Asymptopic solution for a class of semilinear singularly perturbed fractional differential equation*

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This paper considers a class of boundary value problems for the semilinear singularly perturbed fractional differential equation. Under the suitable conditions, first, the outer solution of the original problem is obtained; secondly, using the stretched variable and the composing expansion method the boundary layer is constructed; finally, using the theory of differential inequalities the asymptotic behaviour of solution for the problem is studied and the uniformly valid asymptotic estimation is discussed.

Keywords: semilinear, fractional differential equation, singular perturbation
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1. Introduction

Naturally, many physical problems can be solved by using the idea of the fractional derivative. For instance, in many complicated seepage flow, heat conduction phenomena and so on, they cannot be described by using the idea of normative derivative, but can be solved by using the idea of the fractional derivative.[1] The fractional derivative is an extension to the derivative of integer. Fractional derivative possesses its broad practice sense. However, solving the fractional differential equation is very difficult. In this paper, we construct an asymptotic solution for a class of fractional differential equations by using the singularly perturbed theory, and obtain its valid estimation for asymptotic behaviour by using the theory of differential inequalities.

The nonlinear singularly perturbed problem is a very attractive object in the international academic circles.[2] During the past decade many approximate methods have been developed and refined, including the method of averaging, the boundary layer method, the methods of matched asymptotic expansion and multiple scales. Recently, many scholars such as Guarguaglini and Natalini,[3] Hovhannisyan and Vulanovic,[4] Abid, Jieli and Trabelsi,[5] Graef and Kong[6] and Barbu and Cosma[7] have done a great deal of works. Using the method of differential inequalities and others Mo *et al.* considered also a class of singularly perturbed nonlinear boundary value problems for the reaction diffusion equations,[8] a class of activator inhibitor systems,[9] the ecological environment,[10] the shock wave,[11] the soliton,[12−16] the laser pulse,[17] the ocean science[18,19] and the problems of atmospheric physics.[20,21] In this paper, we constructed asymptotic solution for a class of fractional differential equation, and proved that the asymptotic solution is uniformly valid.

The α-th order fractional derivative $D^{\alpha}y$ of $y(x)$ is defined by

$$D_x^{\alpha}y \equiv \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-t)^{-\alpha}y(t)\mathrm{d}t,$$

where Γ is the Gamma function, α is a positive fraction less than 1.

Consider the following singularly perturbed boundary value problem of 2αth order fractional differential equation:

$$\varepsilon D_x^{\alpha}D_x^{\alpha}y = f(x,y,\varepsilon), \quad 0<x<1, \tag{1}$$

$$y(0,\varepsilon) = A(\varepsilon), \tag{2}$$

$$y(1,\varepsilon) = B(\varepsilon), \tag{3}$$

where ε is a positive parameter.

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We need the following hypotheses:

$[H_1]$ $f(x,y,\varepsilon)$, $A(\varepsilon)$ and $B(\varepsilon)$ are sufficiently smooth with respect to their arguments in corresponding domains;

$[H_2]$ $f_\varepsilon \geq 0$, $f_y \leq -cx^{-\alpha} < 0$, where c is a constant.

2. Outer solution

From the hypotheses, there is a solution $Y_0(x)$ of the reduced equation $f(x,y,0)=0$ for original problem Eqs. (1)–(3).

Now we construct the outer solution Y of the original problem Eqs. (1)–(3). Set

$$Y(x,\varepsilon) \sim \sum_{i=0}^{\infty} Y_i(x)\varepsilon^i. \tag{4}$$

Substituting Eq. (4) into Eq. (1), developing the nonlinear term $f(x,Y,\varepsilon)$ in ε, and equating coefficients of the same powers of ε both sides, by induction we obtain from the solution $Y_0(x)$ of the reduced equation that

$$Y_i(x) = \frac{1}{f_y(x,Y_0,0)}[F_i - D_x^\alpha D_x^\alpha Y_{i-1}],$$
$$0 < x < 1, \quad i = 1,2,\ldots,$$

where F_i, $i=1,2$, are inductively known functions, whose expressions are omitted. Bring $Y_i(x)$ back into Eq. (4), we have obtained the outer solution $Y(x,\varepsilon)$ for the original problem.

We note that the outer solution may not satisfy the boundary conditions (2) and (3), so we need to construct the boundary layer corrections Z near $x=0$ and W near $x=1$, respectively.

3. Boundary layer corrections

Let the solution of the boundary value problem Eqs. (1)–(3) be of the form

$$y \sim Y(x,\varepsilon) + Z(\tau,\varepsilon) \tag{5}$$

with

$$Z(\tau,\varepsilon) \sim \sum_{i=1}^{\infty} Z_i(\tau)\varepsilon^i, \tag{6}$$

where $\tau = x/\sqrt{\varepsilon}$ is a stretched variable.[2]

Substituting Eqs. (5) and (6) into Eqs. (1) and (2), developing the nonlinear term $f(x,Y+Z,\varepsilon)$ and $A(\varepsilon)$ in ε and equating coefficients of the same powers of ε in both sides of the equations, we obtain

$$D_\tau^\alpha D_\tau^\alpha Z_0 = f(x,Y_0+Z_0,0) - f(x,Y_0,0),$$
$$0 < x < 1, \tag{7}$$
$$Z_0(0) = A_0 - Y_0(0), \tag{8}$$
$$D_\tau^\alpha D_\tau^\alpha Z_i = f_y(x,Y_0,0)Z_i + G_i,$$
$$0 < x < 1, \quad i = 1,2,\ldots, \tag{9}$$
$$Z_i(0) = A_i - Y_i(0), \quad i = 1,2,\ldots, \tag{10}$$

where $A_i = \frac{1}{i!}[\partial^i A/\partial\varepsilon^i]_{\varepsilon=0}$, and G_i, $i=1,2$, are inductively known functions, whose constructions are omitted.

From the fractional differential equations (7) and (9), we can obtain the following Volterra integral system:

$$\int_0^\tau [(\tau-t)^{-\alpha}V_0(t)\mathrm{d}t = \Gamma(1-\alpha)\int_0^\tau [f_y(t,Y_0(t)+V_0(t),0) - f_y(t,Y_0(t),0)]\mathrm{d}t + C_{01}, \tag{11}$$
$$\int_0^\tau (\tau-t)^{-\alpha}Z_0(t)\mathrm{d}t = \Gamma(1-\alpha)\int_0^\tau V_0(t)\mathrm{d}t + C_{02}, \tag{12}$$
$$\int_0^\tau [(\tau-t)^{-\alpha}V_i(t)\mathrm{d}t = \Gamma(1-\alpha)\int_0^\tau [f_y(t,Y_0(t),0)V_i(t) + G_i(t)]\mathrm{d}t + C_{i1}, \quad i = 1,2,\ldots, \tag{13}$$
$$\int_0^\tau (\tau-t)^{-\alpha}Z_i(t)\mathrm{d}t = \Gamma(1-\alpha)\int_0^\tau V_i(t)\mathrm{d}t + C_{i2}, \quad i = 1,2,\ldots, \tag{14}$$

where C_{i1}, C_{i2}, $i=0,1,2,\ldots$, are arbitrary constants. From the Volterra integral systems (11)–(12) and (13)–(14) and conditions (8) and (10), we can obtain the solutions $Z_i(\tau)$, $i=0,1,2,\ldots$, successively,[22] and by using the method of the supper and lower solutions ($\alpha = A(\tau)\exp(-k\tau)$, $\beta = B(\tau)\exp(-k\tau)$) and Ref. [22] we can prove inductively that $Z_i(\tau)$, $i=0,1,2,\ldots$ possess boundary layer behaviour near $x=0$

$$Z_i(\tau) = O(\exp(-k_i\tau)) = O\bigg(\exp\bigg(-k_i\frac{x}{\sqrt{\varepsilon}}\bigg)\bigg), \quad 0 < \varepsilon \ll 1, \quad i = 0,1,2,\ldots, \tag{15}$$

where k_i ($\leq k_{i-1}$), $i=1,2,\ldots$, are positive constants.

Substituting $Z_i(\tau)$, $i = 0, 1, 2, \ldots$, into Eq. (6), we can obtain the boundary layer correction $Z(\tau, \varepsilon)$ near $x = 0$ of the solution $y(x, \varepsilon)$ for the original problem.

Let the solution of the boundary value problem Eqs. (1)–(3) be of the form

$$y \sim Y(x, \varepsilon) + Z(\tau, \varepsilon) + W(\sigma, \varepsilon) \tag{16}$$

with

$$W(\sigma, \varepsilon) \sim \sum_{i=1}^{\infty} W_i(\sigma)\varepsilon^i, \tag{17}$$

where $\sigma = (1 - x)/\sqrt{\varepsilon}$ is a stretched variable.

Substituting Eqs. (16) and (17) into Eqs. (1) and (3), developing the nonlinear term $f(Y + Z + W, \varepsilon)$ and $B(\varepsilon)$ in ε and equating coefficients of the same powers of ε in both sides, we obtain

$$D_\tau^\alpha D_\tau^\alpha W_0 = f(x, Y_0 + Z_0 + W_0, 0) - f(x, Y_0 + Z_0, 0), \quad 0 < x < 1, \tag{18}$$

$$W_0(0) = B_0 - Y_0(1) - Z_0(1), \tag{19}$$

$$D_\tau^\alpha D_\tau^\alpha W_i = f_y(x, Y_0 + Z_0, 0)W_i + \bar{G}_i, \quad 0 < x < 1, \ \ i = 1, 2, \ldots, \tag{20}$$

$$W_i(0) = B_i - Y_i(1) - Z_i(1), \quad i = 1, 2, \ldots, \tag{21}$$

where $B_i = \frac{1}{i!}[\partial^i B/\partial\varepsilon^i]_{\varepsilon=0}$, and $\bar{G}_i$, $i = 1, 2$, are inductively known functions, whose constructions are omitted.

From the fractional differential equations (18) and (20), we can obtain the following Volterra integral system:

$$\int_0^\sigma [(\sigma - t)^{-\alpha}]\bar{V}_0(t)\mathrm{d}t = \Gamma(1 - \alpha)\int_0^\sigma [f(t, Y_0(t) + Z_0(0) + W_0(t), 0) - f(t, Y_0(t) + Z_0(0)), 0)]\mathrm{d}t + \bar{C}_{01}, \tag{22}$$

$$\int_0^\sigma (\sigma - t)^{-\alpha}W_0(t)\mathrm{d}t = \Gamma(1 - \alpha)\int_0^\sigma \bar{V}_0(t)\mathrm{d}t + \bar{C}_{02}, \tag{23}$$

$$\int_0^\sigma [(\sigma - t)^{-\alpha}]\bar{V}_i(t)\mathrm{d}t = \Gamma(1 - \alpha)\int_0^\sigma [f_y(t, Y_0 + Z_0, 0)W_i(t) + \bar{G}_i(t)]\mathrm{d}t + \bar{C}_{1i}, \tag{24}$$

$$\int_0^\sigma (\sigma - t)^{-\alpha}W_i(t)\mathrm{d}t = \Gamma(1 - \alpha)\int_0^\sigma \bar{V}_i(t)\mathrm{d}t + \bar{C}_{2i}, \tag{25}$$

where $\bar{C}_{1i}$, $\bar{C}_{2i} = 1, 2$, are arbitrary constants. From the Volterra integral systems (22)–(23) and (24)–(25) and conditions (19) and (21), we can obtain the solutions $W_i(\sigma)$, $i = 0, 1, 2, \ldots$, successively,[22] and we also prove inductively that $W_i(\sigma)$, $i = 0, 1, 2, \ldots$ possess boundary layer behaviour near $x = 1$

$$W_i(\sigma) = O(\exp(-\bar{k}_i\sigma)) = O\left(\exp\left(-\bar{k}_i\frac{1 - x}{\sqrt{\varepsilon}}\right)\right), \quad 0 < \varepsilon \ll 1, \ \ i = 0, 1, 2, \ldots, \tag{26}$$

where $\bar{k}_i$ $(\leq \bar{k}_{i-1})$, $i = 1, 2, \ldots$, are positive constants.

Substituting $W_i(\sigma)$, $i = 0, 1, 2, \ldots$, into Eq. (17), we obtain the boundary layer correction $W(\sigma, \varepsilon)$ near $x = 1$ of the solution $y(x, \varepsilon)$ for the original problem.

Using Y_i, Z_i and W_i, we obtain from Eqs. (4), (6), (16) and (17) the asymptotic expansion of solution for the boundary value problem of 2αth order fractional differential equations (1)–(3):

$$y \sim \sum_{i=0}^{\infty}\left[Y_i(x) + Z_i\left(\frac{x}{\sqrt{\varepsilon}}\right) + W_i\left(\frac{1 - x}{\sqrt{\varepsilon}}\right)\right]\varepsilon^i, \quad x \in [0, 1], \ \ 0 < \varepsilon \ll 1.$$

4. Uniform validity of the asymptotic solution

Now we prove that the asymptotic solution obtained above is a uniformly valid asymptotic expansion in ε.

Definition For two smooth functions $\bar{y}$ and $\underline{y}$, if $\bar{y} \geq \underline{y}$ and they satisfy inequalities respectively:

$$\begin{aligned}
&\varepsilon D_x^\alpha D_x^\alpha \bar{y} - f(x, \bar{y}, \varepsilon) \leq 0,\\
&\bar{y}(0, \varepsilon) \geq A(\varepsilon), \ \bar{y}(1, \varepsilon) \geq B(\varepsilon),\\
&\varepsilon D_x^\alpha D_x^\alpha \underline{y} - f(x, \underline{y}, \varepsilon) \geq 0,\\
&\underline{y}(0, \varepsilon) \leq A(\varepsilon), \ \underline{y}(1, \varepsilon) \leq B(\varepsilon),
\end{aligned}$$

then we say that $\bar{y}$ and $\underline{y}$ are upper and lower solutions of the problem Eqs. (1)–(3), respectively.

Theorem 1 Assume that $[H_1] - [H_2]$ holds. If $\bar{y}(x, \varepsilon)$ and $\underline{y}(x, \varepsilon)$ are upper and lower solutions of the boundary value problem Eqs. (1)–(3) for the singularly perturbed fractional differential equation respectively, then there is a solution $y(x, \varepsilon)$ of the problem Eqs. (1)–(3) such that $\underline{y}(x, \varepsilon) \leq y(x, \varepsilon) \leq \bar{y}(x, \varepsilon)$.

Proof We construct two function sequences which are decided by the following recurrence relations:

$$\varepsilon D_x^\alpha D_x^\alpha y_{n+1} = f(x, y_n, \varepsilon), \quad n = 0, 1, \ldots, \tag{27}$$

$$y_{n+1}(0, \varepsilon) = A(\varepsilon), \ \ y_{n+1}(1, \varepsilon) = B(\varepsilon), \ n = 0, 1, \ldots. \tag{28}$$

Set $\bar{y}_0 = \bar{y}$ and $\underline{y}_0 = \underline{y}$ are initial iteration functions of the relations Eqs. (27) and (28) respectively, and we have $\bar{y}_n$ and $\underline{y}_n$, $n = 1, 2, \ldots$, successively. Thus we obtain two function sequences $\{\bar{y}_n\}$ and $\{\underline{y}_n\}$. Now we consider their convergence.

Set $\bar{u}_0 = \bar{y}_0 - \bar{y}_1$, from the hypothesis $[H_2]$, we have

$$\begin{aligned}\varepsilon D_x^\alpha D_x^\alpha \bar{u}_0 &= \varepsilon D_x^\alpha D_x^\alpha \bar{y}_0 - \varepsilon D_x^\alpha D_x^\alpha \bar{y}_1 \\ &\le f(x, \bar{y}_0, 0) - f(x, \bar{y}_0, \varepsilon) \le 0,\end{aligned}$$

$$\begin{aligned}\bar{u}_0 \,|_{x=0} &= \bar{y}_0 \,|_{x=0} - \bar{y}_1 \,|_{x=0} \ = 0 \ , \\ \bar{u}_0 \,|_{x=1} &= \bar{y}_0 \,|_{x=1} - \bar{y}_1 \,|_{x=1} \ = 0 \ .\end{aligned}$$

Thus from the extremum principle, we have $\bar{u}_0 \ge 0$. That is $\bar{y}_0 \ge \bar{y}_1$.

If $\bar{y}_{n-1} \ge \bar{y}_n$, $n \ge 1$, set $\bar{u}_n = \bar{y}_n - \bar{y}_{n+1}$, then we have

$$\begin{aligned}\varepsilon D_x^\alpha D_x^\alpha \bar{u}_n &= \varepsilon D_x^\alpha D_x^\alpha \bar{y}_n - \varepsilon D_x^\alpha D_x^\alpha \bar{y}_{n+1} \\ &= f(\bar{y}_{n-1}, \varepsilon) - f(\bar{y}_n, \varepsilon) \le 0,\end{aligned}$$

$$\begin{aligned}\bar{u}_n \,|_{x=0} &= \bar{y}_n \,|_{x=0} - \bar{y}_{n-1} \,|_{x=0} \ = 0 \ , \\ \bar{u}_n \,|_{x=1} &= \bar{y}_n \,|_{x=1} - \bar{y}_{n-1} \,|_{x=1} \ = 0.\end{aligned}$$

Thus $\bar{u}_n \ge 0$, that is $\bar{y}_n \ge \bar{y}_{n+1}$, $n \ge 1$.

From inductive method, we know

$$\bar{y} = \bar{y}_0 \ge \bar{y}_1 \ge \ldots \ge \bar{y}_n \ge \bar{y}_{n+1} \ge \ldots.$$

Analogously, we have

$$\underline{y} = \underline{y}_0 \le \underline{y}_1 \le \ldots \le \underline{y}_n \le \underline{y}_{n+1} \le \ldots,$$

and

$$\bar{y}_n \ge \underline{y}_n, \ \ n = 0, 1, \ldots.$$

From the above, and the Arzela-Ascoli theorem, there is a solution $y(x, \varepsilon)$ of the boundary value problem Eqs. (1)–(3) such that $\underline{y}(x, \varepsilon) \le y(x, \varepsilon) \le \bar{y}(x, \varepsilon)$. The proof of the Theorem 1 is completed.

Theorem 2 Under the hypotheses $[H_1] - [H_2]$, there is a solution $y(x, \varepsilon)$ of the boundary value problem Eqs. (1)–(3) for the singularly perturbed fractional differential equation, which possesses the following uniformly valid asymptotic expansion in ε on $x \in [0, 1]$

$$y = \sum_{i=0}^{m} [Y_i(x) + Z_i(x/\varepsilon) + W_i((1-x)/\sqrt{\varepsilon})]\varepsilon^i + O(\varepsilon^{m+1}), \ \ x \in [0, 1], \ \ 0 < \varepsilon \ll 1. \tag{29}$$

Proof Firstly, we construct the auxiliary functions $\alpha(x, \varepsilon)$, $\beta(x, \varepsilon)$:

$$\alpha(x, \varepsilon) = \sum_{i=0}^{m} [Y_i(x) + Z_i(x/\varepsilon) + W_i((1-x)/\sqrt{\varepsilon})]\varepsilon^i - r\varepsilon^{m+1}, \quad x \in [0, 1], \tag{30}$$

$$\beta(x, \varepsilon) = \sum_{i=0}^{m} [Y_i(x) + Z_i(x/\varepsilon) + W_i((1-x)/\sqrt{\varepsilon})]\varepsilon^i + r\varepsilon^{m+1}, \quad x \in [0, 1], \tag{31}$$

where r is a positive constant large enough to be chosen below, m is an arbitrary but fixed positive integer.

Obviously, we have

$$\alpha(x, \varepsilon) \le \beta(x, \varepsilon), \tag{32}$$

and for ε small enough, there is a positive constant δ_1 such that

$$\begin{aligned}\beta(0, \varepsilon) &= \sum_{i=0}^{m} [Y_i(0) + Z_i(0) + W_i((1)/\sqrt{\varepsilon})]\varepsilon^i + r\varepsilon^{m+1} \\ &\ge A(\varepsilon) - \delta_1\varepsilon^{m+1} + r\varepsilon^{m+1} = A(\varepsilon) + (r - \delta_1)\varepsilon^{m+1}.\end{aligned}$$

Thus selecting $r \ge \delta_1$, we have

$$\beta(0, \varepsilon) \ge A(\varepsilon). \tag{33}$$

Analogously, we have

$$\alpha(0, \varepsilon) \le A(\varepsilon), \quad \alpha(1, \varepsilon) \le B(\varepsilon) \le \beta(1, \varepsilon). \tag{34}$$

Now we prove that

$$\varepsilon D_x^\alpha D_x^\alpha \alpha - f(x,\alpha,\varepsilon) \geq 0, \quad 0 < x < 1, \tag{35}$$

$$\varepsilon D_x^\alpha D_x^\alpha \beta - f(x,\beta,\varepsilon) \leq 0, \quad 0 < x < 1. \tag{36}$$

In fact, from the hypotheses and Eqs. (15) and (26), for ε small enough, there is a positive constant δ_2, such that

$$\begin{aligned}
\varepsilon D_x^\alpha D_x^\alpha \beta - f(x,\beta,\varepsilon) &= \varepsilon D_x^\alpha D_x^\alpha \Big(\sum_{i=0}^{m} [Y_i(x) + Z_i(x/\varepsilon) + W_i((1-x)/\sqrt{\varepsilon})]\varepsilon^i \Big) \\
&\quad - f\Big(x, \sum_{i=0}^{m} [Y_i(x) + Z_i(x/\varepsilon) + W_i((1-x)/\sqrt{\varepsilon})]\varepsilon^i + r\varepsilon^{m+1}, \varepsilon \Big) \\
&\leq -f(x,Y_0,0) - \sum_{i=1}^{m} [f_y(x,Y_0,0)Y_i + F_i - D_x^\alpha D_x^\alpha Y_{i-1}]\varepsilon^i \\
&\quad + D_\tau^\alpha D_\tau^\alpha Z_0 - f(x,Y_0+Z_0,0) - f(x,Y_0,0) \\
&\quad + \sum_{i=0}^{m} [D_\tau^\alpha D_\tau^\alpha Z_i - f_y(x,Y_0,0)Z_i + G_i]\varepsilon^i + D_\tau^\alpha D_\tau^\alpha W_0 \\
&\quad - f(x,Y_0+Z_0+W_0,0) + f(x,Y_0+Z_0,0) \\
&\quad + \sum_{i=0}^{m} [D_\tau^\alpha D_\tau^\alpha W_i - f_y(x,Y_0+Z_0,0)W_i - \bar{G}_i]\varepsilon^i \\
&\quad + [f(x, \sum_{i=0}^{m} [Y_i(x) + Z_i(x/\varepsilon) + W_i((1-x)/\sqrt{\varepsilon})]\varepsilon^i, \varepsilon) \\
&\quad - f(x, \sum_{i=0}^{m} [Y_i(x) + Z_i(x/\varepsilon) + W_i((1-x)/\sqrt{\varepsilon})]\varepsilon^i + r\varepsilon^{m+1}, \varepsilon)] \\
&\quad + \delta_2 x^{-\alpha}\varepsilon^{m+1} \leq (-cr\varepsilon^{m+1} + \delta_2)x^{-\alpha}\varepsilon^{m+1}.
\end{aligned}$$

Selecting $r \geq \delta_2/c$, then we have Eq. (36). Analogously, we can prove Eq. (35). From Eqs. (32)–(36), α and β are upper and lower solutions respectively. From the Theorem 1, there is a solution $y(x,\varepsilon)$ of the problem Eqs. (1)–(3) such that $\alpha(x,\varepsilon) \leq y(x,\varepsilon) \leq \beta(x,\varepsilon)$. And from Eqs. (30) and (31), we have the relation Eq. (29). The proof of the Theorem 2 is completed.

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