Energy level formula for two moving charged particles with Coulomb coupling derived via the entangled state representations*

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This paper derives energy level formula for two moving charged particles with Coulomb coupling by making full use of two mutually conjugate entangled state representations. These newly introduced entangled state representations seem to provide a direct and convenient approach for solving certain dynamical problems for two-body systems.

Keywords: two-body dynamic problem, entangled state representation, energy level formula

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1. Introduction

The trapping or scattering of particles is often discussed in particle physics. In treating these problems one often chooses suitable representations to simplify the calculation according to quantum mechanical Hamiltonians. Representation theory in quantum mechanics was invented by Dirac.[1] Dirac emphasizes that choosing appropriate representations for tackling definite dynamic problems could save labour as much as possible. In Ref. [2] following Einstein–Podolsky–Rosen's idea of quantum entanglement,[3] Fan *et al.* have constructed the bipartite entangled state representations, which have many applications in quantum optics (see the review article Ref. [4]). In this work we demonstrate how to use the entangled state representations to solve the dynamic problem of two moving charged particles governed by the Hamiltonian, which includes not only the kinetic coupling $\kappa P_1 P_2$ but also the Coulomb potential $F(Q_1 - Q_2)$, $Q_1 - Q_2 > 0$ between two charged particles

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \kappa P_1 P_2 + F\frac{1}{Q_1 - Q_2}. \quad (1)$$

According to our knowledge, such a Hamiltonian has not been solved in the literature before because the introduction of the term $F/(Q_1 - Q_2)$ increases the complication of solving the corresponding Schrödinger equation. Here we intend to employ the entangled state representations to search for its energy level formula. In Section 2 we briefly introduce two mutually conjugate bipartite entangled state representations. In Section 3 we give an equivalent form of the Hamiltonian in Eq. (1) by using the entangled state $\langle\eta|$ representation. Then in Section 4 we obtain the energy level formula for H by virtue of the entangled state $\langle\xi|$ representation.

2. Two mutually conjugate entangled state representations

Let Q_r be two particles' relative position

$$Q_r = Q_1 - Q_2, \quad (2)$$

and P be the total momentum of two particles

$$P = P_1 + P_2, \quad (3)$$

where Q_i, P_i (i=1, 2) are related to Bose creation and annihilation operators by

$$Q_i = \left(a_i + a_i^\dagger\right)/\sqrt{2},$$
$$P_i = \left(a_i - a_i^\dagger\right)/(\sqrt{2}\,\mathrm{i}), \quad \left[a_i, a_j^\dagger\right] = \delta_{ij}. \quad (4)$$

Due to the fact that the operators Q_r and P are commutative, i.e., $[Q_r, P] = 0$, they have the common

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eigenvector $|\eta\rangle$. At this point we recall that it was Einstein–Podolsky–Rosen who firstly noticed the fact that $[Q_r, P] = 0$ was related to quantum entanglement. In two-mode Fock space the state $|\eta\rangle$ is expressed as[2]

$$|\eta\rangle = \exp\left[-\frac{|\eta|^2}{2} + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle, \quad (5)$$

where $|00\rangle$ is the two-mode vacuum state and η is complex, $\eta = \eta_1 + \mathrm{i}\,\eta_2$. By using Eqs. (2)–(5), we can prove

$$Q_r|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \quad P|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle. \quad (6)$$

Using the technique of integration within an ordered product (IWOP) of operators,[5−7] we can prove the orthonormal and complete relations

$$\langle\eta|\,\eta'\rangle = \pi\delta(\eta-\eta')\,\delta(\eta^*-\eta'^*),$$
$$\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1, \quad \mathrm{d}^2\eta = \mathrm{d}\eta_1\mathrm{d}\eta_2. \quad (7)$$

Since the centre-of-mass coordinate of two particles $Q_{\rm cm} = \mu_1 Q_1 + \mu_2 Q_2$ and the mass-weighted relative momentum $P_r = \mu_2 P_1 - \mu_1 P_2$ are respectively conjugate to P and Q_r, and the operators $Q_{\rm cm}$ and P_r are also commutative, they have the common eigenvector $|\xi\rangle$ conjugated to $|\eta\rangle$. The state $|\xi\rangle$ of course satisfies the eigenvector equations

$$Q_{\rm cm}|\xi\rangle = \sqrt{\frac{\lambda}{2}}\xi_1|\xi\rangle, \quad P_r|\xi\rangle = \sqrt{\frac{\lambda}{2}}\xi_2|\xi\rangle, \quad (8)$$

in two-mode Fock space the explicit form of $|\xi\rangle$ is[2]

$$|\xi\rangle = \exp\left\{-\frac{1}{2}|\xi|^2 + \frac{1}{\sqrt{\lambda}}[\xi + (\mu_1-\mu_2)\xi^*]a_1^\dagger + \frac{1}{\sqrt{\lambda}}[\xi^* - (\mu_1-\mu_2)\xi]a_2^\dagger + \frac{1}{\lambda}\left[(\mu_2-\mu_1)(a_1^{\dagger 2} - a_2^{\dagger 2}) - 4\mu_1\mu_2 a_1^\dagger a_2^\dagger\right]\right\}|00\rangle, \quad (9)$$

where ξ is complex, $\xi = \xi_1 + \mathrm{i}\,\xi_2$ and $\lambda = 2(\mu_1^2 + \mu_2^2)$. The state $|\xi\rangle$ obeys the orthonormal and complete relations

$$\langle\xi|\,\xi'\rangle = \pi\delta(\xi-\xi')\,\delta(\xi^*-\xi'^*),$$
$$\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi| = 1, \quad \mathrm{d}^2\xi = \mathrm{d}\xi_1\mathrm{d}\xi_2. \quad (10)$$

Thus the overlap between states $\langle\eta|$ and $|\xi\rangle$ can be calculated by using Eqs. (6), (7), (9) and (10),

$$\langle\eta\,|\xi\rangle = \sqrt{\frac{\lambda}{4}}\exp\{\mathrm{i}[(\mu_1-\mu_2)(\eta_1\eta_2 - \xi_1\xi_2) + \sqrt{\lambda}(\eta_1\xi_2 - \eta_2\xi_1)]\}, \quad (11)$$

note that we have taken $\hbar = 1$.

3. The equivalent form of the Hamiltonian in Eq. (1)

For the two-body interacting system governed by H, the total momentum P is conservative, i.e.,

$$[H, P] = 0, \quad (12)$$

thus we can introduce the common eigenstate of H and P,

$$H|p, E_n\rangle = E_n|p, E_n\rangle, \quad P|p, E_n\rangle = p|p, E_n\rangle. \quad (13)$$

To obtain the energy level formula, we rewrite the Hamiltonian H in Eq. (1) as

$$H = \left[\left(\frac{1}{2M} + \kappa\mu_1\mu_2\right)P^2 + \left(\frac{1}{2\mu} - \kappa\right)P_r^2 + \kappa(\mu_2-\mu_1)PP_r + F\frac{1}{Q_r}\right], \quad (14)$$

where $\mu = (m_1 m_2)/M$ is the reduced mass. Using Eqs. (6) and (13) we see that, in the entangled state $\langle\eta|$ representation, the energy eigenfunction equation becomes the following form:

$$\langle\eta|H|p, E_n\rangle = \left[\left(\frac{1}{M} + 2\kappa\mu_1\mu_2\right)\eta_2^2\langle\eta\,|p, E_n\rangle + \left(\frac{1}{2\mu} - \kappa\right)\langle\eta|P_r^2|p, E_n\rangle + \kappa(\mu_2-\mu_1)\sqrt{2}\eta_2\langle\eta|P_r|p, E_n\rangle + F\frac{1}{\sqrt{2}\eta_1}\langle\eta\,|p, E_n\rangle\right]. \quad (15)$$

Using Eqs. (8)–(11) we obtain the explicit form of P_r in the $\langle\eta|$ representation, i.e.,

$$\langle\eta|P_r = \langle\eta|\int\frac{\mathrm{d}^2\xi}{\pi}\sqrt{2}\xi_2|\xi\rangle\langle\xi| = -\sqrt{\frac{1}{2}}\left[\mathrm{i}\frac{\partial}{\partial\eta_1} + (\mu_1-\mu_2)\eta_2\right]\langle\eta|, \quad (16)$$

substituting Eq. (16) into Eq. (15) we obtain a new differential equation for $\langle\eta\,|p, E_n\rangle$

$$E_n \langle \eta \, | p, E_n \rangle = \left\{ \frac{1}{2} \left(\kappa - \frac{1}{2\mu} \right) \left[\frac{\partial}{\partial \eta_1} - \mathrm{i} \, (\mu_1 - \mu_2) \, \eta_2 \right]^2 - \mathrm{i} \, \eta_2 \kappa \, (\mu_2 - \mu_1) \left[\frac{\partial}{\partial \eta_1} - \mathrm{i} \, (\mu_1 - \mu_2) \, \eta_2 \right] + \left(\frac{1}{M} + 2\kappa \mu_1 \mu_2 \right) \eta_2^2 + F \frac{1}{\sqrt{2} \eta_1} \right\} \langle \eta \, | p, E_n \rangle \, , \tag{17}$$

which is the differential equation with respect to η_1, while η_2 appears as a parameter.

Making use of an ansatz

$$\langle \eta \, | p, E_n \rangle = \Psi_n \exp[\mathrm{i} \, (\mu_1 - \mu_2) \, \eta_1 \eta_2], \tag{18}$$

and the transformation

$$\exp[-\mathrm{i} \, (\mu_1 - \mu_2) \, \eta_1 \eta_2] \left[\frac{\partial}{\partial \eta_1} - \mathrm{i} \, (\mu_1 - \mu_2) \, \eta_2 \right] \exp[\mathrm{i} \, (\mu_1 - \mu_2) \, \eta_1 \eta_2] = \frac{\partial}{\partial \eta_1}, \tag{19}$$

we obtain the equation for Ψ_n

$$\left\{ -\frac{1}{2} \left(\frac{1}{2\mu} - \kappa \right) \frac{\partial^2}{\partial \eta_1^2} - \mathrm{i} \, \eta_2 \kappa \, (\mu_2 - \mu_1) \, \frac{\partial}{\partial \eta_1} + \left(\frac{1}{M} + 2\kappa \mu_1 \mu_2 \right) \eta_2^2 + F \frac{1}{\sqrt{2} \eta_1} \right\} \Psi_n = E_n \Psi_n. \tag{20}$$

Now let us assume that Ψ_n is taken as

$$\Psi_n = \exp \left[2 \, \mathrm{i} \, \eta_1 \eta_2 \kappa \, (\mu_1 - \mu_2) \, \mu / (1 - 2\mu k) \right] \psi_n = \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} \psi_n, \tag{21}$$

where

$$\rho = 2 \eta_2 \kappa \, (\mu_1 - \mu_2) \, \mu / (1 - 2\mu k), \tag{22}$$

the former two terms of left-hand side in Eq. (20) are converted into

$$\begin{aligned} & -\frac{1}{2} \left(\frac{1}{2\mu} - \kappa \right) \frac{\partial}{\partial \eta_1} \left[\frac{\partial}{\partial \eta_1} + \frac{4\mu}{1 - 2\mu k} \, \mathrm{i} \, \eta_2 \kappa \, (\mu_2 - \mu_1) \right] \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} \psi_n \\ = & -\frac{1}{2} \left(\frac{1}{2\mu} - \kappa \right) \frac{\partial}{\partial \eta_1} \left[\frac{\partial}{\partial \eta_1} - 2 \, \mathrm{i} \, \rho \right] \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} \psi_n. \end{aligned} \tag{23}$$

Furthermore, by using the following transformation

$$\mathrm{e}^{-\mathrm{i} \, \eta_1 \rho} \frac{\partial}{\partial \eta_1} \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} = \frac{\partial}{\partial \eta_1} + \mathrm{i} \, \rho, \tag{24}$$

equation (23) becomes

$$\begin{aligned} & -\frac{1}{2} \left(\frac{1}{2\mu} - \kappa \right) \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} \mathrm{e}^{-\mathrm{i} \, \eta_1 \rho} \frac{\partial}{\partial \eta_1} \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} \mathrm{e}^{-\mathrm{i} \, \eta_1 \rho} \left[\frac{\partial}{\partial \eta_1} - 2 \, \mathrm{i} \, \rho \right] \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} \psi_n \\ = & -\frac{1}{2} \left(\frac{1}{2\mu} - \kappa \right) \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} \left(\frac{\partial}{\partial \eta_1} + \mathrm{i} \, \rho \right) \left(\frac{\partial}{\partial \eta_1} - \mathrm{i} \, \rho \right) \psi_n \\ = & -\frac{1}{2} \left(\frac{1}{2\mu} - \kappa \right) \mathrm{e}^{\mathrm{i} \, \eta_1 \rho} \left(\frac{\partial^2}{\partial \eta_1^2} + \rho^2 \right) \psi_n. \end{aligned} \tag{25}$$

Substituting Eq. (25) into Eq. (20) leads to a new differential equation for ψ_n

$$\left\{ -\frac{1}{2} \left(\frac{1}{2\mu} - \kappa \right) \frac{\partial^2}{\partial \eta_1^2} + \frac{1 - \kappa^2 \mu M}{M (1 - 2\mu\kappa)} \eta_2^2 + F \frac{1}{\sqrt{2} \eta_1} \right\} \psi_n = E_n \psi_n. \tag{26}$$

Since in the $\langle \eta |$ representation there exist the following corresponding relations:

$$P_r \to -\sqrt{\frac{1}{2}} \left[\mathrm{i} \, \frac{\partial}{\partial \eta_1} + (\mu_1 - \mu_2) \, \eta_2 \right], \quad Q_r \to \sqrt{2} \eta_1, \quad P \to \sqrt{2} \eta_2, \tag{27}$$

then using the transformation in Eq. (19), we can convert the relations in Eq. (27) into

$$P_r \to -\mathrm{i} \, \frac{1}{\sqrt{2}} \frac{\partial}{\partial \eta_1}, \quad Q_r \to \sqrt{2} \eta_1, \quad P \to \sqrt{2} \eta_2, \tag{28}$$

so we rewrite the Hamiltonian H in Eq. (26) as follows:

$$H = \left(\frac{1}{2\mu} - \kappa \right) P_r^2 + \frac{1 - \kappa^2 \mu M}{2M (1 - 2\mu\kappa)} P^2 + F \frac{1}{Q_r}, \tag{29}$$

which is also the equivalent presentation of the Hamiltonian in Eq. (1) in $\langle \eta |$ representation.

4. Energy level formula for H obtained via the entangled state $\langle\xi|$ representation

To obtain the energy level formula of H, we first divide H into the following two parts

$$H = H_r + \frac{1-\kappa^2\mu M}{2M(1-2\mu\kappa)}P^2, \tag{30}$$

where

$$H_r = \left(\frac{1}{2\mu}-\kappa\right)P_r^2 + F\frac{1}{Q_r}. \tag{31}$$

Now we consider the energy levels of H_r via the entangled state $\langle\xi|$ representation. Operating $1/Q_r$ on the state $\langle\xi|$ we obtain

$$\begin{aligned}\langle\xi|\frac{1}{Q_r} &= \langle\xi|\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta|\frac{1}{Q_r}\int\frac{\mathrm{d}^2\xi'}{\pi}|\xi'\rangle\langle\xi'| \\ &= \frac{\lambda}{4\sqrt{2}}\int\frac{\mathrm{d}^2\xi'\mathrm{d}^2\eta}{\pi^2}\exp\{\mathrm{i}\left(\mu_1-\mu_2\right)\left(\xi_1\xi_2-\xi_1'\xi_2'\right) \\ &\quad +\mathrm{i}\sqrt{\lambda}\left[\eta_1\left(\xi_2-\xi_2'\right)-\eta_2\left(\xi_1-\xi_1'\right)\right]\}\frac{1}{\eta_1}\left\langle\xi_1',\xi_2'\right|. \end{aligned}\tag{32}$$

Performing the integration in Eq. (32) over η_2 and ξ_1' leads to

$$\langle\xi|\frac{1}{Q_r} = \frac{\lambda}{2\sqrt{2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi_2'}{\pi}\exp\left[\mathrm{i}\left(\mu_1-\mu_2\right)\xi_1\left(\xi_2-\xi_2'\right)\right]\int_{-\infty}^{\infty}\mathrm{d}\eta_1\frac{1}{\eta_1}\exp\left[\mathrm{i}\sqrt{\lambda}\eta_1\left(\xi_2-\xi_2'\right)\right]\left\langle\xi_1,\xi_2'\right|. \tag{33}$$

Using the function[8]

$$\theta\left(\tau\right) = \lim_{\varepsilon\to 0^+}\left(-\frac{1}{2\pi\,\mathrm{i}}\right)\int_{-\infty}^{\infty}\mathrm{e}^{-\,\mathrm{i}\,\tau\omega}\frac{\mathrm{d}\omega}{\omega+\mathrm{i}\,\varepsilon} = \begin{cases}1, & \tau>0,\\ 0, & \tau<0,\end{cases} \tag{34}$$

then integrating over η_1, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\frac{1}{\eta_1}\exp\left[\mathrm{i}\sqrt{\lambda}\eta_1\left(\xi_2-\xi_2'\right)\right] &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\frac{1}{\eta_1+\mathrm{i}\,\varepsilon}\exp\left[\mathrm{i}\sqrt{\lambda}\eta_1\left(\xi_2-\xi_2'\right)\right] \\ &= 2\pi\,\mathrm{i}\begin{cases}1, & \xi_2>\xi_2',\\ 0, & \xi_2<\xi_2'.\end{cases}\end{aligned}\tag{35}$$

So equation (33) becomes

$$\langle\xi|\frac{1}{Q_r} = \mathrm{i}\sqrt{\frac{\lambda}{2}}\int_{-\infty}^{\xi_2}\mathrm{d}\xi_2'\exp\left[\mathrm{i}\left(\mu_1-\mu_2\right)\xi_1\left(\xi_2-\xi_2'\right)\right]\left\langle\xi_1,\xi_2'\right|. \tag{36}$$

The eigenstate of Q_{cm} is $\sqrt{\lambda/2}\,\xi_1$ in the entangled state $|\xi\rangle$ and there is no operator Q_{cm} in H_r, so under these conditions we can take $\langle\xi| = \langle\xi_1=0,\xi_2|$, thus

$$\langle\xi_1=0,\xi_2|\frac{1}{Q_r} = \mathrm{i}\sqrt{\frac{\lambda}{2}}\int_{-\infty}^{\xi_2}\mathrm{d}\xi_2'\left\langle\xi_1=0,\xi_2'\right|, \tag{37}$$

in the $\langle\xi_1=0,\xi_2|$ representation the Schrödinger equation for H_r can be written as

$$\langle\xi_1=0,\xi_2|H_r|\psi\rangle = \left(\frac{1}{2\mu}-\kappa\right)\frac{\lambda}{2}\xi_2^2\psi(0,\xi_2)+\mathrm{i}\,F\sqrt{\frac{\lambda}{2}}\int_{-\infty}^{\xi_2}\mathrm{d}\xi_2'\psi(0,\xi_2') = E\psi(0,\xi_2). \tag{38}$$

Considering the binding energy $E<0$ and letting $f^2 = 4\mu E/(2\mu\kappa-1)\lambda$, we obtain

$$\frac{\psi(0,\xi_2)}{\int_{-\infty}^{\xi_2}\mathrm{d}\xi_2'\psi(0,\xi_2')} = -\,\mathrm{i}\,\frac{F\mu/(1-2\mu k)}{(\xi_2^2+f^2)}\sqrt{\frac{8}{\lambda}}, \tag{39}$$

its solution is

$$\int_{-\infty}^{\xi_2}\mathrm{d}\xi_2'\psi(0,\xi_2') = \exp\left(-\,\mathrm{i}\sqrt{\frac{8}{\lambda}}\frac{F\mu/(1-2\mu k)}{f}\arctan\frac{\xi_2}{f}\right)+C, \tag{40}$$

where C is an integration constant. From Eq. (40) we find that the function $\arctan(\xi_2/f)$ is equal to multiple values for a given value of ξ_2/f, so we only take $f = \sqrt{2/\lambda}(F\mu/(1-2\mu k)n)$ (n is an integer) in order to assure the uniqueness of the wave function $\psi(0,\xi_2')$. Thus the energy levels of H_r are found to be

$$E = \frac{(2\mu\kappa - 1)\lambda}{4\mu} f^2 = \frac{\mu F^2}{2n^2(2\mu k - 1)}, \tag{41}$$

so the energy level formula of H is given as

$$E_n = E + \frac{1-\kappa^2\mu M}{M(1-2\mu\kappa)}\eta_2^2, \tag{42}$$

where the second term of the right-hand side of Eq. (42) stands for the kinetic energy of the centre-of-mass because $\sqrt{2}\eta_2$ is the eigenvalue of P. Through analysing Eq. (42) we clearly see how the kinetic potential and Coulomb coupling potential affect the total energy E_n.

In summary, in this work on the basis of the entangled state $\langle\eta|$ and $\langle\xi|$ representations we have obtained the energy level formula for the Hamiltonian describing two moving charged particles with Coulomb coupling, so the entangled state representations seem to be a good candidate for studying the dynamic problem for a two-body system with mutual interaction. Regarding other applications of the entangled state representations we refer to Refs. [9]–[11].

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