A new (2+1)-dimensional supersymmetric Boussinesq equation and its Lie symmetry study*

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According to the conjecture based on some known facts of integrable models, a new (2+1)-dimensional supersymmetric integrable bilinear system is proposed. The model is not only the extension of the known (2+1)-dimensional negative Kadomtsev–Petviashvili equation but also the extension of the known (1+1)-dimensional supersymmetric Boussinesq equation. The infinite dimensional Kac–Moody–Virasoro symmetries and the related symmetry reductions of the model are obtained. Furthermore, the traveling wave solutions including soliton solutions are explicitly presented.

Keywords: integrable models, supersymmetry, symmetries and symmetry reductions, exact solutions

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1. Introduction

Many phenomena in physics fields are described by modern soliton theory such as the fluid mechanics,[1] field theory,[2] condensed matter physics,[3] optics,[4] particle and nuclear physics,[5] etc. Because integrable models play an important role in the soliton theory, to find as many integrable models as possible is an interesting and important issue. To deal with this problem, some useful methods have been developed. For example, to find the Lax equation by the Kac–Moody algebras is an important research direction.[6,7] Many integrable systems of the same dimensions of the original models are obtained by the asymptotical exact reduction methods based on the Fourier expansion from some known integrable equations (such as the Kadomtsev–Petviashvili (KP), (2+1)-Korteweg-de Vries (KdV) and Nizhnik–Novikov–Veselov (NNV) equations).[8,9] Some special types of higher dimensional integrable models can be found by using the (1+1)-dimensional strong symmetry,[10] the general Virasoro symmetry algebra,[11] the deformation theory,[12] and the conformal invariance of the Schwarzion form of the known integrable equation.[13]

Before going into detail, let us first review a conjecture proposed by Li and Lou[14] according to the following three models[15,16]

$$D_xD_tF\cdot G=0,\quad D_yD_xD_tF\cdot G=0, \tag{1}$$

$$(D_t+D_x^2)F\cdot G=0,\quad D_y(D_t+D_x^2)F\cdot G=0, \tag{2}$$

$$(D_t+D_x^3)F\cdot G=0,\quad D_y(D_t+D_x^3)F\cdot G=0, \tag{3}$$

and some others in Refs. [17]–[19] and [20].

Conjecture: If

$$P(D_t,D_x)F\cdot G=0,\quad Q(D_x)P(D_t,D_x)F\cdot G=0 \tag{4}$$

is integrable, then

$$P(D_t,D_x)F\cdot G=0,\quad R(S)P(D_t,D_x)F\cdot G=0 \tag{5}$$

may be integrable, where F and G are arbitrary functions of (x,t); D_x and D_t are the Hirota's bilinear operators defined below, S is a bilinear operator related to an additional freedom which may be continuous, discrete and even supersymmetric while the function $R(x)$ may be a suitable function of x and in many cases, $R(x)=Q(x)$.

It is stated that even if the conjecture is not correct, it is still believed that many new integrable models can be produced from the ansatz! Moreover, inte-

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grable supersymmetric equations are no doubt interesting and important topics especially in higher dimensions.

Thus, an interesting idea related to higher dimensional integrable models jumps into our minds: we can consider the following (2+1)-dimensional bilinear system

$$(D_t + D_x^2)F \cdot G = 0, \quad S_y(D_t + D_x^2)F \cdot G = 0, \quad (6)$$

as a possible supersymmetric extension of the (2+1)-dimensional negative KP equation[21] ($S_y \to D_y$) and the higher dimensional extension of the integrable (1+1)-dimensional supersymmetric Boussinesq equation

$$(D_t + D_x^2)F \cdot G = 0, \quad S_x(D_t + D_x^2)F \cdot G = 0, \quad (7)$$

where the Hirota's bilinear operators S_y, D_x and D_t are defined as

$$S_y D_x^m D_t^k F \cdot G = (\partial_{\epsilon_4} + \theta\partial_{\epsilon_2})\partial_{\epsilon_1}^m \partial_{\epsilon_3}^k F \cdot G|_{\epsilon_1=0,\epsilon_2=0,\epsilon_3=0,\epsilon_4=0}$$

with θ and ϵ_4 being Grassmannian parameters, ϵ_1, ϵ_2 and ϵ_3 being usual infinitesimal parameters and

$$F \cdot G = F(x + \epsilon_1, y + \epsilon_2, t + \epsilon_3, \theta + \epsilon_4) \cdot G(x - \epsilon_1, y - \epsilon_2, t - \epsilon_3, \theta - \epsilon_4).$$

The main purpose of this paper is to find the Lie symmetries, symmetry reductions and exact solutions of the model (6). In Section 2, applying the classical Lie point symmetry approach, we obtain the Lie point symmetries. In Section 3, four kinds of two dimensional symmetry reductions are derived. In Section 4, some explicit soliton solutions are presented. The last section is devoted to a short summary and discussion.

2. Lie point symmetries

There is still no simple way to solve the symmetry constraint condition for the general higher order symmetries. Thus, as usual, we just use the Lie symmetry approach to obtain the Lie point symmetries. In order to find the Lie symmetries of the new equations (6), we take the following dependent variable transformations

$$u = 2\left(\ln\frac{G}{F}\right)_x, \quad v = 2(\ln FG)_x$$

with $u = p + \theta\psi$ and $v = q + \theta\phi$. Thus, we convert Eqs. (6) into its component form

$$\begin{aligned}
&p_t - 2pp_x - q_{xx} = 0,\\
&q_{yt} - p_{xxy} - 2p_xq_y - 2pq_{xy} - 2\psi_x\phi - 2\psi\phi_x = 0,\\
&\psi_t - 2p_x\psi - 2p\psi_x - \phi_{xx} = 0,\\
&\phi_t - 2p_x\phi - 2p\phi_x - \psi_{xx} = 0,
\end{aligned} \quad (8)$$

where p and q are bosonic fields and ψ and ϕ are fermionic fields.

In this paper we only study symmetries by supposing Eqs.(8) being form invariant under a group action on $(x, y, t, p, q, \psi, \phi)$ space given by their infinitesimals

$$x^* = x + \epsilon X + O(\epsilon^2), \quad (9)$$
$$y^* = y + \epsilon Y + O(\epsilon^2), \quad (10)$$
$$t^* = t + \epsilon T + O(\epsilon^2), \quad (11)$$
$$p^* = p + \epsilon P + O(\epsilon^2), \quad (12)$$
$$q^* = q + \epsilon Q + O(\epsilon^2), \quad (13)$$
$$\psi^* = \psi + \epsilon R + O(\epsilon^2), \quad (14)$$
$$\phi^* = \phi + \epsilon S + O(\epsilon^2), \quad (15)$$

where X, Y, T, P, Q, R and S are functions of the arguments x, y, t, p, q, ψ and ϕ. A symmetry σ of an arbitrary nonlinear system $H(u) = 0$ is defined as a solution of its linearised form through the definition[22,23]

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}H(u + \epsilon\sigma)\Big|_{\epsilon=0} = 0.$$

Therefore, the symmetries of Eqs. (8) are solutions of the linearised system

$$\begin{aligned}
&\sigma_t^p - 2p\sigma_x^p - 2\sigma^p p_x - \sigma_{xx}^q = 0,\\
&\sigma_{yt}^q - \sigma_{xxy}^p - 2p_x\sigma_y^q - 2\sigma_x^p q_y - 2p\sigma_{xy}^p - 2\sigma^p q_{xy} - 2\psi_x\sigma^\phi\\
&- 2\phi\sigma_x^\psi - 2\psi\sigma_x^q - 2\sigma^\psi\phi_x = 0,\\
&\sigma_t^\psi - 2p_x\sigma^\psi - 2\sigma_x^p\psi - 2p\sigma_x^\psi - 2\sigma^p\psi_x - \sigma_{xx}^\psi = 0,\\
&\sigma_t^\phi - 2p_x\sigma^\phi - 2\sigma_x^p\phi - 2p\sigma_x^\phi - 2\sigma^p\phi_x - \sigma_{xx}^\phi = 0,
\end{aligned} \quad (16)$$

where $\{\sigma^p, \sigma^q, \sigma^\psi, \sigma^\phi\}$ is called a Lie symmetry of the model (8). According to the standard Lie symmetry approach,[24–26] the general Lie point symmetry has the form

$$\begin{aligned}
\sigma^p &= Xp_x + Yp_y + Tp_t - P,\\
\sigma^q &= Xq_x + Yq_y + Tq_t - Q,\\
\sigma^\psi &= X\psi_x + Y\psi_y + T\psi_t - R,\\
\sigma^\phi &= X\phi_x + Y\phi_y + T\phi_t - S.
\end{aligned} \quad (17)$$

Substituting Eqs. (17) into Eqs. (16) and then eliminating the quantities $p_t, q_{yt}, \phi_t, \psi_t$, and setting to zero all the coefficients of the independent terms of polynomials p, q, ϕ, ψ and their partial derivatives, an over-determined set of equations are produced for the unknown functions X, Y, T, P, Q, R and S. Solving these over-determined equation, we obtain the explicit forms of the infinitesimal transformations, and thus obtain the Lie point symmetries $\sigma^p, \sigma^q, \sigma^\psi$ and σ^ϕ in the form

$$\begin{aligned}
\sigma^p &= \frac{1}{2}(f_x+2g)p_x + lp_y + fp_t + \frac{1}{2}f_t p + \frac{1}{4}f_{tt}x + \frac{1}{2}g,\\
\sigma^q &= \frac{1}{2}(f_x+2g)q_x + lq_y + fq_t + \frac{1}{2}f_t q + \frac{1}{24}f_{ttt}x^3 + \frac{1}{4}g_{tt}x^2 - hx - k,\\
\sigma^\psi &= \frac{1}{2}(fx+2g)\psi_x + l\psi_y + f\psi_t + \frac{1}{2}(f+l)\psi,\\
\sigma^\phi &= \frac{1}{2}(fx+2g)\phi_x + l\phi_y + f\phi_t + \frac{1}{2}(f+l)\phi,
\end{aligned} \tag{18}$$

where f, g, h, k are arbitrary functions of t, and l is an arbitrary function of y. Because there are five arbitrary functions in Eqs. (18), one can split the symmetry into five generators

$$v_0(k) = k\partial_q, \quad v_1(h) = hx\partial_q, \quad v_2(g) = g\partial_x - \frac{1}{2}g_t\partial_p - \frac{1}{4}g_{tt}x^2\partial_q,$$

$$v_3(f) = \frac{1}{2}f_t\partial_x + f\partial_t - \frac{1}{4}f_{tt}x\partial_p - \frac{1}{24}f_{ttt}x^3\partial_q - \frac{1}{2}f_t p\partial_p - \frac{1}{2}f_t q\partial_q - \frac{1}{2}f_t\phi\partial_\phi - \frac{1}{2}f_t\psi\partial_\psi, \tag{19}$$

$$w(l) = l\partial_y - \frac{1}{2}l_y\phi\partial_\phi - \frac{1}{2}l_y\psi\partial_\psi. \tag{20}$$

The Lie point symmetries of the model are arbitrary combinations of the generators given in Eq. (20). The nonzero commutation relations among the generators are

$$\begin{aligned}
&[v_0(k), v_3(k)] = v_0\Big(-\frac{1}{2}kf_t - fk_t\Big),\\
&[v_1(h), v_2(g)] = v_0(-gh),\\
&[v_1(h), v_3(f)] = v_1(-f_t h - fh_t),\\
&[v_2(g_1), v_2(g_2)] = v_1\Big(-\frac{1}{2}g_1g_{2tt} + \frac{1}{2}g_2g_{1tt}\Big),\\
&[v_2(g), v_3(f)] = v_2\Big(\frac{1}{2}f_t g - fg_t\Big),\\
&[v_3(f_1), v_3(f_2)] = v_3(-f_{1t}f_2 + f_1f_{2t}),\\
&[u(l_1), u(l_2)] = u(l_1l_{2y} - l_2l_{1t}).
\end{aligned} \tag{21}$$

It is clear that the Lie point symmetries of the model possess the Kac–Moody–Virasoro (KMV) structure. Especially, both the symmetry $v_3(f)$ and $w(l)$ constitute an centreless Virasoro symmetry algebra. And then the supersymmetric model (6) is an integrable model under the meaning that it possesses an infinite dimensional KMV symmetry algebra.[24,25,27–29]

3. Two-dimensional symmetry reductions

To find the symmetry reductions, we should solve the following characteristic equations

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}p}{P} = \frac{\mathrm{d}q}{Q} = \frac{\mathrm{d}\psi}{R} = \frac{\mathrm{d}\phi}{S}, \tag{22}$$

then four cases should be considered: (1) $f \neq 0, l \neq 0$, (2)$f \neq 0, l = 0$, (3)$f = 0, l \neq 0$, and (4)$f = l = 0$.

Case 1 $f \neq 0, l \neq 0$.

For the first case, the symmetry reduction solutions read

$$p = -\frac{2g + xf_t - 4P(\xi,\eta)\sqrt{f}}{4f}, \tag{23}$$

$$\begin{aligned}
q = -\frac{1}{48f^4}\{&3xF^2f^3f_t^2 - 24f^{7/2}F\int f^{1/2}g_tF\mathrm{d}t - x^3f^2f_t^2 - 48\int hF\mathrm{d}tf^{7/2}\\
&- 6F\int ff_{ttt}F^2\mathrm{d}tf^{\frac{7}{2}} + 48F\int h\mathrm{d}tf^{7/2} + 12F^2f^4g_t + 12x^2f^3g_t\\
&+ 4F^3f^{9/2}f_{tt} + 2\int ff_{ttt}F^3\mathrm{d}tf^{7/2} - 24xFf^{7/2}g_t + 12\int f^{1/2}g_{tt}F^2\mathrm{d}tf^{7/2}\\
&+ 12xF^2f^{5/2}gf_t - 6x^2f^2gf_t - 48x\int h\mathrm{d}tf^3 + 2x^3f^3f_{ttt} - 6F^2f^3gf_t\\
&+ 24x\int f^{1/2}g_{tt}F\mathrm{d}tf^3 - 2F^3f^{7/2}f_t^2 + 6x\int ff_{ttt}F^2\mathrm{d}tf^3 - 48\int\frac{k}{f^{1/2}}\mathrm{d}tf^{7/2}\\
&- 6xF^2f^4f_{ttt} - 48Q(\xi,\eta)\},
\end{aligned} \tag{24}$$

$$\psi = \frac{\Psi(\xi,\eta)}{\sqrt{fl}}, \qquad \phi = \frac{\Phi(\xi,\eta)}{\sqrt{fl}}, \tag{25}$$

where $\{P,\ Q\}$ and $\{\Psi,\ \Phi\}$ are group invariant Bosonic functions and Fermionic functions of the invariant variables

$$\xi = \frac{x}{\sqrt{f}} - F, \quad \eta = -\int \frac{1}{f}\mathrm{d}t + \int \frac{1}{l}\mathrm{d}y \tag{26}$$

with

$$F = \int f^{-3/2} g \mathrm{d}t. \tag{27}$$

Substituting Eqs. (23)–(25) with Eq. (26) into Eqs. (8) leads to the first type of symmetry reduction equations

$$P_\eta + 2PP_\xi + Q_{\xi\xi} = 0, \tag{28}$$

$$Q_{\eta\eta} + P_{\xi\eta\eta} + 2Q_\eta P_\xi + 2PQ_{\xi\eta} + 2\Psi_\xi\Phi + 2\Psi\Phi_\xi = 0, \tag{29}$$

$$\Psi_\eta - 2P_\xi\Psi - 2P\Psi_\xi - 2\Phi_{\xi\xi} = 0, \tag{30}$$

$$\Phi_\eta + 2P_\xi\Psi + 2P\Psi_\xi - 2\Psi_{\xi\xi} = 0. \tag{31}$$

Case 2 $f \neq 0, l = 0$.

In this case, after integrating (22), the new model system (3) is reduced to

$$P^2 + Q_\xi = C_1, \tag{32}$$

$$P_{\xi\eta} + 2PQ_\eta + 2\Psi\Phi = C_2, \tag{33}$$

$$P\Psi + \Phi_\xi = \theta_1, \tag{34}$$

$$P\Phi + \Psi_\xi = \theta_2, \tag{35}$$

where P and Q are invariant Bosonic functions of the invariant variables $\{\xi,\ \eta\}$, Ψ and Φ are invariant Fermionic functions of the invariant variables, C_1 and C_2 are usual arbitrary constants while θ_1 and θ_2 are arbitrary Grassmannian constants.

$$\xi = \frac{x}{\sqrt{f}} - F, \quad \eta = y \tag{36}$$

with F given by Eq. (27). The corresponding symmetry reduction solutions are

$$\begin{aligned}
p &= -\frac{2g + xf_t - 4P(\xi,\eta)\sqrt{f}}{4f},\\
q &= -\frac{1}{48f^4}\{3xF^2f^3f_t^2 - 24f^{7/2}F\int f^{1/2}g_tF\mathrm{d}t - x^3f^2f_t^2 - 48\int hF\mathrm{d}tf^{7/2}\\
&\quad + 4F^3f^{9/2}f_{tt} + 2\int ff_{ttt}F^3\mathrm{d}tf^{7/2} - 24xFf^{7/2}g_t + 12\int f^{1/2}g_{tt}F^2\mathrm{d}tf^{7/2}\\
&\quad + 12xF^2f^{5/2}gf_t - 6x^2f^2gf_t - 48x\int h\mathrm{d}tf^3 + 2x^3f^3f_{ttt} - 6F^2f^3gf_t\\
&\quad + 24x\int f^{1/2}f_{tt}F\mathrm{d}tf^3 - 2F^3f^{7/2}f_t^2 + 6x\int ff_{ttt}F^2\mathrm{d}tf^3 - 48\int \frac{k}{f^{1/2}}\mathrm{d}tf^{7/2}\\
&\quad - 6F\int ff_{ttt}F^2\mathrm{d}tf^{\frac{7}{2}} + 48F\int h\mathrm{d}tf^{7/2} + 12F^2f^4g_t + 12x^2f^3g_t\\
&\quad - 6xF^2f^4f_{ttt} - 48Q(\xi,\eta)\},\\
\psi &= \frac{\Psi(\xi,\eta)}{\sqrt{fl}}, \quad \phi = \frac{\Phi(\xi,\eta)}{\sqrt{fl}}.
\end{aligned}$$

Case 3 $f = 0, l \neq 0$.

In this case, the symmetry reduction solutions are

$$\begin{aligned}
p &= -\frac{g_t}{2}\int \frac{1}{l}\mathrm{d}y + P(\xi,\eta),\\
q &= -\frac{1}{12}g_{tt}g^2G^3 + \frac{1}{4}(g_{tt}x - 2h)gG^2\\
&\quad - \frac{1}{4}(g_{tt}x^2 - 4hx - 4k)G + Q(\xi,\eta),\\
\psi &= \frac{\Psi(\xi,\eta)}{\sqrt{l}}, \quad \phi = \frac{\Phi(\xi,\eta)}{\sqrt{l}},
\end{aligned}$$

where

$$G = \int \frac{1}{l}\mathrm{d}y;$$

P, Q, Ψ, and Φ, which are invariant functions of the invariant variables ξ and η given by

$$\xi = x - \int \frac{g}{l}\mathrm{d}y, \quad \eta = t$$

satisfy the third type of symmetry reduction equations

$$\begin{aligned}
&P_\xi - 2P_\xi P - Q_{\xi\xi} = 0,\\
&4k_\eta + 4h_\eta\xi - g_{\eta\eta\eta}\xi^2 - 4(2h - g_{\eta\eta})P\\
&+ 2(g_{\eta\eta}\xi^2 - 4h\xi - 4k)P_\xi + 4gP_{\xi\xi\xi} - 8(\Psi\Phi)_\xi\\
&- 4(gQ_\xi)_\eta + 8g(PQ_\xi)_\eta = 0,\\
&\Psi_\eta - 2(P\Psi)_\xi - \Phi_{\xi\xi} = 0,\\
&\Phi_\eta - 2(P\Phi)_\xi - 2\Psi_{\xi\xi} = 0.
\end{aligned}$$

Case 4 $f = l = 0$.

In this case, the new model system (3) is reduced to

$$\begin{aligned} &gP - \int h\mathrm{d}\eta = C_1, \\ &gQ_\xi = C_2, \\ &g\Psi = \theta_1, \\ &g\Phi = \theta_2, \end{aligned}$$

where P, Q, Ψ and Φ are invariant functions of the invariant variables, C_1 and C_2 are usual arbitrary constants and θ_1, and θ_2 are arbitrary Grasmanian constants.

$$\xi = y, \quad \eta = t.$$

The related symmetry reduction solutions read

$$\begin{aligned} p &= -\frac{g_t}{2g}x + \frac{1}{g}\left(C_1 + \int h dt\right), \\ q &= -\frac{g_{tt}}{12g}x^3 + \frac{h}{2g}x^2 + \frac{k}{g}x + \frac{1}{g}C_2, \\ \phi &= \frac{1}{g}\theta_2, \quad \phi = \frac{1}{g}\theta_1. \end{aligned}$$

4. Exact solutions

It is not easy to obtain the general solutions of the system (8). Here, we just present some special solutions. To do so, we set

$$\phi = \theta\phi_1, \quad \psi = \theta\psi_1. \tag{37}$$

Then, substituting Eq. (37) into Eq. (8) and considering $\theta^2 = 0$, we finally obtain

$$p_t - 2pp_x - q_{xx} = 0, \tag{38}$$

$$q_{yt} - p_{xxy} - 2p_xq_y - 2pq_{xy} = 0, \tag{39}$$

$$\psi_{1t} - 2p_x\psi_1 - 2p\psi_{1x} - \phi_{1xx} = 0, \tag{40}$$

$$\phi_{1t} - 2p_x\phi_1 - 2p\phi_{1x} - \psi_{1xx} = 0. \tag{41}$$

It is noted that Eqs. (38) and (39) are the (2+1)-dimensional classical Boussinesq equation. A type of special solutions can be easily obtained by assuming $p = a_0 + a_1\tanh(kx+\beta y+\omega t)$ and $q = b_0 + b_1\tanh(kx+\beta y+\omega t)$, where the constants $a_0, a_1, b_0, b_1, k, \beta$ and ω should be determined by inserting the assumption into Eqs. (38) and (39). Thus, we find the special solutions p and q as

$$\begin{aligned} p &= a_0 + k\tanh(kx + \beta y + 2ka_0t), \\ q &= b_0 + k\tanh(kx + \beta y + 2ka_0t) \end{aligned}$$

with arbitrary constants a_0, b_0, k and β. In this case, we further suppose $\phi_1 = \phi_1(T)$ and $\psi_1 = \psi_1(T)$ with $T = \tanh(kx + \beta y + 2ka_0t)$, and then put them into Eqs. (40) and (41). Solving the resultant equations, we obtain

$$\begin{aligned} \phi_1(T) &= -6c_2T - (1-T^2)\left(c_1 + 3c_2\ln\frac{1+T}{1-T}\right), \\ \psi_1(T) &= -6c_2T - (1-T^2)\left(c_1 + 3c_2\ln\frac{1+T}{1-T}\right), \end{aligned}$$

where c_1 and c_2 are arbitrary constants. Consequently, a type of special travelling wave solutions of the supersymmetric Boussinesq equation (8) is derived as

$$\begin{aligned} p &= a_0 + k\tanh(kx + \beta y + 2ka_0t), \\ q &= b_0 + k\tanh(kx + \beta y + 2ka_0t), \\ \phi(T) &= \left[-6c_2T - (1-T^2)\left(c_1 + 3c_2\ln\frac{1+T}{1-T}\right)\right]\theta, \\ \psi(T) &= \left[-6c_2T - (1-T^2)\left(c_1 + 3c_2\ln\frac{1+T}{1-T}\right)\right]\theta. \end{aligned}$$

5. Summary and discussion

In conclusion, in this paper, three main items of work have been conducted. First, by using the Lie point symmetry approach, we have found Lie point symmetries of the new model (6) which indicates that the model is integrable under the meaning that it possesses an infinitely dimensional centreless Kac–Moody–Virasoro symmetry algebra. Secondly, by solving symmetry constraint equations and their original nonlinear system, the corresponding two-dimensional symmetry reductions of the new model (8) are explicitly obtained. Finally, in Section 4, we obtain a special type of the solitary wave solutions of the supersymmetric Boussinesq equation (16).

The new model (6) is not only a supersymmetric extension of Eq. (2) but also the higher dimensional extension of Eq. (7). The result of this paper is also a further positive example of the conjecture given in Ref. [14] and the conjecture should be checked further because one can find new types of integrable models from known ones via this conjecture.

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