Synchronization of hyperchaotic Chen systems: a class of the adaptive control approach*

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(Received 12 November 2009; revised manuscript received 10 December 2009)

The synchronization of hyperchaotic Chen systems is considered. An adaptive synchronization approach and a cascade adaptive synchronization approach are presented to synchronize a drive system and a response system. By utilizing an adaptive controller based on the dynamic compensation mechanism, exact knowledge of the systems is not necessarily required, and the synchronous speed is controllable by tuning the controller parameters. Sufficient conditions for the asymptotic stability of the two synchronization schemes are derived. Numerical simulation results demonstrate that the adaptive synchronization scheme with four control inputs and the cascade adaptive synchronization scheme with only one control signal are effective and feasible in chaos synchronization of hyperchaotic systems.

Keywords: hyperchaotic Chen systems, adaptive synchronization, cascade adaptive synchronization, synchronous speed

PACC: 0545

1. Introduction

Chaotic systems, a special kind of nonlinear system, exhibit complex and unpredictable dynamic behaviours such as high sensitivity to initial conditions, boundedness in the phase space, and existence of at least one positive Lyapunov exponent. Chaos, a nonlinear phenomenon, has been receiving a great deal of attention from various fields during the past decades. Hyperchaotic systems, compared with ordinary chaotic systems, have more than one positive Lyapunov exponent and clearly possess much more complicated dynamics. Because of their high complexity and pseudo randomness, hyperchaotic systems have been increasingly receiving much attention in secure communication, information processing, biological engineering, chemical processing and other areas.[1−4]

In secure communication, hyperchaos is a powerful tool. In fact, messages can be carried by hyperchaotic signals in a highly secure way. To recover the message injected into the hyperchaotic signals in the transmitter, synchronization of the drive system and the response system is necessary.

Since Pecora and Carroll[5] introduced a way to synchronize two identical chaotic systems, many methods have been proposed to realize the synchronization of chaotic systems, such as the feedback method,[6−9] the adaptive method,[10−14] the neural network method,[15] impulsive synchronization technology,[16] etc. However, broadly speaking, the number of controllers equals the number of system variables, if we realize chaos synchronization by the ways aforementioned. Furthermore, some of the ways are feasible under the hypothesis that all parameters of the drive system and the response system are known as prior knowledge. The parameters of the systems are also the parameters of the controllers. In practice, the value of system parameters cannot be exactly known. The existing uncertainties may destroy the synchronization. In addition, in most cases the control parameters are not directly related to the system response speed. More often than not, synchronization is achieved but the speed of synchronization is not perfect. In other words, the synchronous speed is not controllable.

Therefore, a model free synchronization approach, which makes the synchronous speed controllable, is valuable.

In this paper, an adaptive controller based on the dynamic compensation mechanism is utilized for the synchronization between two identical hyperchaotic Chen systems. First, four control inputs are applied to synchronize the drive system and the response sys-

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tem, and then only one control signal is added to the response system; synchronization between the drive system and the response system is also realized. Numerical simulations are presented to illustrate the feasibility and the effectiveness of the two approaches.

The rest of this paper is organized as follows. The synchronization of hyperchaotic Chen systems is stated and the adaptive synchronization scheme is designed in Section 2. In Section 3, a cascade synchronization scheme with only one control input is proposed, and simulation results are given to demonstrate the scheme presented. Finally, conclusions are drawn in Section 4.

2. Adaptive synchronization of hyperchaotic Chen systems

2.1. Problem statement

The hyperchaotic Chen system[17] is given by

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1) + x_4,\\ \dot{x}_2 &= dx_1 - x_1x_3 + cx_2,\\ \dot{x}_3 &= x_1x_2 - bx_3,\\ \dot{x}_4 &= x_2x_3 + rx_4,\end{aligned} \tag{1}$$

where x_1, x_2, x_3 and x_4 are the state variables; a, b, c, d and r are the real constants.

When $a = 35$, $b = 3$, $c = 12$, $d = 7$, and $r \in (0.085, 0.798]$, system (1) is hyperchaotic. Here we assume that system (1) is the drive system, and the response system is given below

$$\begin{aligned}\dot{y}_1 &= a(y_2 - y_1) + y_4 + u_1,\\ \dot{y}_2 &= dy_1 - y_1y_3 + cy_2 + u_2,\\ \dot{y}_3 &= y_1y_2 - by_3 + u_3,\\ \dot{y}_4 &= y_2y_3 + ry_4 + u_4,\end{aligned} \tag{2}$$

where $U = [u_1, u_2, u_3, u_4]^{\mathrm{T}}$ is such a control input that the two hyperchaotic systems are synchronous. Subtracting system (1) from system (2), we have an error dynamical system between systems (1) and (2), i.e.

$$\begin{aligned}\dot{e}_1 &= a(e_2 - e_1) + e_4 + u_1,\\ \dot{e}_2 &= de_1 + ce_2 - y_1e_3 - x_3e_1 + u_2,\\ \dot{e}_3 &= -be_3 + x_2e_1 + y_1e_2 + u_3,\\ \dot{e}_4 &= re_4 + x_3e_2 + y_2e_3 + u_4,\end{aligned} \tag{3}$$

where $e_i = y_i - x_i, i = 1, 2, 3, 4$. We aim to achieve synchronization between two hyperchaotic Chen systems, i.e., $\lim\limits_{t\to\infty} e_i = 0$ with the help of control inputs u_i, i = 1–4.

2.2. Adaptive synchronization scheme design

In this paper, an adaptive controller based on the dynamic compensation mechanism (proposed by Tornambe, see Ref. [18] for details) is adopted. For error dynamical system (3), $u_1 - u_4$ are adaptive controllers.

When the relative degree is one, the adaptive control law is

$$\begin{aligned}u &= -h_0(y - y_r) - \hat{d},\\ \hat{d} &= \xi + k_0(y - y_r),\\ \dot{\xi} &= -k_0\xi - k_0^2(y - y_r) - k_0u,\end{aligned} \tag{4}$$

in which, $\hat{d}$ is the extended state observer, which estimates and compensates the uncertainties; y is the output of the system; y_r is the desired trajectory; h_0 determines the response speed of the system; h_0 is chosen to be such a value that the root of $s + h_0$ is in the open left half-plane; ξ is the intermediate variable; k_0 is the tunable parameter, which determines the stability of the system.[19,20] u_i, i = 1–4, are given below

$$\begin{aligned}u_i &= -h_{0i}e_i - \hat{d}_i,\\ \hat{d}_i &= \xi_i + k_{0i}e_i,\\ \dot{\xi}_i &= -k_{0i}\xi_i - k_{0i}^2e_i - k_{0i}u_i.\end{aligned} \tag{5}$$

After identical transformation, u_i, i = 1–4, are given below

$$\begin{aligned}u_i &= c_{i1}e_i - \xi_i,\\ \dot{\xi}_i &= c_{i2}e_i,\end{aligned} \tag{6}$$

in which $c_{i1} = -(h_{0i} + k_{0i})$, and $c_{i2} = h_{0i}k_{0i}, i$ = 1–4.

Substituting Eq. (6) into system (3), we derive closed-loop error dynamics as follows:

$$\begin{aligned}\dot{e}_1 &= (-a + c_{11})e_1 + ae_2 + e_4 - \xi_1,\\ \dot{e}_2 &= de_1 + (c + c_{21})e_2 - \xi_2 - y_1e_3 - x_3e_1,\\ \dot{e}_3 &= (-b + c_{31})e_3 - \xi_3 + y_1e_2 + x_2e_1,\\ \dot{e}_4 &= (r + c_{41})e_4 - \xi_4 + y_2e_3 + x_3e_2,\\ \dot{\xi}_1 &= c_{12}e_1,\\ \dot{\xi}_2 &= c_{22}e_2,\\ \dot{\xi}_3 &= c_{32}e_3,\\ \dot{\xi}_4 &= c_{42}e_4.\end{aligned} \tag{7}$$

System (7) can be written in the following compact form:

$$\dot{\varsigma} = A\varsigma + O(\varsigma, t), \tag{8}$$

where $\varsigma = (e_1, e_2, e_3, e_4, \xi_1, \xi_2, \xi_3, \xi_4)^{\mathrm{T}}$,

$$A = \begin{pmatrix} -a + c_{11} & a & 0 & 1 & \\ d & c + c_{21} & 0 & 0 & -\boldsymbol{I}_{4\times 4} \\ 0 & 0 & -b + c_{31} & 0 & \\ 0 & 0 & 0 & r + c_{41} & \\ & \mathrm{diag}(c_{12}, c_{22}, c_{32}, c_{42}) & & & \boldsymbol{0}_{4\times 4} \end{pmatrix}_{8\times 8}$$

$$O(\varsigma, t) = (0, -y_1 e_3 - x_3 e_1, y_1 e_2 + x_2 e_1, y_2 e_3 + x_3 e_2, 0, 0, 0, 0)^{\mathrm{T}}.$$

Here, in matrix A, $\boldsymbol{I}_{4\times 4}$ is a fourth order identity matrix.

Lemma[9] Consider a nonlinear system

$$\dot{x} = A(t)x + O(x, t), \tag{9}$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $O(x, t)$ denotes a nonlinear term, which satisfies $O(0, t) = 0$. The zero solution of system (9) is uniformly asymptotically stable, if the following conditions hold:

1) $\forall t \geq 0$, $\lim\limits_{\|x\|\to 0} \dfrac{\|O(x,t)\|}{||x||} = 0$;

2) $\forall t \geq 0$, $A(t)$ is bounded;

3) The zero solution of the linear system $\dot{x} = A(t)x$ is uniformly asymptotically stable.

According to the Lemma, we have the following theorem.

Theorem 1 For response system (2), the controller parameters are chosen such that the linear system $\dot{x} = A\varsigma$ is uniformly asymptotically stable, i.e. equation (14) is satisfied, the error dynamical system (8) is uniformly asymptotically stable, and thus response system (2) is synchronized with drive system (1).

Proof For closed-loop error dynamical system (8), $O(\varsigma, t)|_{\varsigma=0} = 0$ is satisfied for any t, and we have

$$\frac{\|O(\varsigma, t)\|}{||\varsigma||} \geq 0. \tag{10}$$

Regardless of their initial conditions, the states of a chaotic system are bounded, i.e., there exists a positive constant M such that $|x_i(t)| \leq M, |y_i(t)| \leq M$ for any initial conditions.

Furthermore, we have

$$\begin{aligned} \frac{\|O(\varsigma, t)\|}{||\varsigma||} &= \sqrt{\frac{(-y_1 e_3 - x_3 e_1)^2 + (y_1 e_2 + x_2 e_1)^2 + (y_2 e_3 + x_3 e_2)^2}{e_1^2 + e_2^2 + e_3^2 + e_4^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2}} \\ &\leq \frac{\sqrt{(Me_3 + Me_1)^2 + (Me_2 + Me_1)^2 + (Me_3 + Me_2)^2}}{|e_1|}. \end{aligned} \tag{11}$$

From Eqs. (10) and (11) we have

$$0 \leq \frac{\|O(\varsigma, t)\|}{||\varsigma||} \leq \frac{\sqrt{(Me_3 + Me_1)^2 + (Me_2 + Me_1)^2 + (Me_3 + Me_2)^2}}{|e_1|}, \tag{12}$$

and

$$\lim_{\|\varsigma\|\to 0} \frac{\sqrt{(Me_3 + Me_1)^2 + (Me_2 + Me_1)^2 + (Me_3 + Me_2)^2}}{|e_1|} = 0. \tag{13}$$

According to the Squeeze theorem we have $\lim\limits_{\|\varsigma\|\to 0} \|O(\varsigma, t)\|/||\varsigma|| = 0$, which means that condition 1) is satisfied. The matrix A in Eq. (8) is a constant matrix, so condition 2) is also satisfied. The characteristic polynomial of matrix A in Eq. (8) is

$$f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - (-a + c_{11}) & -a & 0 & -1 & \\ -d & \lambda - (c + c_{21}) & 0 & 0 & \boldsymbol{I}_{4\times 4} \\ 0 & 0 & \lambda - (-b + c_{31}) & 0 & \\ 0 & 0 & 0 & \lambda - (r + c_{41}) & \\ & -\mathrm{diag}(c_{12}, c_{22}, c_{32}, c_{42}) & & & \lambda \boldsymbol{I}_{4\times 4} \end{vmatrix}_{8\times 8}$$

$$= \lambda^8 + \varepsilon_7\lambda^7 + \varepsilon_6\lambda^6 + \varepsilon_5\lambda^5 + \varepsilon_4\lambda^4 + \varepsilon_3\lambda^3 + \varepsilon_2\lambda^2 + \varepsilon_1\lambda + \varepsilon_0,$$

where

$$k_1 = -a + c_{11}, \quad k_2 = -b + c_{31}, \quad k_3 = c + c_{21}, \quad k_4 = r + c_{41},$$
$$\theta_1 = c_{32} + c_{22} + k_1k_3 + c_{12} - ad + k_1k_2 + k_2k_3,$$
$$\theta_2 = -c_{32}(k_1 + k_3) - (c_{22}k_1 + c_{12}k_3) - k_2(c_{22} + k_1k_3 + c_{12} - ad),$$
$$\theta_3 = c_{32}(c_{22} + k_1k_3 + c_{12} - ad) + c_{12}c_{22} + k_2(c_{22}k_1 + c_{12}k_3),$$
$$\theta_4 = -c_{32}(c_{22}k_1 + c_{12}k_3) - c_{22}c_{12}k_2,$$
$$\varepsilon_7 = -(k_1 + k_2 + k_3 + k_4), \quad \varepsilon_6 = k_4(k_1 + k_2 + k_3) + c_{42} + \theta_1, \quad \varepsilon_5 = \theta_2 - k_4\theta_1 - c_{42}(k_1 + k_2 + k_3),$$
$$\varepsilon_4 = \theta_3 - k_4\theta_2 + c_{42}\theta_1, \quad \varepsilon_3 = \theta_4 - k_4\theta_3 + c_{42}\theta_2, \quad \varepsilon_2 = c_{32}c_{22}c_{12} - k_4\theta_4 + c_{42}\theta_3,$$
$$\varepsilon_1 = c_{42}\theta_4 - k_4c_{32}c_{22}c_{12}, \quad \varepsilon_0 = c_{42}c_{32}c_{22}c_{12}.$$

If $\varepsilon_i > 0$, $(i = 0, 1 \ldots, 7)$, according to the Hurwitz criterion, $f(\lambda)$ is of Hurwitz type if and only if the following conditions are satisfied (where $\varepsilon_s = 0$, when $s < 0$ or $s > 8$), i.e.

$$\mu_i = \begin{vmatrix} \varepsilon_1 & \varepsilon_0 & 0 & & \cdots & & 0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \varepsilon_{2i-3} & \varepsilon_{2i-4} & \varepsilon_{2i-5} & \varepsilon_{2i-6} & \varepsilon_{2i-7} & \cdots & \varepsilon_{i-2} \\ \varepsilon_{2i-1} & \varepsilon_{2i-2} & \varepsilon_{2i-3} & \varepsilon_{2i-4} & \varepsilon_{2i-5} & \cdots & \varepsilon_i \end{vmatrix} > 0,$$
$$i = 1, 2, \ldots, 8, \tag{14}$$

that is,

$$\mu_1 = \varepsilon_1 > 0, \quad \mu_2 = \begin{vmatrix} \varepsilon_1 & \varepsilon_0 \\ \varepsilon_3 & \varepsilon_2 \end{vmatrix} > 0,$$

$$\mu_3 = \begin{vmatrix} \varepsilon_1 & \varepsilon_0 & 0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_5 & \varepsilon_4 & \varepsilon_3 \end{vmatrix} > 0,$$

$$\mu_4 = \begin{vmatrix} \varepsilon_1 & \varepsilon_0 & 0 & 0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 \\ \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 \\ \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 \end{vmatrix} > 0,$$

$$\mu_5 = \begin{vmatrix} \varepsilon_1 & \varepsilon_0 & 0 & 0 & 0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & 0 \\ \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 & \varepsilon_3 \\ 0 & 1 & \varepsilon_7 & \varepsilon_6 & \varepsilon_5 \end{vmatrix} > 0,$$

$$\mu_6 = \begin{vmatrix} \varepsilon_1 & \varepsilon_0 & 0 & 0 & 0 & 0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & 0 & 0 \\ \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 \\ \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 \\ 0 & 1 & \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 \\ 0 & 0 & 0 & 1 & \varepsilon_7 & \varepsilon_6 \end{vmatrix} > 0,$$

$$\mu_7 = \begin{vmatrix} \varepsilon_1 & \varepsilon_0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & 0 & 0 & 0 \\ \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & 0 \\ \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ 0 & 1 & \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 & \varepsilon_3 \\ 0 & 0 & 0 & 1 & \varepsilon_7 & \varepsilon_6 & \varepsilon_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & \varepsilon_7 \end{vmatrix} > 0,$$

and

$$\mu_8 = \begin{vmatrix} \varepsilon_1 & \varepsilon_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & 0 & 0 & 0 & 0 \\ \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & 0 & 0 \\ \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 \\ 0 & 1 & \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 \\ 0 & 0 & 0 & 1 & \varepsilon_7 & \varepsilon_6 & \varepsilon_5 & \varepsilon_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & \varepsilon_7 & \varepsilon_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} > 0.$$

Hence, by choosing proper controller parameters condition 3) is satisfied. For system (8), conditions 1), 2) and 3) are all satisfied. According to the lemma, the zero solution of system (8) is uniformly asymptotically stable.

This implies that response system (2) is synchronized with drive system (1). By this we conclude the proof of the theorem 1.

2.3. Numerical simulations

In the numerical simulations, the fourth order Runge–Kutta method is used to solve the systems in time steps of 0.001. The parameters of hyperchaotic systems are chosen to be $a = 35$, $b = 3$, $c = 12$, $d = 7$, $r = 0.5$ and the initial conditions are $(x_1(0), x_2(0), x_3(0), x_4(0))^{\mathrm{T}} = (-0.1, 0.2, -0.6, 0.4)^{\mathrm{T}}$, $(y_1(0), y_2(0), y_3(0), y_4(0))^{\mathrm{T}} = (-1, 0.4, -0.2, 1)^{\mathrm{T}}$ (as in Ref. [8]).

The simulation time is 10 s, control inputs are added at the beginning of the simulation. Controller parameters are chosen to be $k_{01} = 2$, $h_{01} = 0.01$; $k_{02} = 26$, $h_{02} = 0.01$; $k_{03} = 18$, $h_{03} = 0.01$; $k_{04} = 10$, and $h_{04} = 0.01$.

For this selection, $\varepsilon_0 = 0.0001$, $\varepsilon_1 = 0.1981$, $\varepsilon_2 = 43.7261$, $\varepsilon_3 = 3026.3867$, $\varepsilon_4 = 55278.1052$, $\varepsilon_5 = 18578.1313$, $\varepsilon_6 = 2031.0056$, $\varepsilon_7 = 81.54$. $\mu_1 = \varepsilon_1 = 0.1981$, $\mu_2 = 8.3779$, $\mu_3 = 2.32 \times 10^4$, $\mu_4 = 1.27 \times 10^9$, $\mu_5 = 2.35 \times 10^{13}$, $\mu_6 = 4.21 \times 10^{16}$, and $\mu_7 = \mu_8 = 2.99 \times 10^{18}$.

Obviously, conditions given in Eq. (14) are satisfied, i.e. $\mu_i > 0$, $i = 1$–8.

According to Theorem 1, response system (2) is synchronized with the drive system (1).

The synchronization trajectories between the drive system and the response system are given in Figs. 1(a)–1(d). Synchronization errors are shown in Figs. 2(a)–2(d). It is clearly seen that the error dynamical system is driven to the origin immediately, the states of drive and response systems overlap with each other. Figure 3 shows the control inputs of four adaptive controllers. The IAE values of e_1–e_4 are shown below, respectively:

$$\mathrm{IAE}e_1 = 0.0202,\ \mathrm{IAE}e_2 = 0.0048,$$
$$\mathrm{IAE}e_3 = 0.0201,\ \mathrm{IAE}e_4 = 0.0702,$$

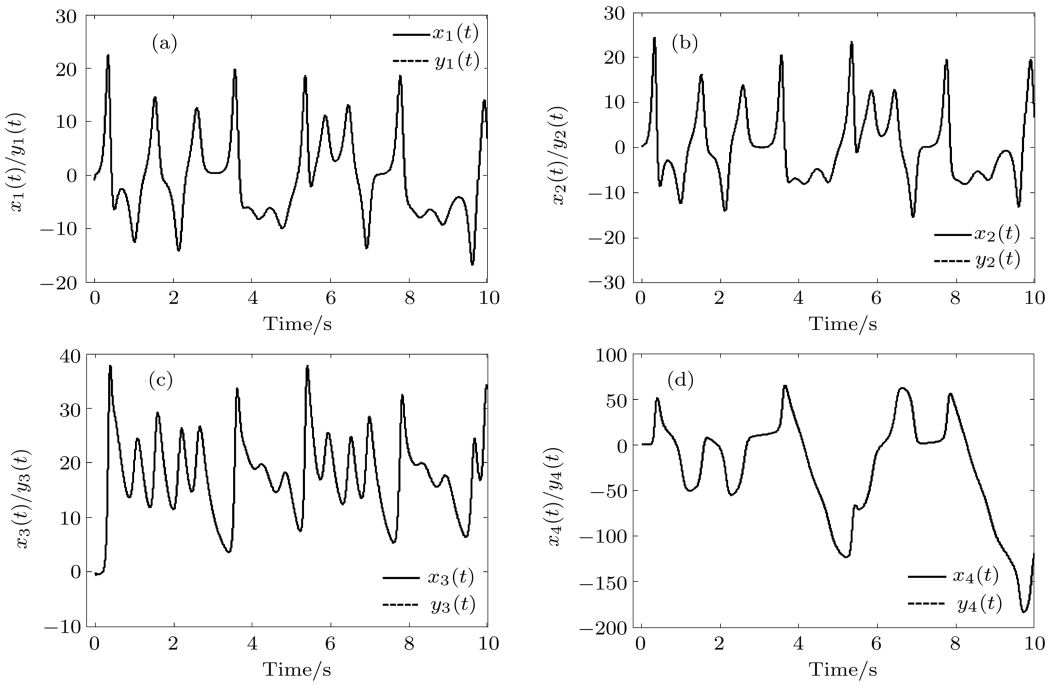
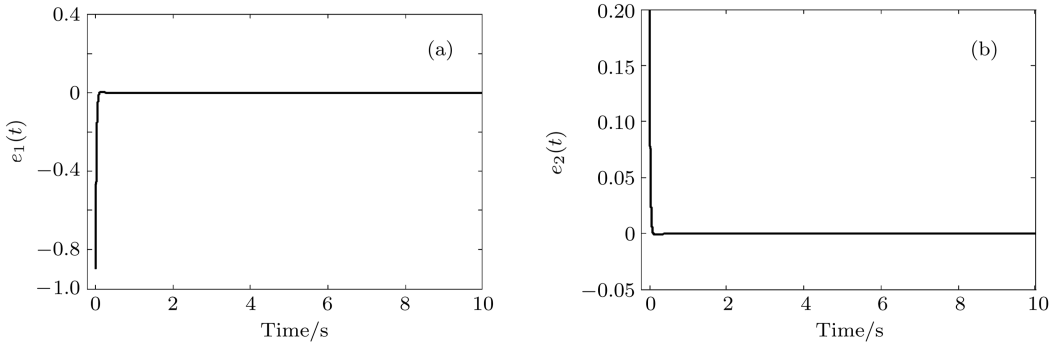


Fig. 1. Synchronization trajectories of the adaptive synchronization scheme.



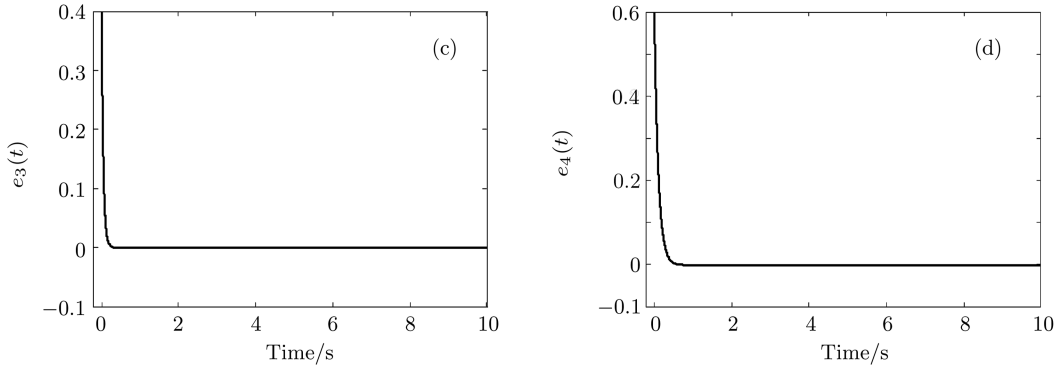


Fig. 2. Synchronization errors of the adaptive synchronization scheme.

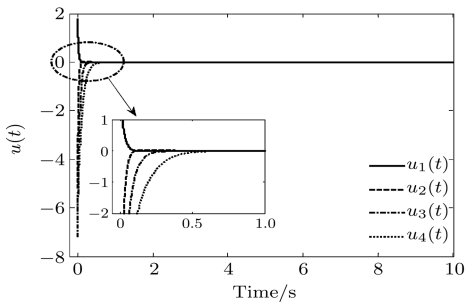


Fig. 3. Adaptive control inputs.

In order to make a comparison, figures 4(a)–4(d) show the synchronization errors given in Ref. [8] (under the same conditions). Active control inputs are presented in Fig. 5. Similarly, the IAE values of e_1–e_4 in Ref. [8] are given below, respectively:

$$\text{IAE}e_1 = 0.0257,\ \text{IAE}e_2 = 0.1994,$$
$$\text{IAE}e_3 = 0.1333,\ \text{IAE}e_4 = 0.6001.$$

Under the same conditions, the synchronization errors of active control adopted in Ref. [8] and control signals are shown below, respectively.

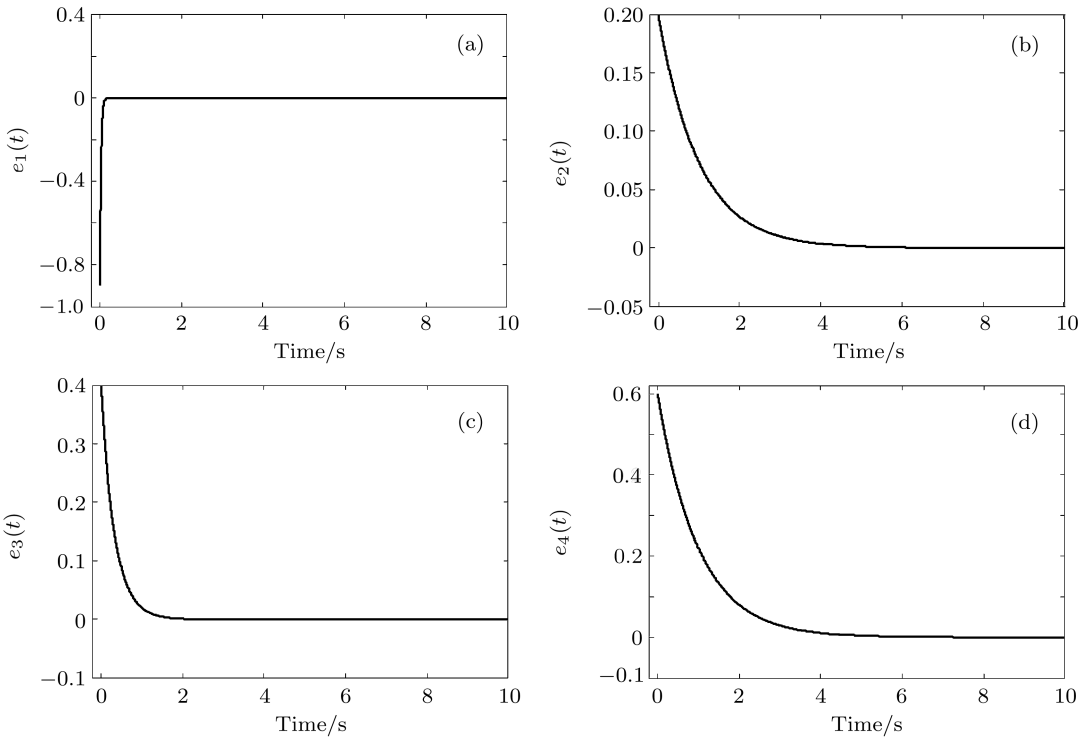


Fig. 4. Synchronization errors of active control in Ref. [8].

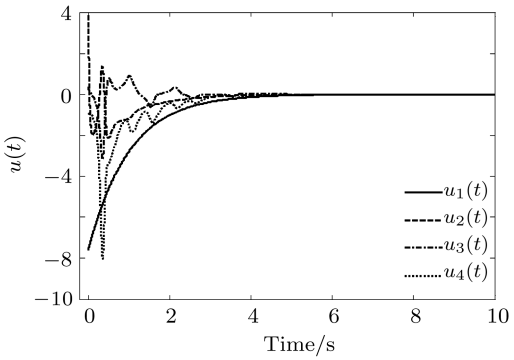


Fig. 5. Active control inputs in Ref. [8].

From the figures, it can be seen that the synchronization errors converge to zero and two hyperchaotic systems have indeed achieved chaos synchronization. From the rate of convergence, the IAE values and the control inputs we may safely arrive at a conclusion that under the same conditions and with the same number of control inputs the adaptive synchronization scheme is superior to the active control scheme used in Ref. [8]. (To exhibit the error dynamics more clearly, the Y-coordinates of Figs. 4(b)–4(d) start from -0.05 and -0.1.)

3. Cascade adaptive synchronization of hyperchaotic Chen systems

3.1. Cascade synchronization scheme

The use of four control inputs does achieve chaos synchronization, but whether less control signals or even one control signals can also work is a meaningful problem in terms of the control theory. In other words, it is of significance to realize synchronization with least control cost.

In this section, a cascade scheme is employed to achieve the synchronization. With the cascade scheme, only one control input is needed.

Here, system (1) is the drive system, and the response system is defined below

$$\begin{aligned}\dot{y}_1 &= a(y_2 - y_1) + y_4,\\ \dot{y}_2 &= dy_1 - y_1y_3 + cy_2 + u,\\ \dot{y}_3 &= y_1y_2 - by_3,\\ \dot{y}_4 &= y_2y_3 + ry_4,\end{aligned} \tag{15}$$

where u is the control input such that the two hyperchaotic systems can be synchronized.

Accordingly, the error dynamical system between systems (1) and (15) is given below

$$\begin{aligned}\dot{e}_1 &= a(e_2 - e_1) + e_4,\\ \dot{e}_2 &= de_1 + ce_2 - y_1e_3 - x_3e_1 + u,\\ \dot{e}_3 &= -be_3 + x_2e_1 + y_1e_2,\\ \dot{e}_4 &= re_4 + x_3e_2 + y_2e_3,\end{aligned} \tag{16}$$

where $e_i = y_i - x_i$, $i = 1, 2, 3, 4$. The control input u is designed such that the drive and response systems are synchronous.

For error dynamical system (16), the dynamics of e_1 and e_3, when e_2 and e_4 are zero, is stable. Let e_4 be the output, i.e. $y = e_4$ and rewrite system (16) as follows:

$$\begin{aligned}\dot{e}_4 &= re_4 + x_3e_2 + y_2e_3,\\ \dot{e}_2 &= de_1 + ce_2 - y_1e_3 - x_3e_1 + u,\\ \dot{e}_1 &= a(e_2 - e_1) + e_4,\\ \dot{e}_3 &= -be_3 + x_2e_1 + y_1e_2,\\ y &= e_4.\end{aligned} \tag{17}$$

For system (17), the cascade synchronization scheme is employed, and the control block diagram is given in Fig. 6 (C_1 and C_2 are both adaptive controllers utilized in Section 2).

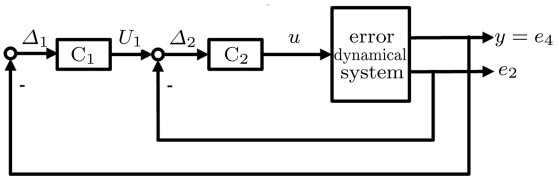


Fig. 6. Cascade adaptive synchronization scheme.

For C_1 and C_2, the control laws, after identical transformation, are given below, respectively:

$$\begin{aligned}U_1 &= c_{11}e_4 - \xi_1,\\ \dot{\xi}_1 &= c_{12}e_4,\end{aligned} \tag{18}$$

$$\begin{aligned}u &= c_{21}(e_2 - U_1) - \xi_2,\\ \dot{\xi}_2 &= c_{22}(e_2 - U_1),\end{aligned} \tag{19}$$

in which

$$\begin{aligned}&c_{11} = -(h_{01} + k_{01}), \quad c_{12} = h_{01}k_{01},\\ &c_{21} = -(h_{02} + k_{02}), \quad \text{and} \quad c_{22} = h_{02}k_{02},\end{aligned}$$

where U_1 is the virtual control input such that e_4 tends to zero. Obviously, U_1 is the desired trajectory of e_2, and u is the real control input, which makes e_2 track U_1.

Substituting Eqs. (18) and (19) into system (16), we derive a closed-loop error dynamical system as follows:

$$\begin{aligned}
\dot{e}_1 &= -ae_1 + ae_2 + e_4,\\
\dot{e}_2 &= de_1 + (c + c_{21})e_2 - c_{21}c_{11}e_4\\
&\quad + c_{21}\xi_1 - \xi_2 - y_1e_3 - x_3e_1,\\
\dot{e}_3 &= -be_3 + y_1e_2 + x_2e_1,\\
\dot{e}_4 &= re_4 + y_2e_3 + x_3e_2,\\
\dot{\xi}_1 &= c_{12}e_4,\\
\dot{\xi}_2 &= c_{22}e_2 - c_{22}c_{11}e_4 + c_{22}\xi_1. \qquad (20)
\end{aligned}$$

As in Section 2, system (20) can be written in the following compact form:

$$\dot{\eta} = A\eta + O(\eta, t), \qquad (21)$$

where $\eta = (e_1, e_2, e_3, e_4, \xi_1, \xi_2)^{\mathrm{T}}$,

$$A = \begin{pmatrix}
-a & a & 0 & 1 & 0 & 0\\
d & c + c_{21} & 0 & -c_{21}c_{11} & c_{21} & -1\\
0 & 0 & -b & 0 & 0 & 0\\
0 & 0 & 0 & r + c_{11} & 0 & 0\\
0 & 0 & 0 & c_{12} & c_{11} & 0\\
0 & c_{22} & 0 & -c_{22}c_{11} & c_{22} & 0
\end{pmatrix}, \quad O(\eta, t) = \begin{pmatrix}
0\\
-y_1e_3 - x_3e_1\\
y_1e_2 + x_2e_1\\
y_2e_3 + x_3e_2 - c_{11}e_4\\
-c_{11}\xi_1\\
0
\end{pmatrix}.$$

Theorem 2 For response system (15), the controller parameters are chosen such that the linear system $\dot{\eta} = A\eta$ is uniformly asymptotically stable, i.e. equation (22) is satisfied, error dynamical system (21) is uniformly asymptotically stable, and thus response system (15) is synchronized with drive system (1).

Proof $\forall t, O(\eta, t)|_{\eta=0} = 0$, and $\|O(\eta, t)\|/||\eta|| \geq 0$. As in Section 2, $|x_i(t)| \leq M, |y_i(t)| \leq M$, we have

$$\begin{aligned}
\frac{\|O(\eta, t)\|}{||\eta||} &= \sqrt{\frac{(-y_1e_3 - x_3e_1)^2 + (y_1e_2 + x_2e_1)^2 + (y_2e_3 + x_3e_2 - c_{11}e_4)^2 + (c_{11}\xi_1)^2}{e_1^2 + e_2^2 + e_3^2 + e_4^2 + \xi_1^2 + \xi_2^2}}\\
&\leq \frac{\sqrt{(Me_3 + Me_1)^2 + (Me_2 + Me_1)^2 + (Me_3 + Me_2 - c_{11}e_4)^2 + (c_{11}\xi_1)^2}}{|e_1|}.
\end{aligned}$$

Therefore,

$$0 \leq \frac{\|O(\eta, t)\|}{||\eta||} \leq \frac{\sqrt{(Me_3 + Me_1)^2 + (Me_2 + Me_1)^2 + (Me_3 + Me_2 - c_{11}e_4)^2 + (c_{11}\xi_1)^2}}{|e_1|},$$

and

$$\lim_{\|\eta\| \to 0} \frac{\sqrt{(Me_3 + Me_1)^2 + (Me_2 + Me_1)^2 + (Me_3 + Me_2 - c_{11}e_4)^2 + (c_{11}\xi_1)^2}}{|e_1|} = 0,$$

then $\lim\limits_{\|\eta\| \to 0} \|O(\eta, t)\|/||\eta|| = 0$. Condition 1) is satisfied. As matrix A is a constant matrix, condition 2) is also satisfied. The characteristic polynomial of matrix A in Eq. (21) is

$$\begin{aligned}
f(\lambda) &= |\lambda I - A|\\
&= \begin{vmatrix}
\lambda + a & -a & 0 & -1 & 0 & 0\\
-d & \lambda - (c + c_{21}) & 0 & c_{21}c_{11} & -c_{21} & 1\\
0 & 0 & \lambda + b & 0 & 0 & 0\\
0 & 0 & 0 & \lambda - (r + c_{11}) & 0 & 0\\
0 & 0 & 0 & -c_{12} & \lambda - c_{11} & 0\\
0 & -c_{22} & 0 & c_{22}c_{11} & -c_{22} & \lambda
\end{vmatrix}\\
&= (\lambda + b)(\lambda - c_{11})[\lambda - (r + c_{11})] \begin{vmatrix}
\lambda + a & -a & 0\\
-d & \lambda - (c + c_{21}) & 1\\
0 & -c_{22} & \lambda
\end{vmatrix}
\end{aligned}$$

$$= \lambda^6 + p_5\lambda^5 + p_4\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0,$$

where

$$\begin{aligned}
p_5 &= b - r - 2c_{11} + a - c - c_{21},\\
p_4 &= c_{22} - ad - ac - ac_{21} + (b - r - 2c_{11})(a - c - c_{21}) - [c_{11}(b - r - c_{11}) + b(r + c_{11})],\\
p_3 &= ac_{22} + (b - r - 2c_{11})(c_{22} - ad - ac - ac_{21}) - [c_{11}(b - r - c_{11}) + b(r + c_{11})](a - c - c_{21})\\
&\quad + c_{11}b(r + c_{11}),\\
p_2 &= -[c_{11}(b - r - c_{11}) + b(r + c_{11})](c_{22} - ad - ac - ac_{21}) + ac_{22}(b - r - 2c_{11})\\
&\quad + c_{11}b(r + c_{11})(a - c - c_{21}),\\
p_1 &= -ac_{22}[c_{11}(b - r - c_{11}) + b(r + c_{11})] + c_{11}b(r + c_{11})(c_{22} - ad - ac - ac_{21}),\\
p_0 &= abc_{22}c_{11}(r + c_{11}).
\end{aligned}$$

If $p_i > 0,\ (i = 0, 1 \ldots, 5)$, according to the Hurwitz criterion, $f(\lambda)$ is of Hurwitz type if and only if the following conditions are satisfied (where $p_s = 0$, when $s < 0$ or $s > 6$), i.e.

$$\rho_i = \begin{vmatrix} p_1 & p_0 & 0 & & \cdots & & 0\\ p_3 & p_2 & p_1 & p_0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ p_{2i-3} & p_{2i-4} & p_{2i-5} & p_{2i-6} & p_{2i-7} & \cdots & p_{i-2}\\ p_{2i-1} & p_{2i-2} & p_{2i-3} & p_{2i-4} & p_{2i-5} & \cdots & p_i \end{vmatrix} > 0, \quad i = 1, 2, \ldots, 6, \tag{22}$$

that is,

$$\rho_1 = p_1 > 0, \quad \rho_2 = \begin{vmatrix} p_1 & p_0\\ p_3 & p_2 \end{vmatrix} > 0, \quad \rho_3 = \begin{vmatrix} p_1 & p_0 & 0\\ p_3 & p_2 & p_1\\ p_5 & p_4 & p_3 \end{vmatrix} > 0, \quad \rho_4 = \begin{vmatrix} p_1 & p_0 & 0 & 0\\ p_3 & p_2 & p_1 & p_0\\ p_5 & p_4 & p_3 & p_2\\ 0 & 1 & p_5 & p_4 \end{vmatrix} > 0,$$

$$\rho_5 = \begin{vmatrix} p_1 & p_0 & 0 & 0 & 0\\ p_3 & p_2 & p_1 & p_0 & 0\\ p_5 & p_4 & p_3 & p_2 & p_1\\ 0 & 1 & \varepsilon_5 & \varepsilon_4 & \varepsilon_3\\ 0 & 0 & 0 & 1 & \varepsilon_5 \end{vmatrix} > 0, \quad \text{and} \quad \rho_6 = \begin{vmatrix} p_1 & p_0 & 0 & 0 & 0 & 0\\ p_3 & p_2 & p_1 & p_0 & 0 & 0\\ p_5 & p_4 & p_3 & p_2 & p_1 & p_0\\ 0 & 1 & p_5 & p_4 & p_3 & p_2\\ 0 & 0 & 0 & 1 & p_5 & p_4\\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} > 0.$$

By choosing proper cascade controller parameters, condition 3) is satisfied. For system (21), conditions 1), 2) and 3) are all satisfied. According to the Lemma, the zero solution of system (21) is uniformly asymptotically stable.

This implies that response system (15) is synchronized with drive system (1). By this we conclude the proof of Theorem 2.

3.2. Numerical simulations

Simulations are performed under the same conditions as those in Section 2.

Simulation time is 10 s, control inputs are added at the beginning of the simulation. Controller parameters are chosen to be $k_{01} = 0.5$, $h_{01} = 0.001$, $k_{02} = 46$ and $h_{02} = 0.01$.

For this selection, $p_0 = 0.0242$, $p_1 = 25.68$, $p_2 = 1481.35$, $p_3 = 3432.29$, $p_4 = 1188.99$, $p_5 = 72.51$. $\rho_1 = p_1 = 25.68$, $\rho_2 = 3.7952 \times 10^4$, $\rho_3 = 1.2948 \times 10^8$, $\rho_4 = 1.4987 \times 10^{11}$, and $\rho_5 = \rho_6 = 1.0423 \times 10^{13}$.

Therefore, the conditions given in Eq. (22) are satisfied, i.e., $\rho_i > 0$, $i = 1$–6. According to Theorem 2, response system (15) and drive system (1) are synchronous.

State response and synchronization errors are shown in Figs. 7 and 8. Figure 9 displays the control signal of the cascade synchronization scheme.

The IAE values of e_1–e_4 are shown below, respectively:

$$\text{IAE}e_1 = 0.1758,\ \text{IAE}e_2 = 0.1685,$$
$$\text{IAE}e_3 = 0.1467,\ \text{IAE}e_4 = 0.2315.$$

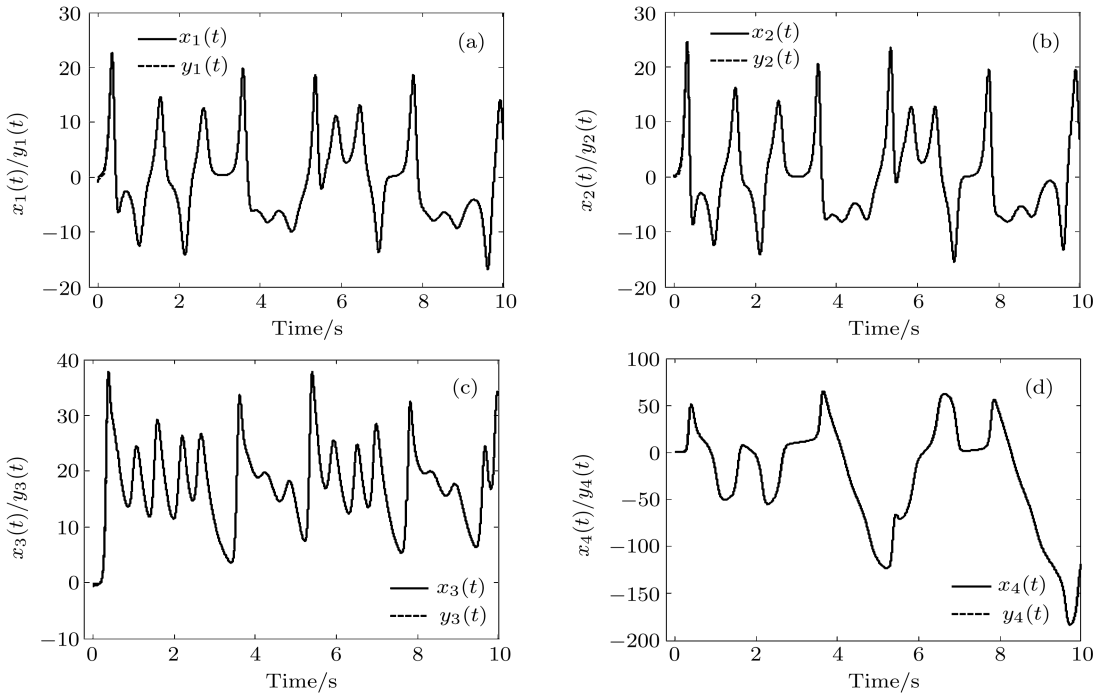


Fig. 7. Synchronization trajectories of the cascade adaptive synchronization scheme.

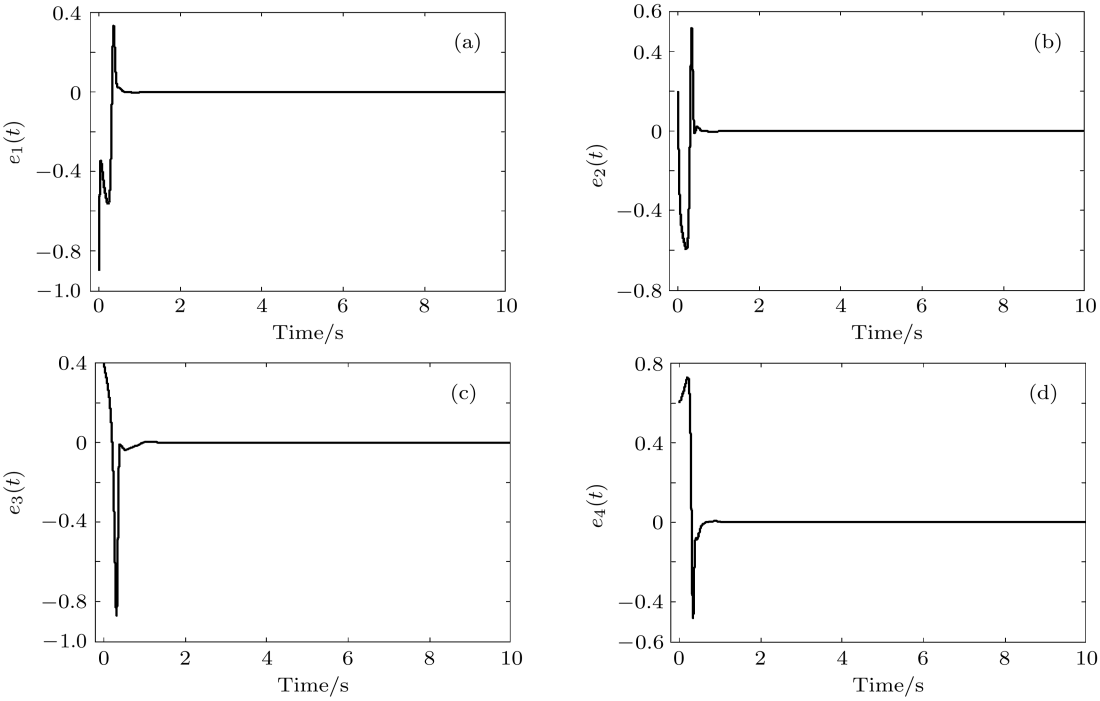


Fig. 8. Synchronization errors of the cascade adaptive synchronization scheme.

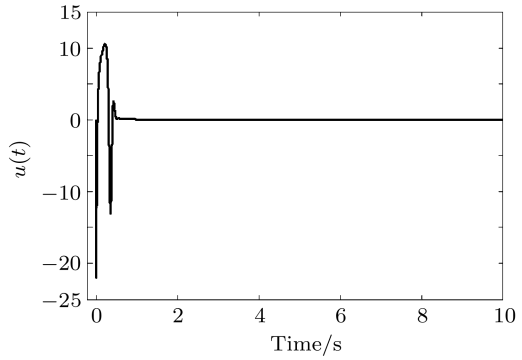


Fig. 9. Cascade adaptive synchronization control signal.

The simulation results demonstrate that the response system (15) synchronizes drive system (1) by only one control signal, if the cascade adaptive synchronization scheme is adopted. To display the initial dynamics more clearly, the Y-coordinates of all figures have been shifted backward by 0.2.

For the cascade synchronization, the response speed of the inner loop should be faster than that of the outer loop. In general, h_{02} is 10 times greater than h_{01} which can guarantee the excellent synchronization performance.

4. Conclusion

In this paper, synchronization between two hyperchaotic Chen systems is investigated. An adaptive synchronization scheme with four control inputs and a cascade adaptive synchronization scheme with only one control input are designed for synchronizing the drive system and the response system. The advantages of the two synchronization approaches are summarized as follows.

(i) Perfect knowledge of the systems is not required. Controller parameters are k_0 and h_0. By tuning the parameters, the systems can be synchronized. This is of crucial importance in practice, since it is not practical or economical to obtain an exact model.

(ii) The synchronous speed of the closed-loop system is controllable. Parameter h_0 determines the response speed of the system. To change the value of h_0, the synchronous speed is tunable.

Simulation results are given for the purposes of illustration and demonstration.

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