Some dynamical property of the Tsallis distribution from a Fokker–Planck equation*

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This paper studies the possible dynamical property of the Tsallis distribution from a Fokker–Planck equation. For the Langevin dynamical system with an arbitrary potential function, Markovian friction and Gaussian white noise, it shows that the current form of Tsallis distribution cannot describe any nonequilibrium dynamics of the system, and it only stands for a simple isothermal situation of the system governed by a potential field. So the form of Tsallis distribution and many existing applications using the Tsallis distribution need to be reconsidered.

Keywords: Tsallis distribution, dynamical property, Fokker–Planck equation

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Since it started in 1988, nonextensive statistical mechanics has obtained very wide applications in many interesting scientific fields. In principle, almost all the formulae and theory using Boltzmann–Gibbs statistics so far could be generalized under this framework[1−4] (for more details, see http://tsallis.cat.cbpf.br/biblio.htm). However, there are long-standing questions,[5−10] such as under what circumstances, e.g. which class of nonextensive systems and under what physical situation, should the nonextensive statistical mechanics be used for their statistical description. The physical meaning of the nonextensive parameter has been investigated in, for example, some astrophysical systems (see Refs. [11] and [12] and references therein), plasma systems[13] and the applications.[14−16] The purpose of this work is to investigate the dynamical property of the current form of Tsallis distribution, $f \sim [1-(1-q)\beta H]^{1/(1-q)}$, in nonextensive statistical mechanics from a Fokker–Planck equation.

Tsallis proposed the entropy in 1988 as a generalization of the Boltzmann–Gibbs entropy,[17] given by

$$S_q = -k\sum_i p_i^q \ln_q p_i, \tag{1}$$

where k is the Boltzmann constant, the set $\{p_i\}$ are the probabilities of the microscopic configurations of the system under investigation, the parameter q is a real number, the q-logarithm is defined as

$$\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x>0; \quad \ln_1 = \ln x), \tag{2}$$

the inverse function, the q-exponential, is

$$\exp_q x \equiv [1+(1-q)x]^{1/(1-q)} \quad (\exp_1 x = \exp x), \tag{3}$$

if $1+(1-q)x>0$ and $\exp_q x = 0$ otherwise. The entropy S_q $(q \neq 1)$ is nonextensive; namely, if a system composed of two probabilistically independent parts A and B, i.e., $p_{ij}^{A\oplus B} = p_i^A p_j^B \forall(ij)$, then the Tsallis entropy of the system is

$$\begin{aligned} &S_q(A\oplus B) \\ &= S_q(A)+S_q(B)+(1-q)k^{-1}S_q(A)S_q(B). \end{aligned} \tag{4}$$

Under this framework, one leads to the basic form of the Tsallis distribution, $f \sim [1-(1-q)\beta H]^{1/(1-q)}$, with the Lagrange parameter, $\beta = 1/kT$, and the Hamiltonian H, in nonextensive statistical mechanics. Clearly, only if taking $q=1$ does the Tsallis distribution have a Boltzmann distribution, $f \sim \exp(-\beta H)$.

To investigate the dynamical property of the Tsallis distribution, we begin with a quite general Langevin equation for the dynamics of a set of variables $\boldsymbol{a} = \{a_i\}$. The abbreviated form is

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \boldsymbol{V}(\boldsymbol{a}) + \boldsymbol{F}(t), \tag{5}$$

where $\boldsymbol{V}(\boldsymbol{a})=\{V_i(\{a_j\})\}$ is a set of some given functions of the variables $\boldsymbol{a}$. One may require that the noise-free part V_i of the dynamics is Markovian and that the noise $\boldsymbol{F}(t) = \{F_i\}$ is a set of white noises and has a Gaussian distribution, with zero mean and delta-correlated second moment matrices,

$$\langle \boldsymbol{F}(t)\boldsymbol{F}(t')\rangle = 2\boldsymbol{B}\delta(t-t'); \quad \langle \boldsymbol{F}(t)\rangle = 0, \tag{6}$$

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where the quantity is $\boldsymbol{B} = \{B_{ij}\}$; $i, j = 1, 2, \ldots, n$.

The question we want to study here is whether or not there is a dynamics of the Langevin equations (5) and (6) which can be described by the Tsallis distribution $f \sim [1-(1-q)\beta H]^{1/(1-q)}$. Rather than looking for a general solution of these equations, we usually ask for the probability distribution $f(\boldsymbol{a}, t)$ of the values of $\boldsymbol{a}$ at time t. What we want is the average of this probability distribution over the noise. The Fokker–Planck equation, corresponding to the dynamics governed by the above Langevin equations, for the noise-averaged distribution function $f(\boldsymbol{a}, t)$, can be written[18] as

$$\frac{\partial}{\partial t} f(\boldsymbol{a}, t) = -\frac{\partial}{\partial \boldsymbol{a}} \cdot \boldsymbol{V}(\boldsymbol{a}) f(\boldsymbol{a}, t) + \frac{\partial}{\partial \boldsymbol{a}} \cdot \boldsymbol{B} \cdot \frac{\partial}{\partial \boldsymbol{a}} f(\boldsymbol{a}, t). \quad (7)$$

The first term on the right-hand side is what one has in the absence of noise; the second term accounts for the averaged effects of the noise. At this point, $\boldsymbol{B}$ can be any function of $\boldsymbol{a}$.

So far nothing has been said about requiring that $f(\boldsymbol{a}, t)$ must approach an equilibrium distribution at long times. In fact, not much is known in general about the long-time stationary-state solution of an arbitrary Fokker–Planck equation. But we can usually guess at a stationary-state solution, put it into the equation, and then search for it if the guess is compatible with $\boldsymbol{V}(\boldsymbol{a})$ and $\boldsymbol{B}$. If it is so, then the stationary-state solution can describe the long-time dynamics governed by the above Langevin equation. Our question therefore becomes finding whether there are possible $\boldsymbol{V}(\boldsymbol{a})$ and $\boldsymbol{B}$ that have a full physical property compatible with the Tsallis distribution.

One assumes the Tsallis distribution function $f \sim [1-(1-q)\beta H]^{1/(1-q)}$ to satisfy the stationary-state Fokker–Planck equation,

$$-\frac{\partial}{\partial \boldsymbol{a}} \cdot \boldsymbol{V}(\boldsymbol{a}) f(\boldsymbol{a}) + \frac{\partial}{\partial \boldsymbol{a}} \cdot \boldsymbol{B} \cdot \frac{\partial}{\partial \boldsymbol{a}} f(\boldsymbol{a}) = 0. \quad (8)$$

Equivalently, it can be written as the following partial differential equation:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial a_i}(V_i f) + \sum_{i,j=1}^{n} \frac{\partial}{\partial a_i}\left(B_{ij} \frac{\partial f}{\partial a_j}\right) = 0, \quad (9)$$

or, furthermore, it is

$$-\sum_{i=1}^{n} \frac{\partial}{\partial a_i}(V_i f) + \sum_{i,j=1}^{n} \left[B_{ij} \frac{\partial^2 f}{\partial a_i \partial a_j} + \left(\frac{\partial B_{ij}}{\partial a_i}\right)\left(\frac{\partial f}{\partial a_j}\right)\right] = 0. \quad (10)$$

The Tsallis distribution function as given in nonextensive statistical mechanics so far is

$$f \sim [1-(1-q)\beta H]^{1/(1-q)} \equiv R^{1/(1-q)}, \quad (11)$$

where the Hamiltonian is $H = H(\boldsymbol{a})$ and the Lagrange parameter $\beta = \beta(\boldsymbol{a})$. Put into Eq. (10), the equation becomes

$$-f \sum_{i=1}^{n} \frac{\partial V_i}{\partial a_i} + f^{-1} \sum_{i=1}^{n} V_i \frac{\partial(\beta H)}{\partial a_i} - f^{-1} \sum_{i,j=1}^{n} \frac{\partial(\beta H)}{\partial a_i} \frac{\partial B_{ij}}{\partial a_i} - \sum_{i,j=1}^{n} B_{ij} \frac{\partial}{\partial a_i}\left[f^{-1} \frac{\partial(\beta H)}{\partial a_j}\right] = 0, \quad (12)$$

or it can be written in terms of R as

$$-R \sum_{i=1}^{n} \frac{\partial V_i}{\partial a_i} + \sum_{i=1}^{n} V_i \frac{\partial(\beta H)}{\partial a_i} - \sum_{i,j=1}^{n} \frac{\partial B_{ij}}{\partial a_i} \frac{\partial(\beta H)}{\partial a_i} - \sum_{i,j=1}^{n} B_{ij} \frac{\partial^2(\beta H)}{\partial a_i \partial a_j} + qR^{-1} \sum_{i,j=1}^{n} \frac{\partial(\beta H)}{\partial a_i} B_{ij} \frac{\partial(\beta H)}{\partial a_j} = 0. \quad (13)$$

If this equation can determine a relation with full physical property between $\boldsymbol{V}(\boldsymbol{a})$ and $\boldsymbol{B}(\boldsymbol{a})$, which may be called a generalized fluctuation–dissipation theorem, then the dynamics of the Langevin equations that can be described by the Tsallis distribution is found.

We consider the dynamics of the two-variable Brownian motion of a particle, with mass m, in an arbitrary potential $\varphi(x)$ (whether long-range or short-range force); the Langevin equations about the coordinate x and the momentum p are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p}{m}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\mathrm{d}\varphi}{\mathrm{d}x} - \zeta \frac{p}{m} + F_p(t), \quad (14)$$

where ζ is the frictional coefficient. The noise is Gaussian and it is a correlated delta-function,

$$\langle F_p(t) F_p(t') \rangle = 2B\delta(t - t'). \quad (15)$$

The quantities that go into the Fokker–Planck equation are

$$\boldsymbol{a} = \begin{bmatrix} x \\ p \end{bmatrix}, \quad \boldsymbol{V}(\boldsymbol{a}) = \begin{bmatrix} p/m \\ -\mathrm{d}\varphi/\mathrm{d}x - \zeta p/m \end{bmatrix}, \quad \boldsymbol{F}(t) = \begin{bmatrix} 0 \\ F_p(t) \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}. \quad (16)$$

Then the Fokker–Planck equation (7) becomes

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\frac{p}{m}f + \frac{\partial}{\partial p}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x} + \zeta\frac{p}{m}\right)f + B\frac{\partial^2 f}{\partial p^2}, \quad (17)$$

and the stationary-state equation (13) for the Tsallis distribution becomes

$$\frac{p}{m}\left(\frac{\mathrm{d}(\beta\varphi)}{\mathrm{d}x} + \frac{p^2}{2m}\frac{\mathrm{d}\beta}{\mathrm{d}x}\right) - \frac{\beta p}{m}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x} + \frac{\zeta p}{m}\right) + \frac{\zeta}{m}R - \frac{\beta B}{m}\left(1 - q\frac{\beta p^2}{m}R^{-1}\right) = 0, \quad (18)$$

or it can be written as

$$\frac{\zeta}{m}R^2 + \left(\frac{1}{2m^2}\frac{\mathrm{d}\beta}{\mathrm{d}x}p^3 - \frac{\zeta\beta}{m^2}p^2 + \frac{\varphi}{m}\frac{\mathrm{d}\beta}{\mathrm{d}x}p - \frac{B\beta}{m}\right)R + q\frac{B\beta^2}{m^2}p^2 = 0. \quad (19)$$

Substituting into $R = 1-(1-q)\beta H$, where the Hamiltonian is $H = p^2/2m + \varphi(x)$, one finds

$$-\frac{(1-q)\beta}{4m^3}\frac{\mathrm{d}\beta}{\mathrm{d}x}p^5 + \frac{(1-q)(3-q)}{4m^3}\zeta\beta^2p^4 + \frac{1}{2m^2}\frac{\mathrm{d}\beta}{\mathrm{d}x}\left[1-2(1-q)\beta\varphi\right]p^3 + \frac{\beta}{m^2}\left[\frac{1+q}{2}B\beta - (2-q)\zeta\left(1-(1-q)\beta\varphi\right)\right]p^2 + \frac{\varphi}{m}\frac{\mathrm{d}\beta}{\mathrm{d}x}\left[1-(1-q)\beta\varphi\right]p - \left[1-(1-q)\beta\varphi\right]\frac{B\beta}{m} + \left[1-(1-q)\beta\varphi\right]^2\frac{\zeta}{m} = 0. \quad (20)$$

Very clearly, from this equation one can obtain the following three identities to be satisfied for the Tsallis distribution $f \sim \left[1-(1-q)\beta\left(p^2/2m+\varphi(x)\right)\right]^{1/(1-q)}$. Namely, if the Tsallis distribution is a stationary-state solution of the Fokker–Planck equation, then it must fulfil the three identities,

$$\frac{\mathrm{d}\beta}{\mathrm{d}x} = 0, \quad (21)$$

$$\zeta = 0, \quad (22)$$

$$B = 0. \quad (23)$$

The three identities can determine possible dynamics compatible with the Tsallis distribution function. Equations (21)–(23) imply the dynamics of an isothermal process, or a thermal equilibrium system, with no friction and no noise. In this case, the corresponding dynamical equation becomes

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\mathrm{d}\varphi}{\mathrm{d}x} + F_p(t), \quad (24)$$

and the noise is irrelated, $\langle F_p(t)F_p(t')\rangle = 0$. Equation (24) is the dynamical evolution equation of the system governed only by the potential field. When we regard the potential function φ as a gravitational one, equation (24) characterizes the dynamical evolution of the particle in a self-gravitating system, e.g. dark matter is such a case. There has been quite a lot of work on the applications of nonextensive statistics to self-gravitating systems, to many-body systems, and to others.[1−4] Our results lead to the conclusion that the current form of the Tsallis distribution cannot describe any nonequilibrium dynamics of the Langevin equation (14) of the systems governed by any potential, except the simple isothermal situation, namely the dynamical system governed by Eq. (24) when it reaches a thermal equilibrium state. So some existing applications using the Tsallis distribution need to be reconsidered. Examples of the Tsallis distribution behaving only as an isothermal distribution in the above dynamics have been reported in the theoretical aspects for self-gravitating systems[11] and also recently in N-body simulations.[19]

Nothing more can be found for the Tsallis distribution from Eq. (20), except the three properties. From this point of view, we cannot find that the Tsallis distribution is necessary for the statistical description of the dynamical system governed by the Langevin equations (14), with any potential function $\varphi(x)$ because one has not specified whether or not the potential is a long-range one. In other words, there is no possible non-equilibrium dynamics of the Langevin equations (14) that should use the current form of Tsallis distribution in nonextensive statistical mechanics for the statistical description.

In conclusion, we have used the Fokker–Planck equation to study the dynamical property of the Tsallis distribution. We start with a general Langevin equation of the dynamical system with an arbitrary potential function, the noise is Gaussian and a correlated delta-function, and the noise-free part is Markovian. We assume the Tsallis distribution is a stationary-state solution of the Fokker–Planck equation and then search for whether it is a physical solution compatible with the dynamical functions in the Langevin equation. We show that no possi-

ble nonequilibrium dynamics of the Langevin system governed by any potential can use the current form of Tsallis distribution for the statistical description, among which dark matter is one example of such a case. The current form of Tsallis distribution only stands for a simple isothermal or thermal equilibrium situation of the system governed by a potential field. We must find a new form of Tsallis distribution function so as to describe the possible nonequilibrium dynamics of systems governed by a potential field.

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