Spatial coherence resonance induced by coloured noise and parameter diversity in a neuronal network*

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Spatial coherence resonance in a two-dimensional neuronal network induced by additive Gaussian coloured noise and parameter diversity is studied. We focus on the ability of additive Gaussian coloured noise and parameter diversity to extract a particular spatial frequency (wave number) of excitatory waves in the excitable medium of this network. We show that there exists an intermediate noise level of the coloured noise and a particular value of diversity, where a characteristic spatial frequency of the system comes forth. Hereby, it is verified that spatial coherence resonance occurs in the studied model. Furthermore, we show that the optimal noise intensity for spatial coherence resonance decays exponentially with respect to the noise correlation time. Some explanations of the observed nonlinear phenomena are also presented.

Keywords: neuronal network, noise, diversity, spatial coherence resonance

PACC: 0540, 0547, 0565

1. Introduction

Stochastic resonance (SR) and coherence resonance (CR) are two prominent examples of constructive effects of noise on nonlinear dynamical systems.[1,2] They are not only widely studied in different kinds of nonlinear dynamical systems[3−6] but also in different kinds of networks.[7−9] Nowadays, many researchers have shifted their interest from one-dimensional networks to two-dimensional spatially extended networks, in which some new concepts emerge, such as spatiotemporal stochastic resonance, spatiotemporal coherence resonance and spatial coherence resonance. Additionally, noise also has other positive effects as stated in Refs. [10] and [11].

Spatiotemporal stochastic resonance was first reported in a two-dimensional excitable medium.[12] It is found that Gaussian white noise can induce excitable media to exhibit noise-sustained rotating spiral waves at some intermediate noise intensities, which indicates the occurrence of spatiotemporal stochastic resonance. Both standard (i.e. temporal) and spatiotemporal coherence resonances were also reported in two-dimensional networks, locally modelled by various kinds of nonlinear systems, such as chaotic oscillators, discrete Rulkov maps and FitzHugh–Nagumo neurons.[13−17] Furthermore, it was found that spatiotemporal coherence resonance can be induced by both additive[13−15] and multiplicative noise.[16,17]

Spatial coherence resonance means that noise is able to extract and optimize an intrinsic spatial scale from a spatially extended medium. Carrillo *et al.*[18] showed that spatial coherence resonance existed close to pattern-forming instabilities in a chlorine dioxide–iodine-malonic acid reaction. Perc *et al.*[19,20] showed that additive and internal noise can induce spatial coherence resonance in locally discrete excitable spatially extended systems and excitable biochemical media. Spatial coherence resonance was also discussed in a two-dimensional lattice network locally modelled by Hodgkin–Huxley (HH) neurons.[21,22] In these articles,[18−22] fluctuations are typically considered as white noise. However, for many systems, white noise is not an accurate approximation of the actual fluctuations. In these cases, coloured noise may be a good choice. Thus, we will discuss the effect of additive Gaussian coloured noise on a two-dimensional neuronal locally modelled by HH neurons in this paper. Also, we will show that the spatial frequency of the two-dimensional network can be extracted both by noise intensity and noise correlation time, indicating the occurrence of spatial coherence resonance. Except for noise, parameter diversity is also a very important factor in neuronal systems. Recently, some reports show that diversity can not only induce neuronal coherence but also enhance the noise-induced coherence

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of the neuronal system.[23,24] The effects of parameter diversity on the studied two-dimensional neuronal network are also discussed here.

This paper is organized as follows. A two-dimensional HH neuronal network with an additive Gaussian coloured noise is introduced in Section 2. Spatial coherence resonances due to additive Gaussian coloured noise and parameter diversity in this network, as well as their ability to extract a particular spatial frequency (wave number) of excitatory waves, are discussed in Sections 3 and 4, respectively. Concluding remarks are given finally in Section 5.

2. Model

The two-dimensional neuronal network considered in this paper is composed of a set of diffusively coupled HH neurons,[25] presented as:

$$\begin{aligned} C\mathrm{d}V_{i,j}/\mathrm{d}t &= -g_{Na}m_{i,j}^3h_{i,j}(V_{i,j}-V_{Na}) \\ &\quad - g_K n_{i,j}^4(V_{i,j}-V_K) - g_L(V_{i,j}-V_L) \\ &\quad + D(V_{i-1,j}+V_{i+1,j}+V_{i,j-1}+V_{i.j+1} \\ &\quad - 4V_{i,j}) + I_{\mathrm{ext}} + \xi_{i,j}(t), \\ \mathrm{d}m_{i,j}/\mathrm{d}t &= \alpha_{m_{i,j}}(V_{i,j})(1-m_{i,j}) - \beta_{m_{i,j}}(V_{i,j})m_{i,j}, \\ \mathrm{d}h_{i,j}/\mathrm{d}t &= \alpha_{h_{i,j}}(V_{i,j})(1-h_{i,j}) - \beta_{h_{i,j}}(V_{i,j})h_{i,j}, \\ \mathrm{d}n_{i,j}/\mathrm{d}t &= \alpha_{n_{i,j}}(V_{i,j})(1-n_{i,j}) - \beta_{n_{i,j}}(V_{i,j})n_{i,j}, \\ &\quad (i,j=1,...,N). \end{aligned} \tag{1}$$

In the above equations $V_{i,j}$ is the membrane potential of each neuron, I_{ext} is the external stimulus current, whereas $m_{i,j}$, $h_{i,j}$, and $n_{i,j}$ are the gating variables describing the ionic transport through the membrane. Moreover, the constants g_{Na}, g_K, and g_L are maximal conductances for ion and leakage channels, while V_{Na}, V_K, and V_L are the corresponding reversal potentials. $\alpha_{m_{i,j}}$, $\alpha_{h_{i,j}}$, $\alpha_{n_{i,j}}$, $\beta_{m_{i,j}}$, $\beta_{h_{i,j}}$, and $\beta_{n_{i,j}}$ are functions of $V_{i,j}$ defined as:

$$\begin{aligned} \alpha_{m_{i,j}}(V_{i,j}) &= \frac{0.1(V_{i,j}+40)}{1-\exp\left[-\dfrac{(V_{i,j}+40)}{10}\right]}, \\ \beta_{m_{i,j}}(V_{i,j}) &= 4.0\exp\left[-\frac{(V_{i,j}+65)}{18}\right], \\ \alpha_{h_{i,j}}(V_{i,j}) &= 0.07\exp\left[-\frac{(V_{i,j}+65)}{20}\right], \\ \beta_{h_{i,j}}(V_{i,j}) &= \left\{1+\exp\left[-\frac{(V_{i,j}+35)}{10}\right]\right\}^{-1}, \\ \alpha_{n_{i,j}}(V_{i,j}) &= \frac{0.01(V_{i,j}+55)}{1-\exp\left[-\dfrac{(V_{i,j}+55)}{10}\right]}, \\ \beta_{n_{i,j}}(V_{i,j}) &= 0.125\exp\left[-\frac{(V_{i,j}+65)}{80}\right]. \end{aligned} \tag{2}$$

The $\xi_{i,j}$ in the first equation of Eq. (1) is an additive Gaussian coloured noise with properties $\langle\xi_{i,j}(t)\rangle = 0$ and $\langle\xi_{i,j}(t)\xi_{m,n}(t')\rangle = \sigma\lambda\exp[-\lambda(t-t')]\delta_{i,m}\delta_{j,n}$, where σ and λ denote noise intensity and the reciprocal of noise correlation time, respectively. Moreover, D is the diffusion coefficient and the other parameters are taken as follows: $C = 1\ \mu\mathrm{F/cm^2}$, $g_{Na} = 120\ \mathrm{ms/cm^2}$, $g_K = 36\ \mathrm{ms/cm^2}$, $g_L = 0.3\ \mathrm{ms/cm^2}$, $V_{Na} = 50$ mV, $V_K = -77$ mV, and $V_L = -54.4$ mV. In the present study the periodic boundary condition is used, $V_{0,j} = V_{N,j}$, $V_{N+1,j} = V_{1,j}$, $V_{i,0} = V_{i,N}$, and $V_{i,N+1} = V_{i,1}$, and the system size is $N = 128$ in each spatial dimension of the two-dimensional lattice. I_{ext} is the external current.

The dynamic behaviour of a single HH neuron with respect to the parameter I_{ext} is described here briefly. When $I_{\mathrm{ext}} < I_0 \approx 6.2$ mA there exists only a globally stable fixed point. The birth of a stable and unstable limit cycle occurs at $I_{\mathrm{ext}} = I_0 \approx 6.2$ mA via a saddle-node bifurcation. For $I_0 < I_{\mathrm{ext}} < I_1 \approx 9.8$ mA there coexist a stable fixed point, a stable limit cycle, and an unstable limit cycle. The unstable limit cycle constitutes the boundary separating the attractive basins corresponding to the fixed point and the limit cycle.

3. Noise-induced spatial coherence resonance

In this section, we will consider noise-induced spatial coherence resonance in system (1). Take the external current $I_{\mathrm{ext}} = 6.1$ mA and initiate all neurons from their excitable steady state. Under this condition, the system would remain quiescent forever without the excitation of noise. Thus, the dynamics of spatial patterns exhibited in this section are evoked solely by additive Gaussian coloured noise.

In Fig. 1, characteristic snapshots of the spatial grid for different values of σ and λ are presented. We can obviously see that there exist intermediate values of σ and λ at which the noise induced patterns exhibit maximally ordered circular excitatory waves, as presented in the middle column of Fig. 1. For both

small σ and λ, the noise is not strong enough to excite the network but just evokes some small fluctuations around the steady state, as shown in the left column of Fig. 1. For large σ and λ, the noise becomes rather violent and makes the beautiful ordered patterns blurred, as shown in the right column of Fig. 1. The ordered patterns, appearing at intermediate noise intensity σ or noise correlation time λ, are the indication of spatial coherence resonance in the studied medium.

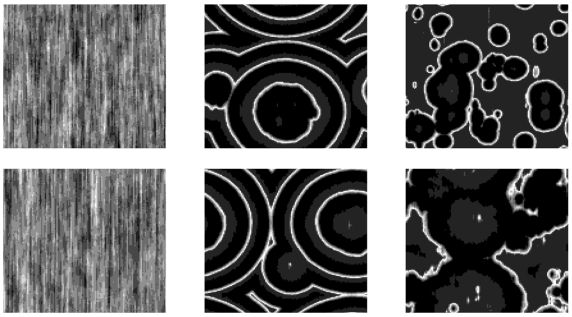


Fig. 1. Characteristic snapshots of the spatial profile of $V_{i,j}$. In the upper panel, $\lambda = 1.4$ and σ is taken as $0.3, 0.5, 1.3$ from left to right. In the lower panel, $\sigma = 0.4$ and λ is taken as $1.0, 2.0, 3.0$ from left to right. Here $D = 0.35$.

To clarify that coloured noise can actually extract spatial frequency from the spatially extended network, we calculate the circular average of the spatial structure function $P(k_x, k_y)$,[18,26] according to

$$p(k) = \int_{\Omega_k} P(\boldsymbol{k}) \mathrm{d}\Omega_k, \tag{3}$$

where $\boldsymbol{k} = (k_x, k_y)$, k_x and k_y are the spatial wave frequencies along the x- and y-directions respectively. Ω_k is a circular shell of radius $k = |\boldsymbol{k}|$, the spatial structure function $P(k_x, k_y)$ of the $V_{i,j}$ field is calculated by

$$P(k_x, k_y) = \langle H^2(k_x, k_y) \rangle, \tag{4}$$

where $H(k_x, k_y)$ is the spatial Fourier transform of the $V_{i,j}$ field at a particular time t, and $\langle \ldots \rangle$ is the ensemble average over noise realizations. More details about the circular average $p(k)$ can be found in Refs. [18] and [26]. The variations of $p(k)$ for different σ and λ are shown in Fig. 2, corresponding to that in Fig. 1.

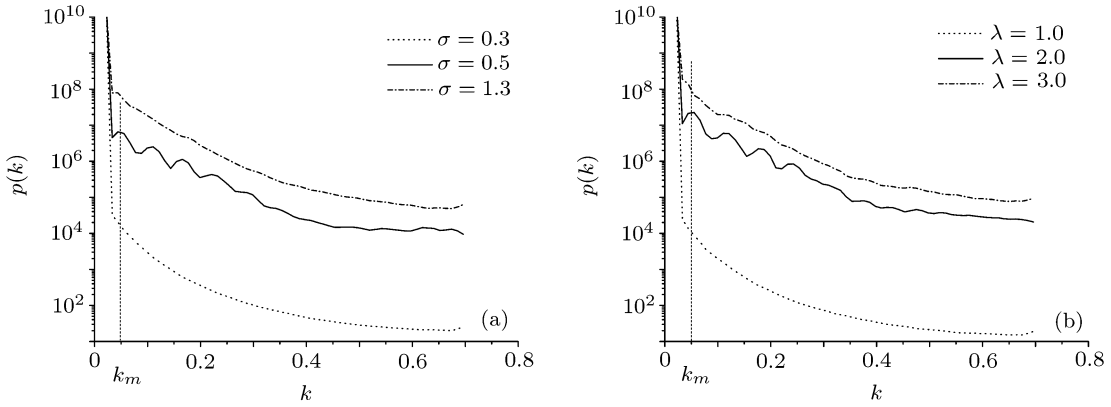


Fig. 2. Circular average of the spatial structure function $P(k_x, k_y)$ for different values of σ (a) and λ (b). The vertical line at $k = k_m$ marks the spatial frequency of excitatory events enhanced resonantly by additive Gaussian noise.

It can be observed that there indeed exists a particular spatial frequency, marked with the vertical line at $k = k_m$, resonantly enhanced at some intermediate level of additive Gaussian coloured noise. To quantify the ability of the coloured noise to extract the characteristic spatial frequency of the medium more precisely, we calculate the signal-to-noise ratio (SNR) as the peak height at k_m normalized with the background fluctuations in the system. This is the spatial counterpart of the measure frequently used for quantifying the constructive effects of noise in the temporal domain of dynamical systems.[1] This quantity was also used for quantifying the effects of noise on the spatial dynamics of spatially extended systems in Ref. [18].

Figure 3 shows how SNR varies with the variations of σ and λ, respectively. It is evident that there exists an optimal level of additive Gaussian coloured noise for which the peak of the circularly averaged structure function is best resolved, indicating clearly the existence of spatial coherence resonance in the studied medium.

Therefore, the noise intensity with the maximal SNR given in Fig. 3(a) can exactly extract the spatial frequency k_m with the maximal $p(k)$ shown in Fig. 2(a).

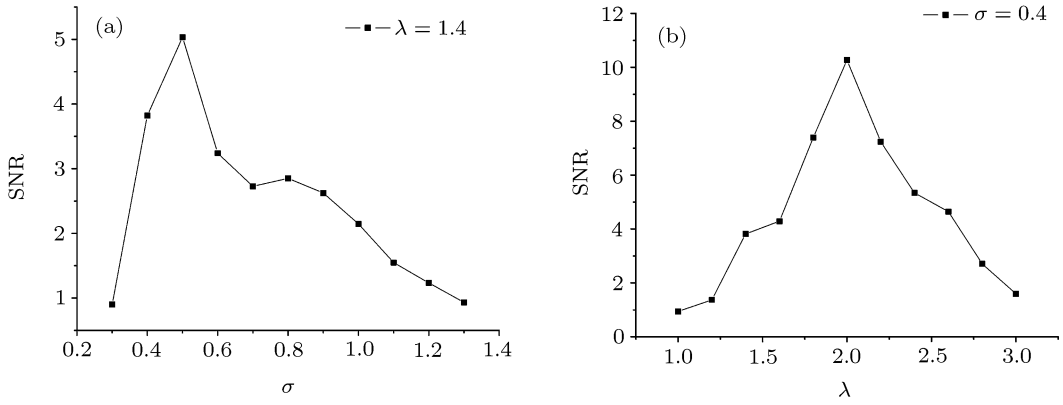


Fig. 3. The signal-to-noise (SNR) ratio vs. σ (a) and λ (b), in which the lines are just guides to the eye.

Within the range of $0.1 < \lambda < 10.0$, figure 4 shows that the optimal noise intensity $\sigma_{\rm opt}$ as a function of the noise correlation time λ follows a power law, i.e., $\sigma_{\rm opt} \propto \lambda^{-\alpha}$ with the characteristic exponent $\alpha \approx 1.22 \pm 0.07$. Thus it follows that $\sigma_{\rm opt}$ will drift to smaller values with λ increasing. Furthermore, this results in the optimal noise strength (that is, $W_{\rm opt} = \sigma_{\rm opt}\lambda$) varying little within this range of λ. This indicates that the oscillators perceive the coloured noise as white, due to their characteristic time scales being much larger than the noise correlation time.

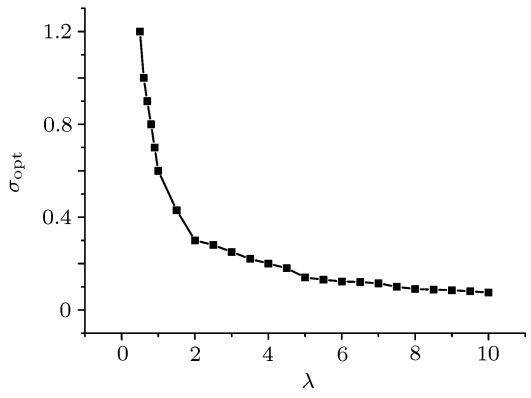


Fig. 4. Variations of the optimal noise intensity $\sigma_{\rm opt}$ with respect to the correlation time λ, in which the line is just a guide to the eye.

Now, we give some explanations of the existence of spatial coherence resonance in the studied excitable medium. It is found that coherence resonance phenomena in temporal excitable systems with no explicit time scales are due to the difference of noise dependence between the activation time ($t_{\rm a}$) and the excursion time ($t_{\rm e}$).[27] Since $t_{\rm e}$ is much more robust to noise perturbation than $t_{\rm a}$, there exists an optimal noise intensity so that $t_{\rm a} \ll t_{\rm e}$ but the fluctuations of $t_{\rm e}$ are moderate, thereby marking the most coherent response of the system independent of σ. The noise-robust excursion time, together with the rate of diffusive spread proportional to D, introduces an eigenfrequency of spatial waves which can be resonantly enhanced at some appropriate level of noise, as shown analytically in a simple excitable model.[28]

4. Diversity-induced spatial patterns

In this section, we will consider diversity-induced spatial coherence resonance in system (1). Let the internal current $I_{\rm ext}$ randomly take values from the uniform distribution at the interval $[6.1 - \delta a,\ 6.1 + \delta a]$, where δa is called the diversity of $I_{\rm ext}$ and is taken as a control parameter in the following study. The noise intensity σ is taken as zero, thus all the phenomena obtained below are induced only by the diversity. Here, the diffusive parameter D is taken as 0.1, and the results shown here are based on realizations of the average of 10 samples; among them there exist some quantitative but not qualitative differences. Figure 5 shows that there exist some intermediate diversities which can induce the two-dimensional network to generate ordered circular patterns. For small δa, the

neuronal network does not fire, so there are no patterns observed; while for large δa, ordered patterns give way to disordered ones. As discussed in Section 3, the phenomenon shown in Fig. 5 implies the occurrence of coherence resonance.



Fig. 5. Characteristic snapshots of the spatial profile of $V_{i,j}$. $D = 0.1$ and δa is taken as $1.0, 2.0, 3.0$ from left to right.

Similar to Section 3, we calculate the circular average $p(k)$ of the spatial structure function to quantify the ability of diversity to extract the characteristic spatial periodicity in the system precisely, and the numerical results are shown in Fig. 6(a). It can be observed that there indeed exists a particular spatial frequency, which is resonantly enhanced by some diversities δa. Although noise is not considered in this case, the diversity can still be viewed as randomness in some sense. Thus, in order to exhibit this phenomenon more precisely, we also use the quantity SNR mentioned above to quantify the ability of some implicit randomness (here refer to a particular diversity parameter) to extract the characteristic spatial periodicity of waves, similar to Ref. [29]. From the numerical results shown in Fig. 6(b), we can observe bell-shape behaviours with respect to the diversity parameter δa. This indicates that diversity can induce spatial coherence resonance in the studied two-dimensional neuronal network.

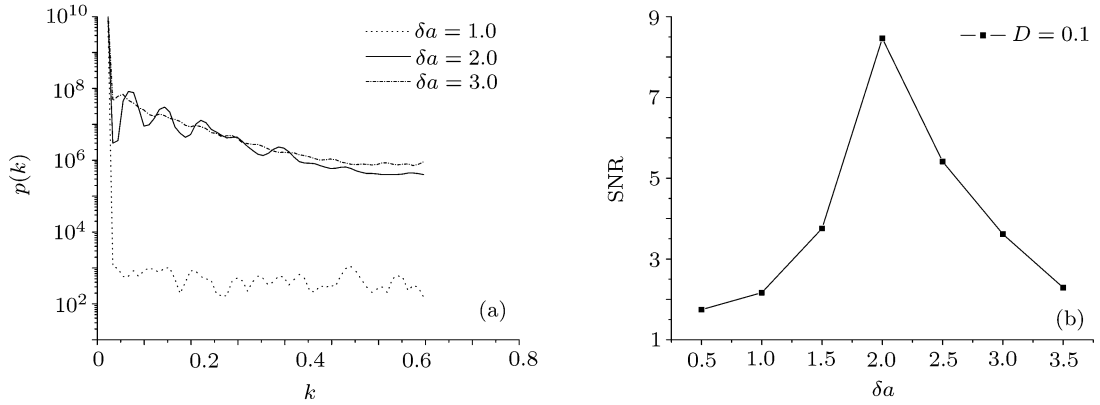


Fig. 6. Variations of the optimal noise intensity $\sigma_{\rm opt}$ (a) and SNR (b) with respect to the correlation time λ, in which the lines are just guides to the eye.

In order to give some explanations for this, we return ourselves to the excitability of a single neuron inside the network. As stated before, neurons with $I_{\rm ext}$ smaller than 6.1 mA will not generate any spikes, and smaller $I_{\rm ext}$ causes less excitation, while neurons with I larger than 6.1 mA will exhibit spikes, and larger $I_{\rm ext}$ leads to more high-rate spikes.

For $\delta a = 1.0$ mA, some neurons with 6.1 mA$< I_{\rm ext} <$ 7.1 mA will generate spikes but with very low firing frequencies. However, the neurons with $I_{\rm ext} <$ 6.1 mA will act as the dominant role and ensure the whole network does not generate any spikes when all neurons are diffusively coupled, as shown in the left of Fig. 5. When δa increases to 2.0 mA, the largest $I_{\rm ext}$ can be taken as 8.1 mA. In this case, the neurons with $I_{ext} >$ 7.1 mA can generate some moderate firing-rate spiking trains. These spiking trains can spread throughout the whole network to make neurons with $I_{\rm ext} <$ 6.1 mA fire too, so we can observe ordered circular patterns for $\delta a = 2.0$ mA, as shown in the middle of Fig. 5. When δa increases further to 3.0 mA, the largest values of $I_{\rm ext}$ become 9.1 mA. In this case, the neurons with $I_{\rm ext} >$ 8.1 mA will generate more high-rate spikes and will activate the neurons with $I_{\rm ext} <$ 6.1 mA to fire easily. This will result in the number of firing neurons inside the network increasing quickly. Meanwhile, the firing neurons transfer the spiking trains through the network and lead to competitions between the firing neighbouring neurons. Then the pattern in this case becomes disordered as shown in the right of Fig. 5. Therefore, only at intermediate diversity δa can the spatial coherence of the studied neuronal network be better pronounced.

5. Conclusions

Spatial coherence resonance in a two-dimensional neuronal network induced by additive Gaussian coloured noise and parameter diversity is studied in this paper. It is shown that additive Gaussian coloured noise is able to induce ordered circular patterns in a resonant manner. By calculating the circular average of the spatial structure function, a characteristic spatial frequency of the excitable waves can be extracted solely by noise in the excitable medium. Therefore, the obtained results offer persuasive evidence for the existence of spatial coherence resonance in the studied HH neuronal network, and in turn successfully answer the question of whether coloured noise is also able to extract spatial scale from a two-dimensional neuronal network. In this sense, we extend the scope of the noise-induced spatial coherence resonance to two-dimensional spatially extended neuronal systems. The effects of parameter diversity are also studied. There exists a particular parameter diversity to extract the spatial frequency similar to the additive noise. Even though the present study is for an HH neuronal network, the results may also have important implications for other neuronal networks.

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