSE) systems.



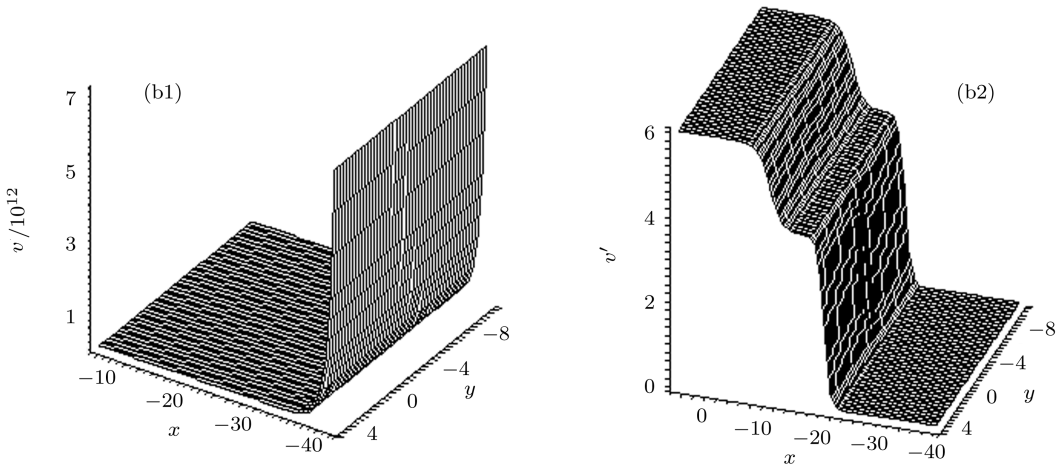


Fig. 2. The plots of functions v and $v' = v - v_0$: Figures 2(a1) and 2(a2) are obtained by using Eqs. (14) and (16) respectively; Figures 2(b1) and 2(b2) determined by Eqs. (29) and (30) respectively where $t = -10$ and the other parameters are the same as in Fig. 1.

One can see from Fig. 2 the seed solution $\{v_0\}$ plays an important role in the function v. For the BKKE and DLSE systems the field v is a divergent solution because $\{v_0\}$ is divergent. If we demand that the seed solution $\{u_0,\ v_0,\ w_0\}$ for NNVE or $\{u_0,\ v_0\}$ for the other three equations is a zero solution, that is $\{u_0,\ v_0,\ w_0\} = \{0,\ 0,\ 0\}$ or $\{u_0,\ v_0\} = \{0,\ 0\}$, substituting the standard form of the three plane solitons of Hirota to the four equation systems we cannot obtain the elastic interacting solution between two dromions about physical field U.

Selecting $p_1 = a_0 + p,\ \ p_2 = 1$ and

$$p_x = -k_1\mathrm{sech}^2(-k_2\xi - k_3t), \tag{34}$$

$$p = -\frac{k_1}{3k_2}(3 + 2k_2k_4 + k_2k_4\mathrm{sech}^2(k_2\xi - k_3t)) \times \tanh(k_2\xi - k_3t), \tag{35}$$

$$x = \xi + k_4\tanh(k_2\xi - k_3t), \tag{36}$$

$$q_y = -l_1\mathrm{sech}^2(l_2\eta), \tag{37}$$

$$q = -\frac{l_1}{3l_2}(3 + 2l_2l_3 + l_2l_3\mathrm{sech}^2(l_2\eta))\tanh(l_2\eta), \tag{38}$$

$$y = \eta + l_3\tanh(l_2\eta) \tag{39}$$

in Eq. (2) and selecting the parameters $k_1,\ k_2,\ k_3,\ k_4,\ l_1,\ l_2$, and l_3 in Eqs. (34)–(39) can satisfy that p is a multi-value function in the small range of $k_2x - k_3t$, and q is a single-value function in the whole range of values y can take. The solution U is a fold solution[17,18] but the other field component may be a divergent solution which can be observed from Fig. 3. Figure 3 illustrates the plots of the fold solution: Figure 3(a) is the field U plot for the four equations systems by using Eqs. (34)–(39). Figures 3(b1) and 3(c1) are the plots of the functions v and w, 3(b2) and 3(c2) are the plots of the functions $v' = v - v_0$ and $w' = w - w_0$ for the NNVE model. Figures 3(d1) and 3(d2) are the plots of the functions v and $v' = v - v_0$ for the DLSE system. From Fig. 3, we can also see that the functions v_0 and w_0 play the determined role. While v or w is likely to be divergent due to the reason of v_0 or w_0. If we take $\{u_0,\ v_0,\ w_0\}$ for NNVE and $\{u_0,\ v_0\}$ for the other three equations as $\{0,\ 0,\ 0\}$ and $\{0,\ 0\}$, the parameters k_2 and k_3 in Eqs. (34)–(39) can only be zero. That is to say, when the seed solution is taken as a zero solution, the physical field U does not possess the special fold solution.

Taking

$$p_1 = 1 + a_1p,\ \ p_2 = a_2 + a_3p, \tag{40}$$

$$p = \exp\left[-x(x + \sin(\ln(x^2)) - \cos(\ln(x^2)))\right], \tag{41}$$

$$q = \exp\left[-y(y + \sin(\ln(y^2)) - \cos(\ln(y^2)))\right] \tag{42}$$

in Eq. (2) with $a_1 = a_2 = 1,\ \ a_3 = 2$, we obtain a nonlocal exact solution about U with fractal structure for small x and y. Figure 4 shows the structure of Eq. (2). Figure 4(a1) is the plot of the field U of the four equations with (40)–(42) and Fig. 4(a2) is the density plot of U in the region $\{x = [-0.032,\ \ 0.032],\ \ y = [-0.032,\ \ 0.032]\}$. Figures 4(b1) and 4(b2) are the plots of the functions v and $v' = v - v_0$ in this case for the BKKE. Obviously, in this case, the seed solution v_0 is the determinant part in the function v. If we demand that the seed solution is a zero solution, the fractal solution about the physical field U also does not exist.

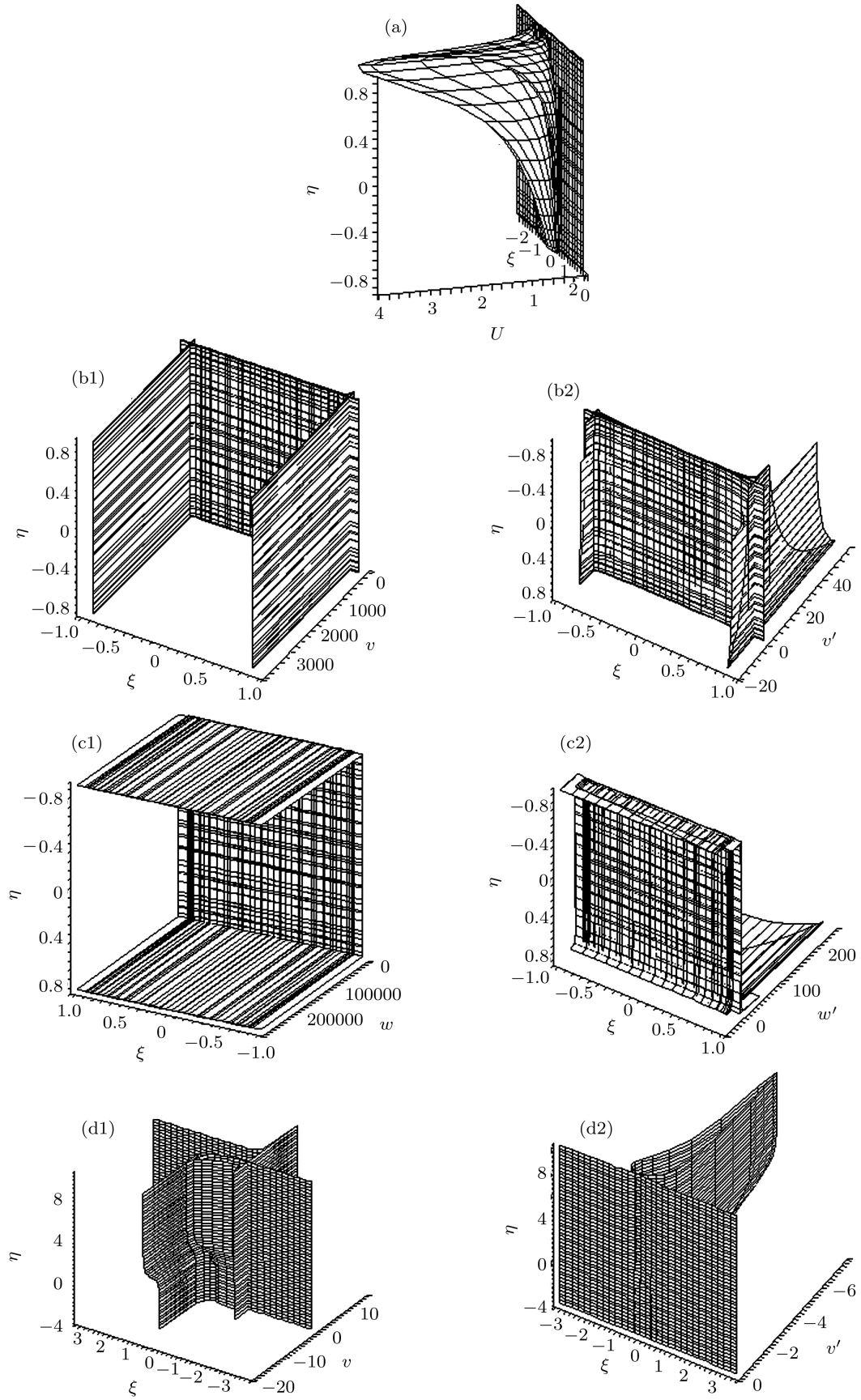


Fig. 3. The plots of functions $U,\ v,\ v'=v-v_0,\ w,\ w'=w-w_0$ determined by Eqs. (2), (14)–(17), (28)–(30), (34)–(39), where $k_1=1,\ k_2=1,\ k_3=1,\ l_1=1,\ l_2=1,\ l_3=0,\ k_4=-2.5,\ a_0=8,\ t=0$.

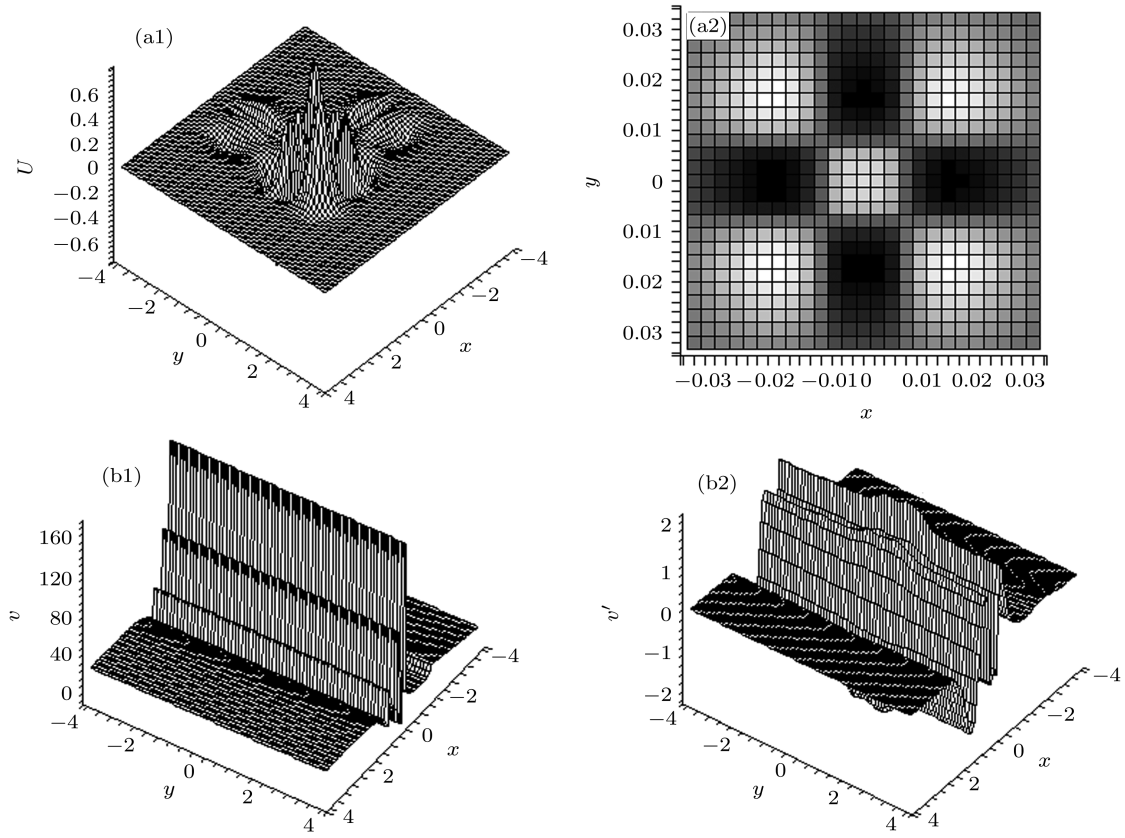


Fig. 4. The fractal plots, figure 4(a1) is the fractal solution for the quantity U given by Eq. (2) with Eqs. (40)–(42); Figure 4(a2) is the density plot of the U in the region $\{x = [-0.032, 0.032], y = [-0.032, 0.032]\}$ for $t = 0$. Figures 4(b1) and 4(b2) are the corresponding solutions of the function v and $v' = v - v_0$ for BKK equation.

4. Summary and discussions

Taking NNVE, ANNVE, BKKE and DLSE as four examples, the solution relationship among different field components of the same equation has been discussed in this paper. Because the nonzero seed solution which can be taken as an arbitrary function of $\{x, \quad t\}$ is introduced into Bäcklund transformation, the physical field U possesses rich solution structures. In fact, these nonzero seed solutions have been added into the other field component of the same equation and they determine the solution form of these components. In most cases, the other field component with the nonzero seed solution is divergent. We show the solution analysis in the three special cases when the physical field U is taken as the interaction solution between two 'dromions', the fold solution and the fractal solution. The forms of the exact solution and the properties of interaction are related to selecting the seed solution. Taking the seed solution as a zero solution, these special solution structures of the physical field U do not exist. For most researches before, the seed solution in the Bäcklund transformation was often taken as zero solution, some significant nonlinear phenomena may be neglected. Recently, many researchers obtained some new localised structures for many (2+1)-dimensional nonlinear models via multi-component variable-separating approach. However, a few of them pay attention to added phenomenon of these localised structures. Although we only give the discussion of four examples, these phenomena should also exist in most of the other multi-component nonlinear models. It is well known that integrable nonlinear models come from some practice nonlinear phenomena and every field component of multi-component nonlinear equation should possess the corresponding physical background for most of nonlinear equations. When someone tries to obtain the solution structure of a special component should attend to the solution expression of the other component, that is to say, the possibility to realise these solutions should be considered.

The dynamics of nonlinear waves in higher-dimensional space is not understood well enough. In order to reveal soliton structure in higher-dimension researchers presented some new approaches included in variable-separating method. However, the added phenomena related to new solutions that come from the new method have been little considered. We think the study of higher-dimensional models should be considered in multi-angle. We hope the analysis shown here is helpful for the deep study of higher-dimensional models.

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