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Perturbation theory of von Neumann entropy*

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In quantum information theory, von Neumann entropy plays an important role; it is related to quantum channel capacities. Only for a few states can one obtain their entropies. In a continuous variable system, numeric evaluation of entropy is not easy due to infinite dimensions. We develop the perturbation theory for systematically calculating von Neumann entropy of a non-degenerate system as well as a degenerate system.

Keywords: von Neumann entropy, perturbation, degenerate state

PACC: 0367, 0365

1. Introduction

In quantum information theory, von Neumann entropy of a state appears in many basic theorems such as quantum source coding,[1] quantum channel coding of classical information,[2,3] quantum channel coding of quantum information.[4,5] The latter two are problems of channel capacities, which are maxima of Holevo quantity and coherent information, respectively (usually the regulation procedure should be taken). The problems can be reduced to the calculations of von Neumann entropy.

The basic method to obtain the von Neumann entropy of a state is to calculate its spectrum. The spectrum can seldom be obtained for a given state. Meanwhile numeric calculation of the spectrum may encounter problems for a continuous variable system which has an infinite dimensional Hilbert space. Hence a perturbation theory for the von Neumann entropy of a state is needed. While quantum mechanics provides us with the theory of lower order perturbation to the energy levels, here we will develop a systematical theory of the entropy perturbation up to any order of precision. The method of von Neumann entropy also appears in many other quantum information problems.[6,7]

2. Perturbation to the entropy of a non-degenerate system

2.1. Perturbation of entropy via eigenvalues

A non-degenerate quantum system is a density matrix ρ_0 with its eigenvalues $E_n \neq E_m$ for all $n \neq m$, where n (or m) is the number (or vector number) specifying the quantum eigenstate. The usual quantum perturbation theory for the non-degenerate systems gives the first and second order eigenvalue perturbations

$$E_n^{(1)} = H_{nn}, \tag{1}$$

$$E_n^{(2)} = \sum_{m\neq n} \frac{|H_{nm}|^2}{E_n - E_m}, \tag{2}$$

when the perturbation to the density matrix is the matrix H, where $H_{nm} = \langle\psi_n| H |\psi_m\rangle$, with $|\psi_n\rangle$ the eigenvector of the density operator ρ_0 corresponding to the eigenvalue E_n. As usual, the perturbed density matrix can be written as $\rho = \rho_0 + \varepsilon H$. Since ρ and ρ_0 are density matrices, we have $H^\dagger = H$, and $\mathrm{Tr}H = 0$. Thus we have $\sum_n E_n^{(1)} = 0$, and it is clear that $\sum_n E_n^{(2)} = 0$. The total eigenvalue up to the second order is $E_n^{(t)} = E_n + \varepsilon E_n^{(1)} + \varepsilon^2 E_n^{(2)}$. The entropy of the state up to the second order perturbation is $S(\rho) = -\mathrm{Tr}\,\rho\log\rho = -\sum_n E_n^{(t)} \log E_n^{(t)} + o(\varepsilon^2)$. Let

$$S^{(n)}(\rho_0) \equiv \left.\frac{\mathrm{d}^n S(\rho)}{\mathrm{d}\varepsilon^n}\right|_{\varepsilon=0},$$

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we have $S(\rho) = S(\rho_0) + \varepsilon S^{(1)}(\rho_0) + \frac{1}{2}\varepsilon^2 S^{(2)}(\rho_0) + o(\varepsilon^2)$ with

$$S^{(1)}(\rho_0) = -\sum_n H_{nn} \log E_n, \tag{3}$$

$$S^{(2)}(\rho_0) = -\sum_n \frac{H_{nn}^2}{E_n} - 2\sum_n E_n^{(2)} \log E_n. \tag{4}$$

2.2. The expansion formula of the entropy

For $a > 0$, we have

$$\log a = \int_0^\infty \frac{at-1}{a+t} \frac{\mathrm{d}t}{1+t^2}.$$

Similarly, for a positive operator A, we have

$$\log A = \int_0^\infty \frac{At-1}{A+t} \frac{\mathrm{d}t}{1+t^2}.^{[8,9]}$$

Thus

$$\log A - \log B = -\lim_{M\to\infty} \int_0^M \left(\frac{1}{A+t} - \frac{1}{B+t}\right)\mathrm{d}t$$

for positive operators A and B.

$$\begin{aligned} S^{(1)}(\rho_0) &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \operatorname{Tr}[\rho \log \rho - \rho_0 \log \rho_0] \\ &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \operatorname{Tr}[\rho_0(\log \rho - \log \rho_0)] \\ &\quad - \operatorname{Tr}[H \log \rho_0]. \end{aligned} \tag{5}$$

Note that

$$\frac{1}{A+t} = \frac{1}{t}\sum_{k=0}^\infty (-1)^k \left(\frac{A}{t}\right)^k,$$

we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\log \rho - \log \rho_0) \\ &= \lim_{M\to\infty,\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^M \sum_{k=0}^\infty (-1)^k [(\rho_0 + \varepsilon H)^k - \rho_0^k] \frac{\mathrm{d}t}{t^{k+1}}. \end{aligned} \tag{6}$$

We only need the terms that are proportional to ε in the further expansion of $(\rho_0 + \varepsilon H)^k$, all higher order ε terms will be nullified when ε tends to 0. So up to the first order of ε, we have

$$\begin{aligned} &(\rho_0 + \varepsilon H)^k = \rho_0^k + \varepsilon H \rho_0^{k-1} + \rho_0 \varepsilon H \rho_0^{k-2} + \cdots \\ &+ \rho_0^{k-1} \varepsilon H + o(\varepsilon^2), \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\log \rho - \log \rho_0) \\ &= \int_0^\infty \sum_{k=1}^\infty (-1)^k \sum_{l=0}^{k-1} \rho_0^l H \rho_0^{k-l-1} \frac{\mathrm{d}t}{t^{k+1}} \\ &= \int_0^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty (-1)^{m+l+1} \rho_0^l H \rho_0^m \frac{\mathrm{d}t}{t^{m+l+2}} \\ &= -\int_0^\infty (\rho_0 + t)^{-1} H (\rho_0 + t)^{-1} \mathrm{d}t. \end{aligned} \tag{8}$$

So

$$\begin{aligned} S^{(1)}(\rho_0) = &\int_0^\infty \operatorname{Tr}\{\rho_0(\rho_0 + t)^{-1} H (\rho_0 + t)^{-1}\}\mathrm{d}t \\ &- \operatorname{Tr}[H \log \rho_0]. \end{aligned} \tag{9}$$

Evaluating in the eigenbasis of ρ_0, it reads

$$\begin{aligned} S^{(1)}(\rho_0) &= \sum_n \int_0^\infty E_n (E_n + t)^{-2} H_{nn} \mathrm{d}t \\ &\quad - \sum_n H_{nn} \log E_n \\ &= -\sum_n H_{nn} \log E_n, \end{aligned} \tag{10}$$

where $\operatorname{Tr} H = 0$ has been used. It coincides with Eq. (3), the result derived from the perturbation of the eigenvalues.

The second derivative of the entropy at ρ_0 is

$$\begin{aligned} S^{(2)}(\rho_0) = -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \operatorname{Tr}[&\rho_+ \log \rho_+ + \rho_- \log \rho_- \\ &- 2\rho_0 \log \rho_0], \end{aligned} \tag{11}$$

where $\rho_\pm = \rho_0 \pm \varepsilon H$. The derivative can be rewritten as $P_1 + P_2$, the two parts are

$$\begin{aligned} P_1 &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \operatorname{Tr} \rho_0 (\log \rho_+ + \log \rho_- - 2\log \rho_0) \\ &= 2\int_0^\infty \operatorname{Tr} \rho_0 (\rho_0 + t)^{-1} H (\rho_0 + t)^{-1} H (\rho_0 + t)^{-1} \mathrm{d}t, \end{aligned} \tag{12}$$

and

$$\begin{aligned} P_2 &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \operatorname{Tr} H (\log \rho_+ - \log \rho_-) \\ &= -2\int_0^\infty \operatorname{Tr} H (\rho_0 + t)^{-1} H (\rho_0 + t)^{-1} \mathrm{d}t. \end{aligned} \tag{13}$$

With the eigenbasis of ρ_0, we have

$$\begin{aligned} S^{(2)}(\rho_0) &= -2\sum_{n,m} \int_0^\infty \frac{t\,|H_{nm}|^2}{(E_n + t)^2 (E_m + t)} \mathrm{d}t \\ &= -\sum_n \frac{|H_{nn}|^2}{E_n} - 2\sum_{n,m\neq n} \frac{|H_{nm}|^2 E_m}{(E_n - E_m)^2} \log \frac{E_m}{E_n} \\ &= -\sum_n \frac{|H_{nn}|^2}{E_n} - 2\sum_{n,m\neq n} \frac{|H_{nm}|^2}{E_n - E_m} \log E_n. \end{aligned} \tag{14}$$

Note that equations (4) and (14) are strictly the same, and the non-degenerate condition is used.

The n^{th} derivative of the entropy can also be carried out. From the definition of the derivative, we have

$$S^{(n)}(\rho_0) = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\sum_{l=0}^{n} S(\rho_0 + l\varepsilon H)\binom{n}{l}(-1)^l = P_3 + P_4 \tag{15}$$

with

$$P_3 = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\sum_{l=0}^{n}\operatorname{Tr}\rho_0\log(\rho_0 + l\varepsilon H)\binom{n}{l}(-1)^l = (-1)^n n!\int_0^\infty \operatorname{Tr}\rho_0(\rho_0+t)^{-1}[H(\rho_0+t)^{-1}]^n \mathrm{d}t \tag{16}$$

for $n > 0$,

$$P_4 = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\sum_{l=0}^{n}\operatorname{Tr}[\varepsilon l H\log(\rho_0 + l\varepsilon)]\binom{n}{l}(-1)^l = -(-1)^n n!\int_0^\infty \operatorname{Tr}[H(\rho_0+t)^{-1}]^n \mathrm{d}t \tag{17}$$

for $n > 1$ (see Appendix A). Thus we have

$$S^{(n)}(\rho_0) = -(-1)^n n!\int_0^\infty \operatorname{Tr}\{t(\rho_0+t)^{-1} \times [H(\rho_0+t)^{-1}]^n\}\mathrm{d}t. \tag{18}$$

The Taylor expansion formula for the entropy of a state $\rho = \rho_0 + \varepsilon H$ will be

$$\begin{aligned} S(\rho) &= \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}S^{(n)}(\rho_0) \\ &= S(\rho_0) - \varepsilon\operatorname{Tr}[H\log\rho_0] \\ &\quad - \sum_{n=2}^{\infty}(-\varepsilon)^n\int_0^\infty \operatorname{Tr}\{t(\rho_0+t)^{-1} \\ &\quad \times [H(\rho_0+t)^{-1}]^n\}\mathrm{d}t. \end{aligned} \tag{19}$$

We calculate the third and the fourth order perturbation for the case of $H_{nn} = 0$, the entropy up to the fourth order perturbation is

$$S(\rho) = S(\rho_0) - \varepsilon^2\sum_{n,m\neq n}\frac{|H_{nm}|^2}{E_n - E_m}\log E_n + \varepsilon^3 Q_3 - \varepsilon^4(Q_{41}+Q_{42}+Q_{43}) + o(\varepsilon^4), \tag{20}$$

where Q_3, Q_{41}, Q_{42}, and Q_{43} are given in Appendix B.

2.3. Perturbation to the one-mode thermal state

Assuming that a one-mode thermal state is perturbed, the perturbed characteristic function is[10]

$$\chi(\mu) = \chi_T(\mu)[1 + \varepsilon(\alpha^*\mu - \alpha\mu^*)], \tag{21}$$

where $\chi_T(\mu) = \exp[-(N+\frac{1}{2})|\mu|^2]$ is the characteristic function of the one-mode thermal state, with N the average photon number of the state, and α is the complex parameter. The density operator can be obtained with $\rho = \int\frac{\mathrm{d}^2\mu}{\pi}\chi(\mu)D(-\mu)$, where $D(\mu) = \exp[\mu a^\dagger - \mu^* a]$ is the displacement operator, $a^\dagger$ and a are the creation and annihilation operators of the system, respectively. By the method of integration within ordered product operator, the perturbed density operator is

$$\rho = \rho_T + \varepsilon(1-v)(\alpha a^\dagger\rho_T + \alpha^*\rho_T a), \tag{22}$$

where $\rho_{\mathrm{T}} = (1-v)\sum_{n=0}^{\infty}v^n|n\rangle\langle n|$ is the unperturbed one-mode thermal state with $v = N/(N+1)$. It is clear that in the eigenbasis of ρ_{T} the diagonal elements of the perturbation $H = (1-v)(\alpha a^\dagger\rho_{\mathrm{T}} + \alpha^*\rho_{\mathrm{T}}a)$ are null, thus in the evaluation of Eq. (19) we only need to consider the off-diagonal elements of H. Evaluating Eq. (19) in the basis of ρ_{T}, up to ε^4, for a null-diagonal H, the entropy will be

$$S(\rho) = S(\rho_{\mathrm{T}}) - \varepsilon^2\sum_{n,m\neq n}\frac{|H_{nm}|^2}{E_n - E_m}\log E_n + \varepsilon^3 Q_3 - \varepsilon^4(Q_{41}+Q_{42}+Q_{43}) + o(\varepsilon^4). \tag{23}$$

In our case, $H_{nm} = (1-v)(\alpha\sqrt{n}E_{n-1}\delta_{n,m+1} + \alpha^* E_n\sqrt{n+1}\delta_{n,m-1})$. Thus $Q_3 = Q_{41} = 0$ due to the structure of H, and

$$Q_{43} = |\alpha|^4\left[\frac{(1+v)^2}{2v} + \frac{1+v}{1-v}\log v\right],$$
$$Q_{42} = -2|\alpha|^4\left[1 + \frac{1+v^2}{1-v^2}\log v\right].$$

The entropy of the perturbed state is

$$S(\rho) = S(\rho_{\mathrm{T}}) - 2\varepsilon^2|\alpha|^2\log\frac{1}{v} - \varepsilon^4|\alpha|^4\left[\frac{(1-v)^2}{2v} + \frac{1-v}{1+v}\log\frac{1}{v}\right] + o(\varepsilon^4). \tag{24}$$

3. Perturbation to the entropy of a degenerate system

3.1. Entropy perturbation up to second order

Note that equation (19) is an overall result regardless of the eigenvalue structure of the state ρ_0. The difference between the degenerate system and non-degenerate system comes when we evaluate the entropy. We now classify the eigenvectors according to the eigenvalues of ρ_0. Suppose the eigenvalue E_n corresponds to the eigenvector set $\{|n,n_i\rangle, |i=1,\dots n_{\rm d}\}$, these eigenvectors span the subspace of dimension $n_{\rm d}$ for E_n. Define $n_{\rm d}\times m_{\rm d}$ matrix $\boldsymbol{H}_{nm}$ with its entries being

$$H_{nn_i,mm_j}=\langle n,n_i|\,H\,|m,m_j\rangle\,. \tag{25}$$

Considering the first order derivative to the entropy, we have

$$\begin{aligned}S^{(1)}(\rho_0)&=\sum_n\int_0^\infty E_n(E_n+t)^{-2}\,{\rm Tr}\,\boldsymbol{H}_{nn}{\rm d}t\\&\quad-\sum_n{\rm Tr}\,\boldsymbol{H}_{nn}\log E_n\\&=-\sum_n{\rm Tr}\,\boldsymbol{H}_{nn}\log E_n.\end{aligned} \tag{26}$$

Since ${\rm Tr}\,\boldsymbol{H}_{nn}=\sum_{i=1}^{n_{\rm d}}\langle n,n_i|\,H\,|n,n_i\rangle$ is an invariant in the subspace, the first order perturbation to the entropy can be written as $-\sum_k H_{kk}\log E_k$, where k is the unified label for all the distinct eigenvectors (for some $k'\neq k$, we may have $E_{k'}=E_k$). Thus the degenerate of the eigenvalues will not affect the expression of the first order perturbation of the entropy. With the notation of $\boldsymbol{H}_{nm}$, it follows the second order derivative to the entropy

$$\begin{aligned}S^{(2)}(\rho_0)&=-\sum_n\frac{{\rm Tr}[\boldsymbol{H}_{nn}\boldsymbol{H}_{nn}^{\rm T}]}{E_n}\\&\quad-2\sum_{n,m\neq n}\frac{{\rm Tr}[\boldsymbol{H}_{nm}\boldsymbol{H}_{nm}^{\rm T}]}{E_n-E_m}\log E_n,\end{aligned} \tag{27}$$

where

$${\rm Tr}[\boldsymbol{H}_{nn}\boldsymbol{H}_{nn}^{\rm T}]={\rm Tr}[\boldsymbol{H}_{nn}\boldsymbol{H}_{nn}^{*}]=\sum_{i,j}\left|H_{nn_i,nn_j}\right|^2$$

is the summation of the absolute square of all entries of the matrix $\boldsymbol{H}_{nn}$, not the summation of the absolute square of diagonal entries of the matrix $\boldsymbol{H}_{nn}$.

3.2. Perturbation to the two-mode thermal state

Let

$$\rho_{\rm T}=(1-v)\sum_{n=0}^{\infty}v^n\,|n\rangle\,\langle n|$$

be a single-mode thermal state, the direct product of $\rho_{\rm T}$ with itself will result in a two-mode $\rho_2=\rho_{\rm T}\otimes\rho_{\rm T}$. The characteristic function of ρ_2 is $\chi_2(\mu_1,\mu_2)=\exp[-(N+\frac{1}{2})(|\mu_1|^2+|\mu_2|^2)]$. We consider the perturbed characteristic function $\chi(\mu_1,\mu_2)=\chi_2(\mu_1,\mu_2)[1+\varepsilon(\alpha\mu_1\mu_2+\alpha^*\mu_1^*\mu_2^*)]$. The perturbed density operator is $\rho=\rho_2+\varepsilon H$, with

$$H=(1-v)^2(\alpha^*a_1^\dagger a_2^\dagger\rho_2+\alpha\rho_2a_1a_2), \tag{28}$$

where $a_i^\dagger$ and a_i are the creation and annihilation operators for the two modes, respectively. The state is

$$\rho_2=(1-v)^2\sum_{n=0}^{\infty}\sum_{j=0}^{n}v^n\,|j,n-j\rangle\,\langle j,n-j|\,.$$

For any given $n>0$, the state is $(n+1)$-fold degenerate. In the eigenbasis of ρ_2, the entries of matrix $\boldsymbol{H}_{nm}$ are

$$\begin{aligned}&\langle j,n-j|\,H\,|k,m-k\rangle\\&=(1-v)^2[\alpha^*\delta_{j,k+1}\delta_{n,m+2}E_{n-2}\sqrt{j(n-j)}\\&\quad+\alpha\delta_{j,k-1}\delta_{n,m-2}E_n\sqrt{(j+1)(n-j+1)}],\end{aligned} \tag{29}$$

where $E_n=(1-v)^2v^n$ is the eigenvalue of ρ_2. Clearly, $\boldsymbol{H}_{nn}$ now is a zero matrix, so that ${\rm Tr}\,\boldsymbol{H}_{nn}={\rm Tr}[\boldsymbol{H}_{nn}\boldsymbol{H}_{nn}^{\rm T}]=0$. We only need to evaluate the second term of Eq. (27). We have

$$\begin{aligned}&\sum_{n,m\neq n}\frac{{\rm Tr}[\boldsymbol{H}_{nm}\boldsymbol{H}_{nm}^{\rm T}]}{E_n-E_m}\log E_n\\&=(1-v)^4\,|\alpha|^2\sum_n\Bigg[\frac{E_{n-2}^2}{E_n-E_{n-2}}\sum_{j=0}^{n}j(n-j)\\&\quad+\frac{E_n^2}{E_n-E_{n+2}}\sum_{j=0}^{n}(j+1)(n-j+1)\Bigg]\log v^n\\&=-\frac{2\,|\alpha|^2}{1+v}\log\frac{1}{v}.\end{aligned} \tag{30}$$

Thus the entropy of the perturbed state will be

$$S(\rho)=2S(\rho_{\rm T})-\frac{\varepsilon^2\,|\alpha|^2}{1+v}\log\frac{1}{v}+o(\varepsilon^2). \tag{31}$$

4. Density operator expansion

In the calculation of the coherent information, even if we assume that the input state undergoes

a simple form perturbation $\rho = \rho_0 + \varepsilon H$, the output state $\mathcal{E}(\rho)$ and the combined output state $(\mathcal{I} \otimes \mathcal{E}) |\Psi\rangle \langle\Psi|$ may have a more complicated perturbation structure by the application of channel map $\mathcal{E}$ and the purification from ρ to $|\Psi\rangle$. So we need to consider a more generic form of perturbation

$$\rho = \rho_0 + \varepsilon H^{(1)} + \varepsilon^2 H^{(2)} + \varepsilon^3 H^{(3)} + \cdots. \tag{32}$$

It is reasonable to require that $\operatorname{Tr} H^{(n)} = 0$ is true for each n. The entropy perturbation can be obtained in due course, which is

$$S(\rho) = S(\rho_0) - \sum_{n=1}^{\infty} \varepsilon^n \operatorname{Tr}[H^{(n)} \log \rho_0] - \sum_{n=2}^{\infty} \varepsilon^n \sum_{j=2}^{n} (-1)^j \sum_{i_1 i_2 \cdots i_j} \int_0^{\infty} t(\rho_0 + t)^{-1} H^{(i_1)} (\rho_0 + t)^{-1} \times H^{(i_2)} (\rho_0 + t)^{-1} \cdots H^{(i_j)} (\rho_0 + t)^{-1} \delta_{i_1 + i_2 + \cdots + i_j, n}. \tag{33}$$

We may express the second and the third order derivatives of the entropy in more explicit forms

$$\frac{1}{2!} S^{(2)}(\rho_0) = -\operatorname{Tr} \int_0^{\infty} t(\rho_0 + t)^{-1} H^{(1)} (\rho_0 + t)^{-1} H^{(1)} (\rho_0 + t)^{-1} \mathrm{d}t - \operatorname{Tr}[H^{(2)} \log \rho_0], \tag{34}$$

$$\frac{1}{3!} S^{(3)}(\rho_0) = \operatorname{Tr} \int_0^{\infty} t(\rho_0 + t)^{-1} [H^{(1)} (\rho_0 + t)^{-1}]^3 \mathrm{d}t - 2 \operatorname{Tr} \int_0^{\infty} t(\rho_0 + t)^{-1} H^{(1)} (\rho_0 + t)^{-1} H^{(2)} (\rho_0 + t)^{-1} \mathrm{d}t - \operatorname{Tr}[H^{(3)} \log \rho_0]. \tag{35}$$

It is known that the von Neumann entropy is invariant under unitary transformation of the state. Let us consider the displacement of thermal state $\rho_{\mathrm{d}} = D(\varepsilon\alpha)\rho_{\mathrm{T}} D^{\dagger}(\varepsilon\alpha)$. Since the displacement operator is unitary, the entropy will not be changed by the displacement operation. Thus $S(\rho_{\mathrm{d}}) = S(\rho_{\mathrm{T}})$, the entropy $S(\rho_{\mathrm{d}})$ is not a function of ε. If we expand the entropy of ρ_{d} with respect to ε, then the perturbation up to any order should be zero. The characteristic function of the displaced state is

$$\begin{aligned} \chi_{\mathrm{d}}(\mu) &= \operatorname{Tr}[\rho_{\mathrm{d}} D(\mu)] \\ &= \operatorname{Tr}[\rho_{\mathrm{T}} D^{\dagger}(\varepsilon\alpha) D(\mu) D(\varepsilon\alpha)] \\ &= \chi_{\mathrm{T}}(\mu) \exp[\varepsilon(\alpha^* \mu - \alpha \mu^*)]. \end{aligned}$$

Expanding with respect to ε, the characteristic function is

$$\chi_{\mathrm{d}}(\mu) = \chi_{\mathrm{T}}(\mu) \Big[1 + \varepsilon(\alpha^* \mu - \alpha \mu^*) + \frac{1}{2!} \varepsilon^2 (\alpha^* \mu - \alpha \mu^*)^2 + \cdots \Big]. \tag{36}$$

The state is $\rho_{\mathrm{d}} = \rho_{\mathrm{T}} + \varepsilon H^{(1)} + \varepsilon^2 H^{(2)} + \cdots$, with

$$H^{(2)} = \frac{1}{2!} (1 - v)^2 [\alpha^2 a^{\dagger 2} \rho_{\mathrm{T}} + \rho_{\mathrm{T}} \alpha^{*2} a^2 + 2 |\alpha|^2 a^{\dagger} \rho_{\mathrm{T}} a] - (1 - v) |\alpha|^2 \rho_{\mathrm{T}}. \tag{37}$$

The $H^{(1)}$ term has been treated in the former section, which gives rise to the first term of the right-hand side of Eq. (34), the value $2 |\alpha|^2 \log v$. By direct calculation we have

$$-\operatorname{Tr}[H^{(2)} \log \rho_0] = -2 |\alpha|^2 \log v. \tag{38}$$

So we have $S^{(2)}(\rho_0) = 0$. The first and the third derivatives of the entropy should be 0, considering the symmetry of the system. Thus up to the third order perturbation, $S(\rho_{\mathrm{d}}) = S(\rho_{\mathrm{T}})$ is verified.

5. Conclusion

We have derived the entropy perturbation formula for degenerate and non-degenerate systems. Up to any given order of perturbation of the density operator, the entropy perturbation can be written in a concise form via an operator integral, the integral can then be evaluated in the eigenbasis of the unperturbed density operator of the system. We can anticipate that the perturbation formula may be used for calculating the entropy of some non-Gaussian states that are not far from Gaussian states in a continuous variable system.

Appendix A: The n-th order derivative of entropy

$S^{(n)}(\rho_0) = P_3 + P_4$, with

$$\begin{aligned}
P_3 &= -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\sum_{l=0}^{n}\mathrm{Tr}\,\rho_0\log(\rho_0 t+l\varepsilon)\binom{n}{l}(-1)^{n-l} = -\lim_{\varepsilon\to 0}\frac{(-1)^n}{\varepsilon^n}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\mathrm{Tr}\,\rho_0\int_0^\infty\frac{-\mathrm{d}t}{\rho_0+l\varepsilon+t}\\
&= \lim_{\varepsilon\to 0}\frac{(-1)^n}{\varepsilon^n}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\mathrm{Tr}\,\rho_0\int_0^\infty\frac{\mathrm{d}t}{t}\sum_{m=0}^{\infty}\Big(-\frac{\rho_0+l\varepsilon H}{t}\Big)^m\\
&= \lim_{\varepsilon\to 0}\frac{(-1)^n}{\varepsilon^n}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\\
&\quad\times\mathrm{Tr}\,\rho_0\int_0^\infty\frac{\mathrm{d}t}{t}\sum_{m=n}^{\infty}\sum_{i_1,\cdots i_n}\Big(-\frac{\rho_0}{t}\Big)^{i_1}\frac{l\varepsilon H}{t}\cdots\Big(-\frac{\rho_0}{t}\Big)^{i_n}\frac{l\varepsilon H}{t}\Big(-\frac{\rho_0}{t}\Big)^{m-n-\sum_{j=0}^{n}i_j}(-1)^n\\
&= \sum_{l=0}^{n}(-1)^l\binom{n}{l}l^n\int_0^\infty\mathrm{Tr}\,\rho_0(\rho_0+t)^{-1}[H(\rho_0+t)^{-1}]^n\mathrm{d}t\\
&= (-1)^n n!\int_0^\infty\mathrm{Tr}\,\rho_0(\rho_0+t)^{-1}[H(\rho_0+t)^{-1}]^n\mathrm{d}t.
\end{aligned}$$

The last equation comes from the identity $\sum_{l=0}^{n}(-1)^l\binom{n}{l}l^n = (-1)^n n!$. In the fourth equation we have used the fact that $\sum_{l=0}^{n}(-1)^l\binom{n}{l}l^m = 0$ for all $m < n$. We now prove these two results. Assuming $g(x) = (1-x)^n = \sum_{l=0}^{n}(-1)^l\binom{n}{l}x^l$, let

$$f_m(x) = \Big(x\frac{\mathrm{d}}{\mathrm{d}x}\Big)^m g(x) = \sum_{l=0}^{n}(-1)^l\binom{n}{l}l^m x^l,$$

then

$$f_m(x) = \sum_{k=1}^{m}c_k^{(m)}x^k\frac{\mathrm{d}^k g(x)}{\mathrm{d}x^k},$$

where $c_k^{(m)}$ is the coefficient which obeys the recursive relation $c_k^{(m+1)} = kc_k^{(m)} + c_{k-1}^{(m)}$. We always have $c_m^{(m)} = 1$. Note that $\mathrm{d}^k g(0)/\mathrm{d}x^k = 0$, for all $k < n$, and $\mathrm{d}^k g(0)/\mathrm{d}x^k = (-1)^n n!$, for $k = n$. Thus we have $\sum_{l=0}^{n}(-1)^l\binom{n}{l}l^n = f_n(1) = (-1)^n n!$ and $\sum_{l=0}^{n}(-1)^l\binom{n}{l}l^m = f_m(1) = 0$ for all $m < n$.

P_4 can be obtained in a very similar way.

Appendix B: The third and fourth order off-diagonal perturbations

When the perturbation is off-diagonal in the eigenbasis of the unperturbed state, we have $H_{nn} = 0$. The contribution to the third order perturbation should be

$$Q_3 = \int_0^\infty\mathrm{Tr}\{t(\rho_0+t)^{-1}[H(\rho_0+t)^{-1}]^3\}\mathrm{d}t.$$

Evaluating in the basis of ρ_0, we have

$$Q_3 = \sum_{n,m,k}\int_0^\infty\frac{tH_{nm}H_{mk}H_{kn}}{(E_n+t)^2(E_m+t)(E_k+t)}\mathrm{d}t.$$

The summation should be taken for the case that n, m and k are all different since $H_{nn} = 0$. After integrating and considering the permutation symmetry, we have

$$Q_3 = \sum_{n<m<k}\frac{2\,\mathrm{Re}(H_{nm}H_{mk}H_{kn})}{E_m-E_k}\Big[\frac{\log(E_n/E_k)}{(E_n-E_k)} - \frac{\log(E_n/E_m)}{E_n-E_m}\Big].$$

The contribution to the fourth order perturbation is

$$Q_4 = \sum_{n,m,l,k}\int_0^\infty\frac{tH_{nm}H_{ml}H_{lk}H_{kn}}{(E_n+t)^2(E_m+t)(E_l+t)(E_k+t)}\mathrm{d}t.$$

When all n, m, l, k are different from each other, the integral will be

$$Q_{41} = \sum_{n<m<l<k} 2\,\mathrm{Re}(H_{nm}H_{ml}H_{lk}H_{kn} + H_{nm}H_{ml}H_{lk}H_{kn} + H_{nm}H_{ml}H_{lk}H_{kn}) \times \left[\frac{\log(E_m/E_n)}{(E_m-E_n)(E_m-E_l)(E_m-E_k)} + \frac{\log(E_l/E_n)}{(E_l-E_n)(E_l-E_m)(E_l-E_k)} + \frac{\log(E_k/E_n)}{(E_k-E_n)(E_k-E_m)(E_k-E_l)}\right],$$

where permutation symmetry are considered. When two of n, m, l, k are equal, that is (i) $n = l \neq m \neq k$ and (ii) $m = k \neq n \neq l$, the integral will be

$$Q_{42} = \sum_{n,m,k} \int_0^\infty \frac{t\,|H_{nm}H_{nk}|^2}{(E_n+t)^3(E_m+t)(E_k+t)}\mathrm{d}t + \sum_{n,l,m} \int_0^\infty \frac{t\,|H_{mn}H_{ml}|^2}{(E_n+t)^2(E_m+t)^2(E_l+t)}\mathrm{d}t,$$

which is

$$Q_{42} = \sum_{n<m<k} (|H_{nm}H_{nk}|^2 + |H_{mn}H_{mk}|^2 + |H_{kn}H_{km}|^2) \times \left[\frac{1}{E_n(E_n-E_m)(E_n-E_k)} + \frac{\log(E_m/E_n)}{(E_m-E_n)^2(E_m-E_k)} + \frac{\log(E_k/E_n)}{(E_k-E_n)^2(E_k-E_m)}\right].$$

When $n = l$ and $m = k$, we have the integral

$$Q_{43} = \sum_{n,m} \int_0^\infty \frac{t\,|H_{nm}|^4}{(E_n+t)^3(E_m+t)^2}\mathrm{d}t,$$

which is

$$Q_{43} = \sum_{n<m} |H_{nm}|^4 \left[\frac{1}{2(E_n-E_m)^2}\left(\frac{1}{E_n}+\frac{1}{E_m}\right) - \frac{1}{(E_n-E_m)^3}\log\frac{E_n}{E_m}\right].$$

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