Elastic analysis of an elliptic notch in quasicrystals of point group 10 subjected to shear loading*

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(Received 14 April 2009; revised manuscript received 9 July 2009)

Based on the stress potential and complex variable function method, this paper makes an elastic analysis of an elliptic notch subjected to uniform shear stress at infinity in quasicrystals with point group 10. With the aid of conformal transformation, an exact solution for the elliptic notch of the quasicrystals is obtained. The solution of the mode II Griffith crack as a special case is constructed. The stress intensity factor and energy release rate have been also obtained as a direct result of the crack solution.

Keywords: quasicrystals, plane elasticity, elliptic notch, stress potential

PACC: 6140M, 6220M, 4630N

1. Introduction

The quasicrystalline state of solid matter was first discovered in Al–Mn alloys in 1984.[1] The mechanical properties have been the subject of numerous studies since its discovery. Among them, the theoretical research on elasticity and defects has made great progress.[2−8] According to these theories, many methods such as the iterative method,[9] the Green function method,[10] the displacement potential and Fourier transform method,[7] the perturbation method,[11] and the generalized perturbation method[12] were developed to solve the relevant elasticity and defects problems. It is well known that the methods mentioned above cannot solve the notch problems. The analysis of the notch problem in quasicrystals becomes more complicated than conventional materials due to the introduction of the phason field. To solve the problem it is necessary to develop an effective theory and method. By means of the complex variable function method and conformal transformation technique, the elliptic notch problem of one-dimensional hexagonal quasicrystals has been solved successfully by Fan.[13] Afterward, Liu and Fan[14] further developed the method for plane elasticity of other one-dimensional quasicrystals. The complex variable function method based on stress potential for the quasicrystals of point group 10 was developed in Ref. [15].

Shibuya *et al.*[16] demonstrated first that quasicrystals could be plastically deformed at high temperatures. Since then, more and more researchers have paid attention to the plastic deformation of quasicrystals.[16−19] In general, shear deformation is more interesting for investigating the plastic progress of materials. This paper makes an elastic analysis of an elliptic notch subjected to uniform shear stress at infinity in quasicrystals with point group 10; this analysis is helpful to understand quantitatively the influence of the elliptic notch and crack on the mechanical behaviour of quasicrystals.

2. General solution

The physical problem considered in this paper is shown in Fig. 1. The elliptic notch is supposed to

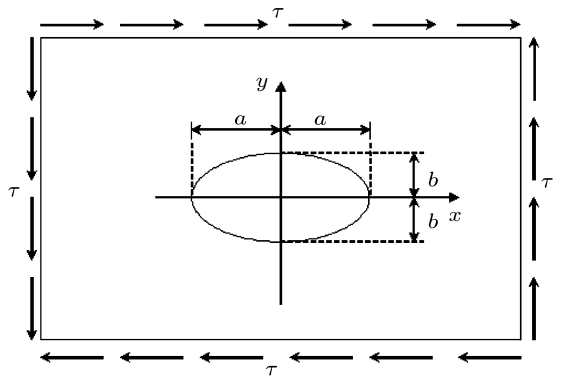


Fig. 1. An elliptic notch penetrating a quasicrystal with point group 10 along the periodic direction.

*Project supported by the National Natural Science Foundation of China (Grant No. 10761005), by the Natural Science Foundation of Inner Mongolia of China (Grant Nos. 2009MS0102, 2009BS0101 and 2009BS0104), the Natural Science Foundation of Inner Mongolia Normal University (Grant No. QN07034) and the Natural Science Foundation of Inner Mongolia Department of Public Education (Grant No. NJzy08024).

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penetrate through a two-dimensional quasicrystal with point group 10, which is subjected to uniform shear stress at infinity. The periodic direction is assumed to be orientated in the z-axis and the quasi-periodic plane is parallel to the xy-plane.

In this case, it is evident that all the field variables are independent of variable z. According to quasicrystal elasticity theory,[4] we have the deformation geometry equations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \quad w_{ij} = \frac{\partial w_i}{\partial x_j}, \tag{1}$$

equilibrium equations (if the body force is neglected)

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \frac{\partial H_{ij}}{\partial x_j} = 0, \tag{2}$$

and the generalized Hooke's law

$$\begin{aligned}
\sigma_{xx} &= L(\varepsilon_{xx} + \varepsilon_{yy}) + 2M\varepsilon_{xx} + R_1(w_{xx} + w_{yy}) + R_2(w_{xy} - w_{yx}),\\
\sigma_{yy} &= L(\varepsilon_{xx} + \varepsilon_{yy}) + 2M\varepsilon_{yy} - R_1(w_{xx} + w_{yy}) - R_2(w_{xy} - w_{yx}),\\
\sigma_{xy} &= \sigma_{yx} = 2M\varepsilon_{xy} + R_1(w_{yx} - w_{xy}) + R_2(w_{xx} + w_{yy}),\\
H_{xx} &= K_1 w_{xx} + K_2 w_{yy} + R_1(\varepsilon_{xx} - \varepsilon_{yy}) + 2R_2\varepsilon_{xy},\\
H_{xy} &= K_1 w_{xy} - K_2 w_{yx} - 2R_1\varepsilon_{xy} + R_2(\varepsilon_{xx} - \varepsilon_{yy}),\\
H_{yx} &= K_1 w_{yx} - K_2 w_{xy} + 2R_1\varepsilon_{xy} - R_2(\varepsilon_{xx} - \varepsilon_{yy}),\\
H_{yy} &= K_1 w_{yy} + K_2 w_{xx} + R_1(\varepsilon_{xx} - \varepsilon_{yy}) + 2R_2\varepsilon_{xy},
\end{aligned} \tag{3}$$

where $L = C_{12}, M = (C_{11} - C_{12})/2 = C_{66}$. σ_{ij}, ε_{ij}, u_i and C_{ij} are stress, strain, displacement, and elastic constants of phonon fields, respectively, H_{ij}, w_{ij}, w_i and K_i are the stress, strain, displacement, and elastic constants of phason fields respectively, R_i are the phonon–phason coupling elastic constants.

By the deformation geometry equations (1), the deformation compatibility equations can be obtained as follows:

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y},\\
&\frac{\partial w_{xy}}{\partial x} = \frac{\partial w_{xx}}{\partial y}, \quad \frac{\partial w_{yx}}{\partial y} = \frac{\partial w_{yy}}{\partial x}.
\end{aligned} \tag{4}$$

Based on the generalized Hooke's law, Eq. (3), all strain components can be expressed by relevant stress components. So the deformation compatibility equations (4) can be rewritten by the stresses σ_{ij}, H_{ij}. If we introduce stress potential functions $\phi(x,y), \psi_1(x,y), \psi_2(x,y)$ such as

$$\begin{aligned}
&\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2},\\
&\sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y},\\
&H_{xx} = \frac{\partial \psi_1}{\partial y}, \quad H_{xy} = -\frac{\partial \psi_1}{\partial x},\\
&H_{yx} = -\frac{\partial \psi_2}{\partial y}, \quad H_{yy} = \frac{\partial \psi_2}{\partial x}
\end{aligned} \tag{5}$$

with

$$\begin{aligned}
\phi &= c_1 \nabla^2 \nabla^2 G,\\
\psi_1 &= -\left(R_1 \frac{\partial}{\partial y} \Pi_1 + R_2 \frac{\partial}{\partial x} \Pi_2\right) \nabla^2 G,\\
\psi_2 &= \left(R_1 \frac{\partial}{\partial x} \Pi_2 - R_2 \frac{\partial}{\partial y} \Pi_1\right) \nabla^2 G,
\end{aligned} \tag{6}$$

then equilibrium equations (2) and deformation compatibility equations (4) expressed by stresses will be satisfied if

$$\nabla^2 \nabla^2 \nabla^2 \nabla^2 G = 0, \tag{7}$$

where

$$\begin{aligned}
&\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Pi_1 = 3\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2},\\
&\Pi_2 = 3\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2},\\
&c = M(K_1 + K_2) - 2(R_1^2 + R_2^2),\\
&c_1 = \frac{c}{K_1 - K_2} + M.
\end{aligned}$$

The general solution of Eq. (7) is

$$G = 2\mathrm{Re}\left[g_1(z) + \overline{z} g_2(z) + \frac{1}{2}\overline{z}^2 g_3(z) + \frac{1}{6}\overline{z}^3 g_4(z)\right], \tag{8}$$

where $g_i(z)$ $(i = 1, 2, 3, 4)$ are analytic functions of $z = x + \mathrm{i}\, y$. The bar denotes the complex conjugate hereinafter.

Equation (7) is the final governing equation based on the stress potential of quasicrystals of point group 10, see Ref. [15] for details.

3. The complex representation of stresses and displacements

In view of Eqs. (5), (6) and (8), the stress can be written as follows:

$$\begin{aligned}
\sigma_{xx} &= -32c_1\mathrm{Re}(\Omega(z) - 2g_4'''(z)),\\
\sigma_{yy} &= 32c_1\mathrm{Re}(\Omega(z) + 2g_4'''(z)),\\
\sigma_{xy} &= \sigma_{yx} = 32c_1\mathrm{Im}\Omega(z),\\
H_{xx} &= 32R_1\mathrm{Re}(\Theta'(z) - \Omega(z))\\
&\quad - 32R_2\mathrm{Im}(\Theta'(z) - \Omega(z)),\\
H_{xy} &= -32R_1\mathrm{Im}(\Theta'(z) + \Omega(z))\\
&\quad - 32R_2\mathrm{Re}(\Theta'(z) + \Omega(z)),\\
H_{yx} &= -32R_1\mathrm{Im}(\Theta'(z) - \Omega(z))\\
&\quad - 32R_2\mathrm{Im}(\Theta'(z) - \Omega(z)),\\
H_{yy} &= -32R_1\mathrm{Re}(\Theta'(z) + \Omega(z))\\
&\quad + 32R_2\mathrm{Im}(\Theta'(z) + \Omega(z)),
\end{aligned} \tag{9,10}$$

where

$$\begin{aligned}
\Theta(z) &= g_2^{(\mathrm{IV})}(z) + \bar{z}g_3^{(\mathrm{IV})} + \frac{1}{2}\bar{z}^2 g_4^{(\mathrm{IV})}(z),\\
\Omega(z) &= g_3^{(\mathrm{IV})}(z) + \bar{z}g_4^{(\mathrm{IV})}(z),
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
g_i'(z) &= \mathrm{d}g_i(z)/\mathrm{d}z,\\
g_i'''(z) &= \mathrm{d}^3 g_i(z)/\mathrm{d}z^3,\\
g_i^{(\mathrm{IV})}(z) &= \mathrm{d}^4 g_i(z)/\mathrm{d}z^4,\\
\Theta'(z) &= \mathrm{d}\Theta(z)/\mathrm{d}z.
\end{aligned}$$

Similar to Muskhelishvili,[20] complex representations of the displacement components can be obtained as follows (here we have omitted the rigid body displacements):

$$\begin{aligned}
u_x + \mathrm{i}\,u_y &= 32(4c_1c_2 - c_3 - c_1c_4)g_4''(z)\\
&\quad - 32(c_1c_4 - c_3)(\overline{g_3'''(z)} + z\overline{g_4'''(z)}),\\
w_x + \mathrm{i}\,w_y &= \frac{32(R_1 - \mathrm{i}\,R_2)}{K_1 - K_2}\overline{\Theta(z)},
\end{aligned} \tag{12}$$

where

$$g_4''(z) = \mathrm{d}^2 g_4(z)/\mathrm{d}z^2, \quad c_2 = \frac{c + (L+M)(K_1+K_2)}{4(L+M)c},$$
$$c_3 = \frac{R_1^2 + R_2^2}{c}, \quad c_4 = \frac{K_1 + K_2}{c}.$$

4. Elliptic notch problem

The boundary conditions of the physical problem considered in Section 2 can be written as follows:

$$\sigma_{xx}l + \sigma_{xy}m = X_n, \quad \sigma_{yy}m + \sigma_{xy}l = Y_n, \tag{13}$$
$$H_{xx}l + H_{xy}m = X^h, \quad H_{yy}m + H_{yx}l = Y^h, \tag{14}$$

where $l = \mathrm{d}y/\mathrm{d}s$, $m = -\mathrm{d}x/\mathrm{d}s$. X_n, Y_n denote the components of surface traction, X^h and Y^h are components of generalized surface traction. Since the measurement of generalized traction has not been reported so far, for simplicity, we assume that $X^h = 0$, $Y^h = 0$.

If we introduce the following new symbols:

$$\begin{aligned}
&g_2^{(\mathrm{IV})}(z) = f_2(z), \quad g_3'''(z) = f_3(z),\\
&g_4''(z) = f_4(z),
\end{aligned} \tag{15}$$

then we find that equation (9) and the boundary condition (13) are identical to the classical elasticity except the coefficient. Use the conformal mapping

$$z = \omega(\zeta) = R_0\left(\frac{1}{\zeta} + m\zeta\right) \tag{16}$$

to transform the exterior of the ellipse at the z-plane onto the interior of the unit circle γ at the ζ-plane, in which $R_0 = (a+b)/2, m = (a-b)/(a+b)$. Let

$$g_i(z) = g_i[\omega(\zeta)] = \Phi_i(\zeta), \quad (i = 2, 3, 4). \tag{17}$$

By Eqs. (9), (13), (16) and (17), we have

$$\begin{aligned}
\Phi_3(\zeta) &= \frac{\mathrm{i}\,\tau R_0}{32c_1}\frac{1+\zeta^4}{\zeta(1-m\zeta^2)},\\
\Phi_4(\zeta) &= \frac{\mathrm{i}\,\tau R_0}{32c_1}\zeta.
\end{aligned} \tag{18}$$

From Eqs. (10) and (14), one obtains

$$\begin{aligned}
&R_1\mathrm{Im}\Theta(z) + R_2\mathrm{Re}\Theta(z) = 0,\\
&-R_1\mathrm{Re}\Theta(z) + R_2\mathrm{Im}\Theta(z) = 0,
\end{aligned} \tag{19}$$

where Re and Im denote the real and imaginary parts, respectively. Multiplying the second formula of Eqs. (19) by i and adding it to the first yields

$$\Theta(z) = 0. \tag{20}$$

Substituting Eq. (17) into Eq. (20), then multiplying both sides of the equations by $\mathrm{d}\sigma/[2\pi\,\mathrm{i}(\sigma - \zeta)]$, and integrating around the unit circle γ, we have

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma \frac{\Phi_2(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma + \frac{1}{2\pi\,\mathrm{i}}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_3'(\sigma)\mathrm{d}\sigma}{\sigma-\zeta} + \frac{1}{4\pi\,\mathrm{i}}\int_\gamma \left[\frac{\overline{\omega(\sigma)}^2\Phi_4''(\sigma)}{\omega'(\sigma)^2} - \frac{\overline{\omega(\sigma)}^2\omega''(\sigma)\Phi_4'(\sigma)}{\omega'(\sigma)^3}\right]\frac{\mathrm{d}\sigma}{\sigma-\zeta} = 0. \tag{21}$$

Utilizing the Cauchy integral formula, analytic extension of the complex variable function theory and Eqs. (18), (21), we have

$$\Phi_2(\zeta) = \frac{\mathrm{i}\,\tau R_0}{32c_1}\frac{(\zeta^2+m)(m\zeta^6-2\zeta^4-2m\zeta^2+1)}{\zeta(m\zeta^2-1)^3}. \tag{22}$$

The elliptic notch problem has been solved.

5. Elastic field induced by a mode II crack

The solutions of the mode II crack can be obtained as a special case of the above solution. Inserting $m=1$, $R_0=a/2$ into Eqs. (18), (22) and making use of Eq. (15), one gets

$$g_2^{(\mathrm{IV})}(z) = \frac{\mathrm{i}\,\tau}{64c_1}\left(\frac{a^2z^2}{2\sqrt{(z^2-a^2)^3}} - \frac{2z^2}{\sqrt{z^2-a^2}}\right), \tag{23}$$

$$g_3'''(z) = \frac{\mathrm{i}\,\tau}{64c_1}\frac{2z^2-a^2}{\sqrt{z^2-a^2}}, \tag{24}$$

$$g_4''(z) = \frac{\mathrm{i}\,\tau}{64c_1}(z-\sqrt{z^2-a^2}). \tag{25}$$

Employing Eqs. (9), (10) and (23)–(25), we can express the complex representation of stress and displacement as follows:

$$\sigma_{xx} = \mathrm{Re}\left[\frac{-\mathrm{i}\,\tau}{2}\left(\frac{4z}{\sqrt{z^2-a^2}} - \frac{a^2(z-\overline{z})}{\sqrt{(z^2-a^2)^3}} - 2\right)\right], \tag{26}$$

$$\sigma_{yy} = \mathrm{Re}\left[\frac{\mathrm{i}\,\tau}{2}\left(2 - \frac{a^2(z-\overline{z})}{\sqrt{(z^2-a^2)^3}}\right)\right], \tag{27}$$

$$\sigma_{xy} = \sigma_{yx} = \mathrm{Im}\left[\frac{\mathrm{i}\,\tau}{2}\left(\frac{2z}{\sqrt{z^2-a^2}} - \frac{a^2(z-\overline{z})}{\sqrt{(z^2-a^2)^3}}\right)\right], \tag{28}$$

$$H_{xx} = \mathrm{Re}\left[\frac{\mathrm{i}\,\tau R_1}{2c_1}\Gamma(z)\right] - \mathrm{Im}\left[\frac{\mathrm{i}\,\tau R_2}{2c_1}\Gamma(z)\right], \tag{29}$$

$$H_{xy} = \mathrm{Im}\left[\frac{-\mathrm{i}\,\tau R_1}{2c_1}\Psi(z)\right] - \mathrm{Re}\left[\frac{\mathrm{i}\,\tau R_2}{2c_1}\Psi(z)\right], \tag{30}$$

$$H_{yx} = \mathrm{Im}\left[\frac{-\mathrm{i}\,\tau R_1}{2c_1}\Gamma(z)\right] - \mathrm{Re}\left[\frac{\mathrm{i}\,\tau R_2}{2c_1}\Gamma(z)\right], \tag{31}$$

$$H_{yy} = \mathrm{Re}\left[\frac{-\mathrm{i}\,\tau R_1}{2c_1}\Psi(z)\right] + \mathrm{Im}\left[\frac{\mathrm{i}\,\tau R_2}{2c_1}\Psi(z)\right], \tag{32}$$

where

$$\Gamma(z) = \frac{4a^2(z-\overline{z})}{\sqrt{(z^2-a^2)^3}} - \frac{4z}{\sqrt{z^2-a^2}} - \frac{3a^2z(z-\overline{z})^2}{2\sqrt{(z^2-a^2)^5}},$$

$$\Psi(z) = \frac{2a^2(z-\overline{z})}{\sqrt{(z^2-a^2)^3}} - \frac{3a^2z(z-\overline{z})^2}{2\sqrt{(z^2-a^2)^5}},$$

$$u_x + \mathrm{i}\,u_y = \frac{\mathrm{i}\,\tau}{2c_1}\left[(4c_1c_2-c_3-c_1c_4)(z-\sqrt{z^2-a^2}) + (c_1c_4-c_3)\left(z+\overline{\sqrt{z^2-a^2}} - \frac{\overline{z}(z-\overline{z})}{\sqrt{z^2-a^2}}\right)\right], \tag{33}$$

$$w_x + \mathrm{i}\,w_y = -\frac{(R_2+\mathrm{i}\,R_1)\tau}{2(K_1-K_2)c_1}\left[\frac{2\overline{z}(z-\overline{z})}{\overline{\sqrt{z^2-a^2}}} + \frac{a^2}{2}\frac{(z-\overline{z})^2}{\overline{\sqrt{(z^2-a^2)^3}}}\right]. \tag{34}$$

Similar to the classical elasticity theory, we introduce three pairs of polar coordinates (r,θ), (r_1,θ_1), (r_2,θ_2), and with the origin at the crack centre, right crack tip and left tip, i.e.,

$$z = r\mathrm{e}^{\mathrm{i}\,\theta}, \quad z-a = r_1\mathrm{e}^{\mathrm{i}\,\theta_1}, \quad z+a = r_2\mathrm{e}^{\mathrm{i}\,\theta_2}$$

respectively, analytic expressions for the stress and displacement fields can be obtained (see Appendix A for details), from which one can obtain asymptotic solutions for the crack tip $r_1/a \ll 1$. When $R_2=0$, $R_1=R$, the material reduces to point $10mm$ quasicrystals, and the results reduce to those for point group $10mm$ given by Li and Fan.[21]

As a result, the stress intensity factor can be calculated:

$$K_{\mathrm{II}} = \tau\sqrt{\pi a}. \tag{35}$$

In quasicrystals, we assume that under applied loads the crack tip advances along the crack plane from $x=a$ to $x=a+\delta$. In this case the energy release rate is defined by

$$\begin{aligned}\mathscr{G}_{\mathrm{II}} &= \lim_{\delta\to 0}\frac{1}{2\delta}\int_0^{\delta}[\sigma_{xy}(r)\Delta u_x(\delta-r) \\ &\quad + H_{xy}(r)\Delta w_x(\delta-r)]\mathrm{d}r \\ &= \left(c_2 - \frac{c_3}{2c_1}\right)(K_{\mathrm{II}})^2,\end{aligned} \tag{36}$$

where r denotes the distance from the crack tip, Δu_x and Δw_x are the difference between displacements across the crack faces corresponding to phonon and phason fields, respectively.

6. Conclusion and discussion

A quasicrystal with point group 10 containing an elliptic notch which penetrates the medium along the periodic axis is considered. Complex and real expressions for the entire elasticity field of the mode II crack are derived. When $R_1 = R$, $R_2 = 0$, the results of Li and Fan[21] are given. The stress intensity factor and energy release rate are given, which are all important criteria of fracture mechanics.

Although the general solution (8) is expressed by four analytic functions, we need only three of them to express the stress and displacement components of phonon and phason fields; in particular, the stress expression for phonon fields is consistent with the classical elasticity theory. This greatly simplifies the solving procedures including those for more complicated problems. It is concluded that the stress potential and complex variable function method is powerful for the complicated stress boundary value problem of quasicrystals.

Appendix A Details of the real representation of the crack solution

By Eqs. (26)–(32), the stress fields can be obtained as follows:

$$\sigma_{xx} = \tau[2r(r_1r_2)^{-1/2}\sin(\theta-\overline{\theta}) - a^2r(r_1r_2)^{-3/2}\sin\theta\cos 3\overline{\theta}],$$
$$\sigma_{yy} = \tau[a^2r(r_1r_2)^{-3/2}\sin\theta\cos 3\overline{\theta}],$$
$$\sigma_{xy} = \sigma_{yx} = \tau[r(r_1r_2)^{-1/2}\cos(\theta-\overline{\theta}) - a^2r(r_1r_2)^{-3/2}\sin\theta\sin 3\overline{\theta}],$$
$$\begin{aligned} H_{xx} = {} & \frac{R_1\tau}{c_1}[2r(r_1r_2)^{-1/2}\sin(\theta-\overline{\theta}) - 4a^2r(r_1r_2)^{-3/2}\sin\theta\cos 3\overline{\theta} - 3a^2r^3(r_1r_2)^{-5/2}\sin^2\theta\sin(\theta-5\overline{\theta})] \\ & - \frac{R_2\tau}{c_1}[-2r(r_1r_2)^{-1/2}\cos(\theta-\overline{\theta}) + 4a^2r(r_1r_2)^{-3/2}\sin\theta\sin 3\overline{\theta} + 3a^2r^3(r_1r_2)^{-5/2}\sin^2\theta\cos(\theta-5\overline{\theta})], \end{aligned}$$
$$\begin{aligned} H_{xy} = {} & -\frac{R_1\tau}{c_1}[2a^2r(r_1r_2)^{-3/2}\sin\theta\sin 3\overline{\theta} + 3a^2r^3(r_1r_2)^{-5/2}\sin^2\theta\cos(\theta-5\overline{\theta})] \\ & + \frac{R_2\tau}{c_1}[2a^2r(r_1r_2)^{-3/2}\sin\theta\cos 3\overline{\theta} + 3a^2r^3(r_1r_2)^{-5/2}\sin^2\theta\sin(\theta-5\overline{\theta})], \end{aligned}$$
$$\begin{aligned} H_{yx} = {} & -\frac{R_1\tau}{c_1}[-2r(r_1r_2)^{-1/2}\cos(\theta-\overline{\theta}) + 4a^2r(r_1r_2)^{-3/2}\sin\theta\sin 3\overline{\theta} + 3a^2r^3(r_1r_2)^{-5/2}\sin^2\theta\cos(\theta-5\overline{\theta})] \\ & - \frac{R_2\tau}{c_1}[2r(r_1r_2)^{-1/2}\sin(\theta-\overline{\theta}) - 4a^2r(r_1r_2)^{-3/2}\sin\theta\cos 3\overline{\theta} - 3a^2r^3(r_1r_2)^{-5/2}\sin^2\theta\sin(\theta-5\overline{\theta})], \end{aligned}$$
$$\begin{aligned} H_{yy} = {} & \frac{R_1\tau}{c_1}[2a^2r(r_1r_2)^{-3/2}\sin\theta\cos 3\overline{\theta} + 3a^2r^3(r_1r_2)^{-5/2}\sin^2\theta\sin(\theta-5\overline{\theta})] \\ & + \frac{R_2\tau}{c_1}[2a^2r(r_1r_2)^{-3/2}\sin\theta\sin 3\overline{\theta} + 3a^2r^3(r_1r_2)^{-5/2}\sin^2\theta\cos(\theta-5\overline{\theta})], \end{aligned}$$

where $\overline{\theta} = (\theta_1+\theta_2)2$.

Using Eqs. (33) and (34), we can calculate the displacement fields and obtain

$$u_x = \frac{-\tau}{2}\left[\left(4c_2 - 2\frac{c_3}{c_1}\right)(r\sin\theta - (r_1r_2)^{1/2}\sin\overline{\theta}) - 2\left(c_4 - \frac{c_3}{c_1}\right)r^2(r_1r_2)^{-1/2}\sin\theta\cos(\theta+\overline{\theta})\right],$$
$$u_y = \frac{\tau}{2}\left[\left(4c_2 - 2\frac{c_3}{c_1}\right)(r\cos\theta - (r_1r_2)^{1/2}\cos\overline{\theta}) + 2\left(c_4 - \frac{c_3}{c_1}\right)r^2(r_1r_2)^{-1/2}\sin\theta\sin(\theta+\overline{\theta})\right],$$
$$\begin{aligned} w_x = {} & \frac{R_1\tau}{(K_1-K_2)c_1}[2r^2(r_1r_2)^{-1/2}\sin\theta\cos(\theta-\overline{\theta}) - a^2r^2(r_1r_2)^{-3/2}\sin^2\theta\sin 3\overline{\theta}] \\ & - \frac{R_2\tau}{(K_1-K_2)c_1}[2r^2(r_1r_2)^{-1/2}\sin\theta\sin(\theta-\overline{\theta}) - a^2r^2(r_1r_2)^{-3/2}\sin^2\theta\cos 3\overline{\theta}], \end{aligned}$$
$$w_y = \frac{R_1\tau}{(K_1-K_2)c_1}[a^2r^2(r_1r_2)^{-3/2}\sin^2\theta\cos 3\overline{\theta} - 2r^2(r_1r_2)^{-1/2}\sin\theta\sin(\theta-\overline{\theta})]$$

$$+\frac{R_2\tau}{(K_1-K_2)c_1}[a^2r^2(r_1r_2)^{-3/2}\sin^2\theta\sin 3\overline{\theta}-2r^2(r_1r_2)^{-1/2}\sin\theta\cos(\theta-\overline{\theta})].$$

References

[1] Shechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951
Shechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951

[2] Bak P 1985 *Phys. Rev.* B **32** 5764

[3] Levine D and Steinhardt P J 1984 *Phys. Rev. Lett.* **53** 2477

[4] Hu C Z, Wang R H and Ding D H 2000 *Rep. Prog. Phys.* **63** 1

[5] Hu C Z, Wang R H, Yang W G and Ding D H 1996 *Acta Cryst.* A **52** 251

[6] Fan T Y and Mai Y W 2004 *Appl. Mech. Rev.* **57** 325

[7] Li X F and Fan T Y 1998 *Chin. Phys. Lett.* **15** 278

[8] Gao Y, Xu S P and Zhao B S 2007 *Pramana J. Phys.* **68** 803

[9] De P and Pelcovits A 1987 *Phys. Rev.* B **35** 8609

[10] Ding D H, Wang R H, Yang W G and Hu C Z 1995 *J. Phys.: Condens. Matter* **7** 5423

[11] Peng Y Z and Fan T Y 2000 *Chin. Phys.* **9** 764

[12] Ricker M, Bachteler J and Trebin H R 2001 *Eur. Phys. J.* B **23** 351

[13] Fan T Y 1999 *The Mathematical Theory of Elasticity of Quasi-crystals and Applications* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[14] Liu G T, Fan T Y and Guo R P 2004 *Int. J. Solids and Structures* **41** 3949

[15] Li L H and Fan T Y 2006 *J. Phys.: Condens. Matter* **18** 10631

[16] Shibuya T, Hashimoto T and Takeuchi S 1990 *Jpn. J. Appl. Phys.* **29** L349

[17] Tei–Ohkawa T, Edagawa K and Takeuchi S 1995 *J. Non. Cryst. Solids* **189** 25

[18] Fan T Y, Trebin H R, Messerschmidt U and Mai Y W 2004 *J. Phys.: Condens. Matter* **16** 5229

[19] Feuerbacher M, Metzmacher C, Wollgarten M, Urban K, Baufeld B, Bartsch M and Messerschmidt U 1997 *Mater. Sci. Engin.* A **233** 103

[20] Muskhelishvili N I 1963 *Some Basic Problems of the Mathematical Theory of Elasticity* (Noordhoff: Groningen)

[21] Li X F and Fan T Y 2002 *Chin. Phys.* **11** 266