Effect of a barrier potential on soliton dynamical characteristics in condensates

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(Received 16 August 2009; revised manuscript received 3 September 2009)

By using the multiple-scale method, this paper analytically studies the effect of a barrier potential on the dynamical characteristics of the soliton in Bose–Einstein condensates. It is shown that a stable soliton is exhibited at the top of the barrier potential and the region of the absence of the barrier potential. Meanwhile, it is found that the height of the barrier potential has an important effect on the dark soliton dynamical characteristics in the condensates. With the increase of height of the barrier potential, the amplitude of the dark soliton becomes smaller, its width is narrower, and the soliton propagates more slowly.

Keywords: Bose–Einstein condensates, solitons, barrier potential

PACC: 0530J, 0290, 1110L

1. Introduction

The observation of Bose–Einstein condensates (BECs) in gases of weakly interacting alkali-metal atoms has stimulated intensive studies of nonlinear matter waves.[1,2] One of the central questions in this field is how to control the dynamical characteristics of the soliton in the BECs. In a realistic experimental setting, an external electromagnetic field is used to produce, trap, and manipulate the BEC.[1,2] Theoretically, the physics of BECs can be described by the Gross–Pitaevskii (GP) equation, whose mathematical form is equal to the nonlinear Schrödinger equation.[1,2] To our knowledge, only two parameters in the GP equation can be changed. One is the external potential, and the other is the interatomic interaction strength. It is possible that the technologies of controlling the dynamical characteristics of the soliton are to vary the trapped external potential by virtue of magnetic fields or (and) to adjust the inter-atomic interaction strength via the process of Feshbach resonance.[3−5] To date, the effect of the interatomic interaction on the nonlinear dynamics of the BECs has been intensively investigated.[6−18] Meanwhile, physicists have also investigated some types of external potentials influencing a number of the BEC properties, including static and dynamical properties.[19−30] When BECs are in a harmonic potential,[29,30] there exist dark (bright) solitons for the repulsive (attractive) interaction. When the condensates are trapped in a tanh-shaped external potential,[27] the BECs exhibit bright (dark) solitons for the repulsive (attractive) interaction.

So far, related theoretical studies have focused on the adiabatic propagation of a BEC in the presence of obstacles.[31] Particularly interesting in this context is the propagation of a Bose–Einstein condensate through a double barrier potential, which can be seen as a Fabry–Perot interferometer for matter waves.[31] In the field of atom chips, such a bosonic quantum dot could be realized by suitable geometries of microfabricated wires.[32,33] This phenomenon is well known for nonlinear optics.[34]

In this paper, we study the effect of a barrier potential on the soliton dynamical characteristics in the condensates. It is shown that with the increase of height of the barrier potential, the amplitude of the soliton becomes smaller, its width is narrower, and the soliton propagates more slowly.

2. Asymptotic expansion and variable coefficient Korteweg–de vries (VCKdV) equation

2.1. Hydro-dynamical model

We here consider a soliton of BEC atoms propagating through a double barrier potential in a magnetic waveguide. In the presence of a strong cylindrical confinement with trapping frequency $\omega_\perp$, the mean field dynamics of the condensate can be described by

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the effective one-dimensional (1D) GP equation in the dimensionless variables[1,2]

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi + g_{1\mathrm{D}}\,|\psi|^2\,\psi \qquad (1)$$

with $g_{1\mathrm{D}} = 2\,|a_\mathrm{s}|\,/a_0$. Here a_s is the s-wave scattering length and $a_0 = \sqrt{\hbar/m\omega_\perp}$ (transverse harmonic-oscillator length) with the atomic mass m. For the sake of definiteness, a symmetrical double barrier potential can be given by[31]

$$V(x) = V_0\left\{\exp\left[-(x+L/2)^2/\sigma^2\right] + \exp\left[-(x-L/2)^2/\sigma^2\right]\right\}, \qquad (2)$$

(see Fig. 1). As discussed in Ref. [31], our numerical calculations were performed for ^{87}Rb atoms ($a_\mathrm{s} = 5.77$ nm) with $\omega_\perp = 2\pi\times 10^3$ μm, $V_0 = \hbar\omega_\perp$, and $L = 10\sigma = 5$ $\mu\mathrm{m}\simeq 14.7a_0$. So, the corresponding dimensionless parameters are $V_0 = 1.0$, $L = 14.7$, and $\sigma = 1.47$. Expressing the wave function in terms of modulus and phase, we set $\psi(x,t) = A(x,t)\exp[-\mathrm{i}\mu t + \mathrm{i}\varphi(x,t)]$. Substituting it into Eq. (1) and then separating the real and imaginary parts, we obtain the following two equations:

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{1}{2}A\frac{\partial^2\varphi}{\partial x^2} = 0, \qquad (3)$$

$$-\frac{1}{2}\frac{\partial^2 A}{\partial x^2} + \left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \mu + \frac{\partial\varphi}{\partial t} + V(x)\right]A + g_{1\mathrm{D}}A^3 = 0. \qquad (4)$$

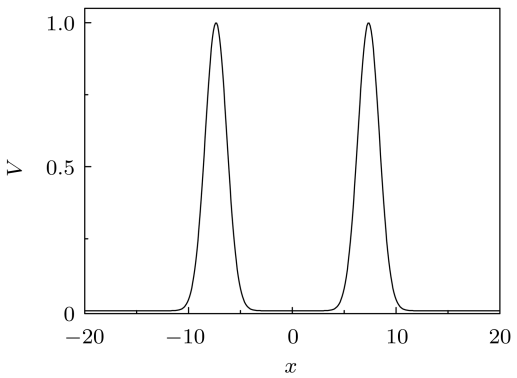


Fig. 1. A double barrier potential. The parameters used are $V_0 = 1.0$, $L = 14.7$ and $\sigma = 1.47$.

2.2. Linear stability condition of soliton formation

To obtain the effect of the barrier potential on the dynamical characteristics of the soliton in the condensates, we here study the linear stability condition of soliton formation. Due to the strong confinement in the transverse direction, the system is similar to a wave guide, in which the excitation from the ground state propagates in the elongated direction.[9,10] The strong confinement also ensures the dynamical stability of a soliton excitation.[9,10] Therefore, we set $A = u_0(x) + \alpha(x,t)$ (without loss of generality, we assume that $u_0(x)$ is positive, which characterizes the condensate background) with $\alpha(x,t) = \alpha_0\exp(\mathrm{i}\beta) +$ c.c. and $\varphi = \varphi_0\exp(\mathrm{i}\beta) +$ c.c. Here, c.c. is complex conjugate and $\beta = kx - \omega t$ where k is the wave number and ω indicates the eigen-frequency. Considering that α_0 and φ_0 are small constants, from the linearization of Eqs. (3) and (4) we obtain

$$\mathrm{i}\omega\alpha_0 = \mathrm{i}k\varphi_0\frac{\partial u_0}{\partial x} - \frac{1}{2}k^2u_0\varphi_0, \qquad (5)$$

$$\mu\,u_0 = -\frac{1}{2}\frac{\partial^2}{\partial x^2}u_0 + V(x)\,u_0 + g_{1\mathrm{D}}u_0^3, \qquad (6)$$

$$\mu\,\alpha_0 = \frac{1}{2}k^2\alpha_0 - \mathrm{i}\omega\varphi_0u_0 + V(x)\,\alpha_0 + 3g_{1\mathrm{D}}u_0^2\alpha_0. \qquad (7)$$

For the linear case, i.e., $g_{1\mathrm{D}} = 0$, equation (6) is turned into

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_0 + 2\left\{\mu - V_0\left[\mathrm{e}^{-(x+L/2)^2/\sigma^2} + \mathrm{e}^{-(x-L/2)^2/\sigma^2}\right]\right\}u_0 = 0. \qquad (8)$$

Under the case of $g_{1\mathrm{D}} \neq 0$, utilizing Eqs. (5), (6) and (7), we have

$$\omega^2 = \left(\frac{1}{2u_0}\frac{\partial^2u_0}{\partial x^2} + \frac{1}{2}k^2 + 2g_{1\mathrm{D}}u_0^2\right)\times\left(\frac{1}{2}k^2 - \frac{\mathrm{i}k}{u_0}\frac{\partial u_0}{\partial x}\right). \qquad (9)$$

Subsequently, we discuss the stability problem of soliton formation. To demonstrate that the soliton is stable or unstable, we set $\omega = \omega_\mathrm{r} + \mathrm{i}\omega_\mathrm{i}$ where the subscripts denote the real and imaginary parts.[35] The main reason is that if $\omega_\mathrm{i} \neq 0$ then it implies exponential growth of the mode and hence the state ψ is dynamically unstable.[35] If the eigen-frequency is real, then it means that the soliton in the BEC is stable. From Eq. (9), we see the dependence of the instability growth rate ω_i on k and $(1/u_0)(\partial u_0/\partial x)$. On the one hand, when $k = 0$, one finds $\omega = 0$, which is an inessential solution. On the other hand, we get that $(1/u_0)(\partial u_0/\partial x) = 0$. Combining with Eq. (8) and Fig. 1, we find that at the top (maximum) of the barrier potential and in the region of the absence of barrier potential the wave function ψ possesses dynamical stability. We therefore conclude that there is a stable soliton at the top of the barrier potential and in the region of the absence of the barrier potential.

2.3. VCKdV equation and its soliton solution

To understand the effect of the barrier potential on soliton dynamical characteristics in BECs, we here make use of a multiple-scale method[36−38] to derive an explicit expression for the wave function of the BECs. First, we should introduce two "slow" variables $\xi = \varepsilon(x - V_g t)$ and $\tau = \varepsilon^3 t$ to characterize the slow variation of soliton dynamics and a "fast" variable $x_0 = x$ to represent the direction of soliton propagation. Moreover, we assume that $A(x,t)$ and $\varphi(x,t)$ can be expanded into a polynomial form in terms of ε in the forms of $A = u_0 + \varepsilon^2\left[a^{(0)}(\xi,\tau) + \varepsilon^2 a^{(1)}(\xi,\tau) + \ldots\right]$, and $\varphi = \varepsilon\left[\varphi^{(0)}(\xi,\tau) + \varepsilon^2\varphi^{(1)}(\xi,\tau) + \ldots\right]$, where ε is a smallness parameter characterizing the relative amplitude of the excitation. Inserting them into Eqs. (3) and (4), and then separating each order in terms of the power of ε, respectively, from Eq. (3), we have

$$V_g\frac{\partial a^{(0)}}{\partial\xi} - \frac{1}{2}u_0\frac{\partial^2\varphi^{(0)}}{\partial\xi^2} = 0, \tag{10}$$

$$\begin{aligned} &V_g\frac{\partial a^{(1)}}{\partial\xi} - \frac{1}{2}u_0\frac{\partial^2\varphi^{(1)}}{\partial\xi^2} \\ &= \frac{\partial a^{(0)}}{\partial\xi}\frac{\partial\varphi^{(0)}}{\partial\xi} + \frac{1}{2}a^{(0)}\frac{\partial^2\varphi^{(0)}}{\partial\xi^2} + \frac{\partial a^{(0)}}{\partial\tau}, \end{aligned} \tag{11}$$

and from Eq. (4), we have

$$-\mu u_0 + V(x_0)u_0 + g_{1D}u_0^3 = 0, \tag{12}$$

$$\begin{aligned} &-\mu a^{(0)} + V(x_0)a^{(0)} + 3g_{1D}u_0^2 a^{(0)} \\ &= V_g u_0\frac{\partial\varphi^{(0)}}{\partial\xi}, \end{aligned} \tag{13}$$

$$\begin{aligned} &-\mu a^{(1)} + V(x_0)a^{(1)} + 3g_{1D}u_0^2 a^{(1)} \\ &= \frac{1}{2}\frac{\partial^2 a^{(0)}}{\partial\xi^2} - \frac{1}{2}u_0\left(\frac{\partial\varphi^{(0)}}{\partial\xi}\right)^2 - 3g_{1D}u_0\left(a^{(0)}\right)^2 \\ &\quad + V_g u_0\frac{\partial\varphi^{(1)}}{\partial\xi} + V_g a^{(0)}\frac{\partial\varphi^{(0)}}{\partial\xi} - u_0\frac{\partial\varphi^{(0)}}{\partial\tau}. \end{aligned} \tag{14}$$

From Eq. (12), we obtain that the chemical potential of the system is $\mu = V(x_0) + g_{1D}u_0^2$. Comparing Eq. (10) with Eq. (13), we get $V_g = \pm\sqrt{g_{1D}}u_0$. Subsequently, from Eqs. (11) and (14), we obtain

$$\frac{\partial\varphi^{(0)}}{\partial\tau} + \frac{3}{4}\left(\frac{\partial\varphi^{(0)}}{\partial\xi}\right)^2 - \frac{1}{8V_g}\frac{\partial^3\varphi^{(0)}}{\partial\xi^3} = 0. \tag{15}$$

Equation (15) can also be called the VCKdV equation. Making the transformation $\eta = \varepsilon\varphi^{(0)}$, $\xi = \varepsilon X = \varepsilon(x - V_g t)$, and $\tau = \varepsilon^3 t$, from Eq. (15) we obtain

$$\frac{\partial\eta}{\partial t} + \frac{3}{4}\left(\frac{\partial\eta}{\partial X}\right)^2 - \frac{1}{8V_g}\frac{\partial^3\eta}{\partial X^3} = 0. \tag{16}$$

The single-soliton solution of the VCKdV equation can be given by

$$\begin{aligned} \eta = &-\frac{k}{V_g}\exp(kX - \omega t + \delta_0) \\ &\times \operatorname{sech}(kX - \omega t + \delta_0), \end{aligned} \tag{17}$$

where k and δ_0 are arbitrary constants, and $\omega = -k^3/(2V_g)$. Setting $k^2 = 2V_g^2 A_0$ (A_0 is a positive integrating constant) and using $\xi = \varepsilon X = \varepsilon(x - V_g t)$, from the single soliton solution of Eq. (1) we obtain

$$\begin{aligned} \psi = u_0\Big\{1 - A_0\operatorname{sech}^2\Big[V_g\sqrt{2A_0}(X + A_0V_g t - V_g t - x_{00})\Big]\Big\}\exp\Big\{-\mathrm{i}\mu t - \sqrt{2A_0}\mathrm{i} \\ \times \exp\Big[V_g\sqrt{2A_0}(x + A_0V_g t - V_g t - x_{00})\Big]\operatorname{sech}\Big[V_g\sqrt{2A_0}(X + A_0V_g t - V_g t - x_{00})\Big]\Big\}, \end{aligned} \tag{18}$$

where $x_{00} = -V_g\sqrt{2A_0}\delta_0$ is another constant denoting the initial position of the soliton on the pedestal background.

3. Numerical results and discussions

To observe the effect of the barrier potential on the dynamical characteristics of the soliton in the BECs, we further give numerical calculation of the dynamical properties.

First, we probe the stability problem of the soliton. Figure 2 shows the density distribution at different times of the BECs trapped in the absence of the barrier potential. We see that there exists a sinus in the amplitude function of the density distribution at $x = 0$ at the initial time, which can be associated with a dark soliton. As time goes on, the dark soliton propagates rightward without attenuation and changes in shape (including its height and width). This illustrates that the dark soliton is a stable propagating solitary wave, which is a result of the interplay between the nonlinearity and atomic dispersion.

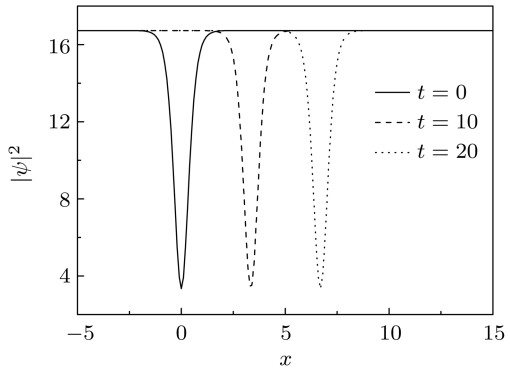


Fig. 2. In the absence of a barrier potential, the density distribution of the condensate at different times $t = 0$ (solid line), 10 (dashed line), 20 (dotted line). The parameters used are $g_{1D} = 0.1$, $A_0 = 0.8$, $x_{00} = 0$, $\mu = 2.8$, and $V(x) = 0$.

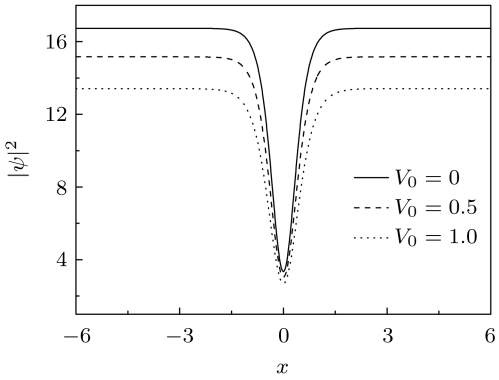


Fig. 3. At the top ($x = \pm L/2$) of the barrier potential, the density distribution of the condensate with the different heights $V_0 = 0$ (solid line), 0.5 (dashed line), and 1.0 (dotted line). The parameters used are the same as those in Figs. 2 and 1.

As we have already mentioned, the stable condition of soliton formation is at the top of the barrier potential and the region of the absence of the barrier potential. So, we may conclude that the dark soliton propagates stably rightward in the region of the absence of the barrier potential. When it passes through a barrier potential, the soliton possesses dynamical instability. When it passes the top of the barrier potential, the soliton can also be stable.

Finally, we discuss the effect of the height of the barrier potential on the dynamical characteristics of the soliton in the condensates. Figure 3 presents the density distribution of the condensates at the initial stage under consideration of the different level heights of a barrier potential. We find that the amplitude of the soliton decreases with the increasing height of a barrier potential. So, we conclude that a barrier potential has a damping effect for a propagating soliton. Meanwhile, we depict the width of the soliton versus the height of the barrier potential in Fig. 4. When the barrier height increases, the width of the soliton slowly decreases. In addition, the relation between the velocity of the soliton and the barrier height is shown in Fig. 5. One may see that the velocity of the soliton decreases with increasing the height of the barrier potential. This might be from the effect of the barrier potential when the soliton propagates in BEC. Based on the above discussion, we find that the dynamical characteristics of the solitons in the BEC can be effectively managed by means of the height of the barrier potential.

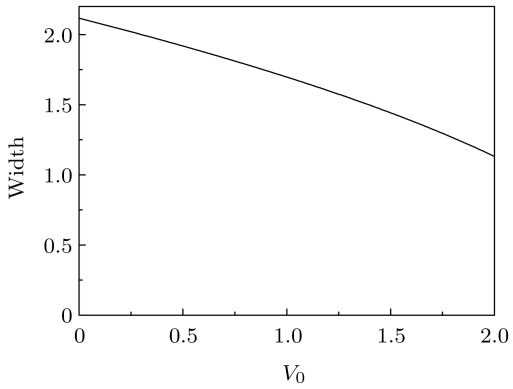


Fig. 4. At the top of the barrier potential, the width of the soliton versus the barrier height V_0. The parameters used are the same as those in Fig. 3.

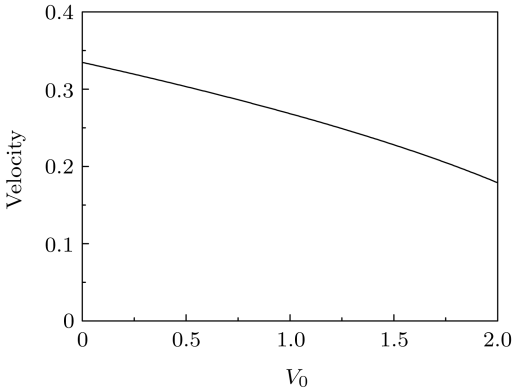


Fig. 5. At the top of the barrier potential, the velocity of the soliton as a function of the barrier height parameter V_0. The parameters used are the same as those in Fig. 3.

4. Conclusions

In summary, we have used the multiple-scale method to analytically study the effect of the barrier potential on the dynamical characteristics in a BEC. In the linear case, we have obtained that there

are stable solitons at the top of the barrier potential and in the region of the absence of the barrier potential. For the nonlinear case, the amplitude and phase of the wave function of the condensates are governed by the VCKdV equation. Our numerical calculation showed that the height of the barrier potential has an important effect on the dynamical characteristics of the dark soliton. With the increase of height of the barrier potential, the amplitude of the dark soliton becomes smaller, its width is narrower, and the soliton propagates more slowly. These results showed that the amplitude, velocity, and width of material solitons can be effectively managed by means of the height of the barrier potential.

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