New two-fold integration transformation for the Wigner operator in phase space quantum mechanics and its relation to operator ordering*

Fan Hong-Yi(范洪义)†

Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, *China*

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Using the Weyl ordering of operators expansion formula (Hong-Yi Fan, *J. Phys.* A **25** (1992) 3443) this paper finds a kind of two-fold integration transformation about the Wigner operator $\Delta(q',p')$ (q-number transform) in phase space quantum mechanics,

$$\iint_{-\infty}^{\infty}\frac{\mathrm{d}p'\mathrm{d}q'}{\pi}\Delta\left(q',p'\right)\mathrm{e}^{-2\,\mathrm{i}(p-p')(q-q')}=\delta\left(p-P\right)\delta\left(q-Q\right),$$

and its inverse

$$\iint_{-\infty}^{\infty}\mathrm{d}q\mathrm{d}p\delta\left(p-P\right)\delta\left(q-Q\right)\mathrm{e}^{2\,\mathrm{i}(p-p')(q-q')}=\Delta\left(q',p'\right),$$

where Q, P are the coordinate and momentum operators, respectively. We apply it to study mutual converting formulae among Q–P ordering, P–Q ordering and Weyl ordering of operators. In this way, the contents of phase space quantum mechanics can be enriched. The formula of the Weyl ordering of operators expansion and the technique of integration within the Weyl ordered product of operators are used in this discussion.

Keywords: Wigner operator, Weyl ordering, two-fold integration transformation

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1. Introduction

Phase space quantum mechanics (PSQM) pioneered by Wigner[1] and Weyl[2] has received more and more attention since the foundation of quantum mechanics, because it has wide applications in quantum statistics, quantum optics, and quantum chemistry. In PSQM observables and states are replaced by functions on classical phase space so that expected values are calculated, as in classical statistical physics, by averaging over the phase space. The phase-space approach provides valuable physical insight and allows us to describe similar classical and quantum processes using similar language. Development of PSQM[3−5] is always accompanied by solving operator ordering problems. Weyl proposed a scheme for quantizing classical coordinate and momentum quantity q^mp^n (c-number) as the quantum operators (q-number) in the following way

$$q^mp^n\to\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\binom{m}{l}Q^{m-l}P^nQ^l, \tag{1}$$

where Q, P are the coordinate and momentum operators, respectively, $[Q,P]=\mathrm{i}\hbar$. (Later in this work we set $\hbar=1$). The Q's eigenvector $|q\rangle$ and P's eigenvector $|p\rangle$ satisfy the relation $|q\rangle\langle q|=\delta\left(q-Q\right)$, $|p\rangle\langle p|=\delta\left(p-P\right)$, respectively. The right-hand side of Eq. (1) is in Weyl ordering, so we introduce the symbol $\vdots\,\vdots$ to characterize it,[6,7] and

$$\begin{aligned}q^mp^n&\to\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\binom{m}{l}Q^{m-l}P^nQ^l\\&=\vdots\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\binom{m}{l}Q^{m-l}P^nQ^l\vdots\\&=\vdots Q^mP^n\vdots,\end{aligned} \tag{2}$$

where in the second step we have used the property that Bose operators are commutative within $\vdots\,\vdots$. This

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†Corresponding author. E-mail: fhym@ustc.edu.cn

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is like the fact that Bose operators are commutative within the normal ordering symbol : :. The Weyl quantization rule between an operator $H(P,Q)$ and its classical correspondence is

$$H(P,Q)=\iint_{-\infty}^{\infty}\mathrm{d}q\mathrm{d}p h(p,q)\,\Delta(q,p), \tag{3}$$

where $\Delta(q,p)$ is the Wigner operator.[2−5,8] Using $\vdots\,\vdots$ we have invented an integration technique within the Weyl ordered product of operators with which we have constructed a Weyl ordering of operators expansion formula (see Eq. (21) below, which is the same as Eq. (53) in Ref. [7]). In this work we shall use this formula to find a kind of two-fold q-number integration transformation about the Wigner operator $\Delta(q',p')$ in PSQM (see Eqs. (33) and (34) below), which helps to convert P–Q ordering and Q–P ordering to Weyl ordering, and *vice versa*. The ordering rules of the Heisenberg–Weyl (HW) algebra generators have been subjects of interest for decades, and have a huge literature devoted to them. Cohen introduced a ("Cohen") function which provides all possible ordering schemes for the HW algebra generators;[9] Cahill and Glauber[10] described all different orderings by a single complex parameter, called the ordering parameter. The quantizations of q^mp^n and q–p exponentials in all orderings, for example, can be found in Wolf's book.[11,12] In the present work we shall adopt another different approach for tackling three ordering schemes: Weyl, P–Q and Q–P, and will present a well-ordered compendium of formulae of this type. The work is arranged as follows: In Section 2 we briefly review the Weyl ordered form of the Wigner operator. In Section 3 we discuss the transition from the density operator ρ's Weyl ordering expansion to ρ's Weyl correspondence. In Section 4 we derive the Weyl ordering forms of $\delta(p-P)\,\delta(q-Q)$ and $\delta(q-Q)\,\delta(p-P)$, their transformation to the Wigner operator is shown in Section 5. Sections 6 and 7 are devoted to deriving mutual converting formulae among Q–P ordering, P–Q ordering and Weyl ordering of operators. In this way, the contents of PSQM can be enriched.

2. The Weyl ordered form of the Wigner operator

According to Eq. (3) we can rewrite Eq. (2) as

$$\vdots Q^mP^n\vdots=\iint_{-\infty}^{\infty}\mathrm{d}q\mathrm{d}p q^mp^n\Delta(q,p), \tag{4}$$

which implies that the integration kernel (the Wigner operator) is[6,7]

$$\begin{aligned}\Delta(q,p)&=\vdots\delta(q-Q)\,\delta(p-P)\vdots\\&=\vdots\delta(p-P)\,\delta(q-Q)\vdots.\end{aligned} \tag{5}$$

Substituting Eq. (5) into Eq. (3) yields $H(P,Q)=\vdots h(P,Q)\vdots$, where $\vdots h(P,Q)\vdots$ is just the result of replacing $p\to P, q\to Q$ in $h(p,q)$ and then putting it within $\vdots\,\vdots$. Further, using $Q=(a+a^\dagger)/\sqrt{2}$, $P=(a-a^\dagger)/\sqrt{2}\mathrm{i}$, $\alpha=(q+\mathrm{i}p)/\sqrt{2}$, $\left[a,a^\dagger\right]=1$, we can express

$$\begin{aligned}\Delta(q,p)&\to\Delta(\alpha,\alpha^*)\\&=\frac{1}{2}\vdots\delta(\alpha-a)\,\delta\left(\alpha^*-a^\dagger\right)\vdots.\end{aligned} \tag{6}$$

It then follows

$$\begin{aligned}\vdots K\left(a^\dagger,a\right)\vdots&=\int\mathrm{d}^2\alpha K(\alpha^*,\alpha)\vdots\delta(\alpha-a)\,\delta\left(\alpha^*-a^\dagger\right)\vdots\\&=2\int\mathrm{d}^2\alpha K(\alpha^*,\alpha)\,\Delta(\alpha,\alpha^*).\end{aligned} \tag{7}$$

Thus the neat expression of $\Delta(q,p)$ in Dirac's delta function form is very useful; one of its uses is that the marginal distributions of the Wigner operator can be clearly shown, since the coordinate and momentum projectors are respectively

$$|q\rangle\langle q|=\delta(q-Q)=\vdots\delta(q-Q)\vdots, \tag{8}$$

$$|p\rangle\langle p|=\delta(p-P)=\vdots\delta(p-P)\vdots, \tag{9}$$

we immediately know that the following marginal integration

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}q\Delta(q,p)&=\int_{-\infty}^{\infty}\mathrm{d}q\vdots\delta(q-Q)\,\delta(p-P)\vdots\\&=\vdots\delta(p-P)\vdots=|p\rangle\langle p|,\end{aligned} \tag{10}$$

similarly,

$$\int_{-\infty}^{\infty}\mathrm{d}p\Delta(q,p)=\vdots\delta(q-Q)\vdots=|q\rangle\langle q|. \tag{11}$$

It then follows the completeness of $\Delta(q,p)$,

$$\iint_{-\infty}^{\infty}\mathrm{d}q\mathrm{d}p\Delta(q,p)=1, \tag{12}$$

so the Weyl rule for $H(P,Q)$ in Eq. (3) can also be viewed as H's expansion in terms of $\Delta(q,p)$. When $H(P,Q)$ is Weyl ordered, which means $H(P,Q)=$

$\vdots H(P,Q) \vdots$, then using the completeness Eq. (12) we see

$$\vdots H(P,Q) \vdots = \vdots H(P,Q) \vdots \iint_{-\infty}^{\infty} \mathrm{d}q\mathrm{d}p \Delta(q,p) = \iint_{-\infty}^{\infty} \mathrm{d}q\mathrm{d}p H(q,p) \Delta(q,p), \tag{13}$$

as if $\Delta(q,p)$ was the "eigenvector" of $\vdots H(P,Q) \vdots$. On the other hand, due to the normally ordered forms of $|q\rangle\langle q|$ and $|p\rangle\langle p|$[8]

$$|q\rangle\langle q| = \frac{1}{\sqrt{\pi}} {:}\mathrm{e}^{-(q-Q)^2}{:}, \tag{14}$$

$$|p\rangle\langle p| = \frac{1}{\sqrt{\pi}} {:}\mathrm{e}^{-(p-P)^2}{:}, \tag{15}$$

we know the normally ordered form of $\Delta(q,p)$[13]

$$\begin{aligned}\Delta(q,p) &= \frac{1}{\pi} {:}\mathrm{e}^{-(q-Q)^2-(p-P)^2}{:} \\ &= \frac{1}{\pi} {:}\mathrm{e}^{-2\left(\alpha^*-a^\dagger\right)(\alpha-a)}{:} \\ &= \Delta(\alpha,\alpha^*),\end{aligned} \tag{16}$$

where $a = (Q+\mathrm{i}P)/\sqrt{2}$, $\left[a, a^\dagger\right] = 1$, and

$$\begin{aligned}&\mathrm{Tr}\left[\Delta(\alpha,\alpha^*)\Delta(\alpha',\alpha'^*)\right] \\ &= \frac{1}{4\pi}\delta(\alpha-\alpha')\delta(\alpha^*-\alpha'^*).\end{aligned} \tag{17}$$

3. From the density operator ρ's Weyl ordering expansion to ρ's Weyl correspondence

In Refs. [6] and [7] we have presented a Weyl ordering of operators expansion formula

$$\rho = 2\int \frac{\mathrm{d}^2\beta}{\pi} \vdots \langle -\beta|\rho|\beta\rangle \exp\left[2\left(\beta^* a - a^\dagger\beta + a^\dagger a\right)\right] \vdots, \tag{18}$$

where $|\beta\rangle = \exp[-|\beta|^2/2 + \beta a^\dagger]|0\rangle$ is the coherent state,[14,15] its completeness relation is

$$\int \frac{\mathrm{d}^2\beta}{\pi} |\beta\rangle\langle\beta| = 1, \quad a|\beta\rangle = \beta|\beta\rangle. \tag{19}$$

For the pure coherent state density operator $|\alpha\rangle\langle\alpha|$, using Eq. (18) and the overlap

$$\langle\alpha|\beta\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right) + \alpha^*\beta\right], \tag{20}$$

we derive

$$\begin{aligned}|\alpha\rangle\langle\alpha| &= 2\vdots \int \frac{\mathrm{d}^2\beta}{\pi} \langle -\beta|\alpha\rangle\langle\alpha|\beta\rangle \\ &\quad \times \exp[2\left(\beta^* a - a^\dagger\beta + a^\dagger a\right)]\vdots \\ &= 2\vdots \exp\left[-2(\alpha-a)\left(\alpha^*-a^\dagger\right)\right]\vdots \\ &= 2\vdots \exp\left[-(p-P)^2 - (q-Q)^2\right]\vdots,\end{aligned} \tag{21}$$

thus the Weyl ordered form of pure coherent state $|\alpha\rangle\langle\alpha|$ is a Gaussian in p–q space. Combining Eqs. (7), (17) and (18) yields

$$\begin{aligned}&2\pi\mathrm{Tr}\left[\rho\Delta(\alpha,\alpha^*)\right] \\ =&4\int \mathrm{d}^2\beta \langle -\beta|\rho|\beta\rangle \mathrm{Tr}\left\{\vdots \mathrm{e}^{2\left(\beta^* a - a^\dagger\beta + a^\dagger a\right)} \vdots \Delta(\alpha,\alpha^*)\right\} \\ =&4\int \mathrm{d}^2\beta \langle -\beta|\rho|\beta\rangle \mathrm{Tr}\Big[2\int \mathrm{d}^2\alpha' \Delta(\alpha',\alpha'^*)\Delta(\alpha,\alpha^*) \\ &\times \exp\left[2\left(\beta^*\alpha' - \alpha'^*\beta + \alpha'^*\alpha'\right)\right]\Big] \\ =&2\int \frac{\mathrm{d}^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle \int \mathrm{d}^2\alpha' \mathrm{e}^{2\left(\beta^*\alpha' - \alpha'^*\beta + \alpha'^*\alpha\right)} \\ &\times \delta(\alpha-\alpha')\delta(\alpha^*-\alpha'^*) \\ =&2\int \frac{\mathrm{d}^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle \mathrm{e}^{2(\beta^*\alpha - \alpha^*\beta + \alpha^*\alpha)},\end{aligned} \tag{22}$$

which is just an alternate expression of the Wigner function of ρ, comparing Eq. (18) with Eq. (22) we see that the latter is just the result of replacing $a \to \alpha$, $a^\dagger \to \alpha^*$, in the former; this is because the right-hand side of Eq. (18) is in Weyl ordering, the classical Weyl correspondence $2\pi\mathrm{Tr}\left[\rho\Delta(\alpha,\alpha^*)\right]$ of the right-hand side of Eq. (18) is just the right-hand side of Eq. (22).

4. Weyl ordering of $\delta(p-P)\,\delta(q-Q)$ and $\delta(q-Q)\delta(p-P)$

Now we examine what the Weyl ordering of $\delta(p-P)\,\delta(q-Q)$ is. Using the completeness relation of $|q\rangle$, the coordinate eigenstate, and the completeness relation of the momentum eigenstate $|p\rangle$, $\langle q|p\rangle = (1/\sqrt{2\pi})\mathrm{e}^{\mathrm{i}pq}$, we have

$$\begin{aligned}&\delta(p-P)\,\delta(q-Q) \\ &= \int \mathrm{d}p'|p'\rangle\langle p'|\,\delta(p-P)\,\delta(q-Q)\int \mathrm{d}q'|q'\rangle\langle q'| \\ &= \int \frac{\mathrm{d}p'}{\sqrt{2\pi}}|p'\rangle \int \mathrm{d}q'\langle q'|\,\delta(p-p')\,\delta(q-q')\,\mathrm{e}^{-\mathrm{i}p'q'} \\ &= \frac{1}{\sqrt{2\pi}}|p\rangle\langle q|\,\mathrm{e}^{-\mathrm{i}pq}.\end{aligned} \tag{23}$$

The overlap between $\langle q|$ and the coherent state is

$$\langle q|\ \beta\rangle = \pi^{-1/4}\exp\left\{-\frac{q^2}{2}+\sqrt{2}q\beta-\frac{1}{2}\beta^2-\frac{1}{2}|\beta|^2\right\}, \tag{24}$$

and

$$\langle -\beta\ |p\rangle = \pi^{-1/4}\exp\left\{-\frac{p^2}{2}-\sqrt{2}\,\mathrm{i}\,p\beta^*+\frac{1}{2}\beta^{*2}-\frac{1}{2}|\beta|^2\right\}. \tag{25}$$

Substituting Eq. (23) into Eq. (18) and using Eqs. (24) and (25) lead to

$$\begin{aligned}
&\delta(p-P)\,\delta(q-Q)\\
&=\frac{\sqrt{2}}{\pi}\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\langle-\beta|\ p\rangle\langle q|\,\mathrm{e}^{-\mathrm{i}pq}\,|\beta\rangle\\
&\quad\times\exp\left[2\left(\beta^*a-a^\dagger\beta+a^\dagger a\right)\right]\vdots\\
&=\frac{\sqrt{2}}{\pi}\mathrm{e}^{-\frac{q^2+p^2}{2}-\mathrm{i}pq}\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\mathrm{e}^{-|\beta|^2+\sqrt{2}q\beta-\sqrt{2}\mathrm{i}p\beta^*}\\
&\quad\times\mathrm{e}^{2\left(\beta^*a-a^\dagger\beta+a^\dagger a\right)-\frac{\beta^2}{2}+\frac{\beta^{*2}}{2}}\vdots\\
&=\frac{1}{\pi}\vdots\exp\left[\sqrt{2}q\left(a-a^\dagger\right)+\sqrt{2}\mathrm{i}p\left(a+a^\dagger\right)\right]\\
&\quad\times\exp\left[-2\mathrm{i}pq+a^{\dagger2}-a^2-a^\dagger a\right]\vdots\\
&=\frac{1}{\pi}\vdots\exp[-2\mathrm{i}\,(q-Q)\,(p-P)]\vdots.
\end{aligned}\tag{26}$$

Similarly, we can derive

$$\begin{aligned}
&\delta(q-Q)\,\delta(p-P)\\
&=2\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\langle-\beta|\ q\rangle\langle p|\,\mathrm{e}^{\mathrm{i}pq}\,|\beta\rangle\,\mathrm{e}^{2\left(\beta^*a-a^\dagger\beta+a^\dagger a\right)}\vdots\\
&=\frac{1}{\pi}\vdots\exp[2\mathrm{i}\,(q-Q)\,(p-P)]\vdots.
\end{aligned}\tag{27}$$

Equations (26) and (27) have been derived in another approach in Ref. [16].

5. New transformation of the Wigner operator

By noticing Eq. (5) we see

$$\begin{aligned}
&\frac{1}{\pi}\vdots\exp[-2\mathrm{i}\,(q-Q)\,(p-P)]\vdots\\
&=\frac{1}{\pi}\iint\mathrm{d}p'\mathrm{d}q'\mathrm{e}^{-2\mathrm{i}\left(q-q'\right)\left(p-p'\right)}\vdots\delta(q'-Q)\,\delta(p'-P)\vdots\\
&=\frac{1}{\pi}\iint\mathrm{d}p'\mathrm{d}q'\Delta(q',p')\,\mathrm{e}^{-2\mathrm{i}\left(p-p'\right)\left(q-q'\right)}.
\end{aligned}\tag{28}$$

It then follows from Eqs. (28) and (26) that

$$\begin{aligned}
&\frac{1}{\pi}\iint\mathrm{d}p'\mathrm{d}q'\Delta(q',p')\,\mathrm{e}^{-2\mathrm{i}\left(p-p'\right)\left(q-q'\right)}\\
&=\delta(p-P)\,\delta(q-Q).
\end{aligned}\tag{29}$$

Similarly, we can derive

$$\frac{1}{\pi}\iint\mathrm{d}p'\mathrm{d}q'\Delta(q',p')\,\mathrm{e}^{2\mathrm{i}\left(p-p'\right)\left(q-q'\right)}=\delta(q-Q)\,\delta(p-P), \tag{30}$$

so $\mathrm{e}^{\pm2\mathrm{i}\left(p-p'\right)\left(q-q'\right)}/\pi$ can be considered as the classical Weyl correspondence of $\delta(q-Q)\,\delta(p-P)$ and $\delta(p-P)\,\delta(q-Q)$, respectively. Moreover, the inverse transform of Eq. (28) is

$$\begin{aligned}
&\iint\frac{\mathrm{d}q\mathrm{d}p}{\pi}\vdots\exp[-2\mathrm{i}\,(q-Q)\,(p-P)]\vdots\mathrm{e}^{2\mathrm{i}\left(p-p'\right)\left(q-q'\right)}\\
&=\iint\frac{\mathrm{d}q\mathrm{d}p}{\pi}\iint\mathrm{d}p''\mathrm{d}q''\Delta(q'',p'')\,\mathrm{e}^{-2\mathrm{i}\left(p-p''\right)\left(q-q''\right)+2\mathrm{i}\left(p-p'\right)\left(q-q'\right)}\\
&=\iint\mathrm{d}p''\mathrm{d}q''\Delta(q'',p'')\,\mathrm{e}^{-2\,\mathrm{i}\left(p''q''-p'q'\right)}\delta(q'-q'')\,\delta(p'-p'')\\
&=\Delta(q',p'),
\end{aligned}\tag{31}$$

which means

$$\iint\mathrm{d}q\mathrm{d}p\delta(p-P)\,\delta(q-Q)\,\mathrm{e}^{2\mathrm{i}\left(p-p'\right)\left(q-q'\right)}=\Delta(q',p'), \tag{32}$$

or

$$\iint\mathrm{d}q\mathrm{d}p\delta(q-Q)\,\delta(p-P)\,\mathrm{e}^{-2\mathrm{i}\left(p-p'\right)\left(q-q'\right)}=\Delta(q',p'). \tag{33}$$

Equations (29)–(33) are new transformations of the Wigner operator in q–p phase space.

Further, multiplying both sides of Eq. (31) from the left by $\iint\mathrm{d}q'\mathrm{d}p'h(p',q')$ we obtain

$$\begin{aligned}
&\iint \mathrm{d}q'\mathrm{d}p' h\left(p',q'\right) \Delta\left(q',p'\right) \\
&= \iint \mathrm{d}q'\mathrm{d}p' h\left(p',q'\right) \iint \frac{\mathrm{d}q\mathrm{d}p}{\pi} \mathrm{e}^{2\mathrm{i}(p-p')(q-q')} \\
&\quad \times \vdots \exp[-2\,\mathrm{i}\,(q-Q)\,(p-P)] \vdots \\
&= \iint \frac{\mathrm{d}q\mathrm{d}p}{\pi} \vdots \mathrm{e}^{-2\mathrm{i}(q-Q)(p-P)} \vdots G\left(p,q\right), \qquad (34)
\end{aligned}$$

where we have introduced

$$G\left(p,q\right) \equiv \frac{1}{\pi} \iint \mathrm{d}q'\mathrm{d}p' h\left(p',q'\right) \mathrm{e}^{2\mathrm{i}(p-p')(q-q')}, \quad (35)$$

this is a new interesting transformation, because when $h\left(p',q'\right)=1$,

$$\begin{aligned}
&\frac{1}{\pi} \iint \mathrm{d}q'\mathrm{d}p' \mathrm{e}^{2\mathrm{i}(p-p')(q-q')} \\
&= \int_{-\infty}^{\infty} \mathrm{d}q' \delta\left(q-q'\right) \mathrm{e}^{2\mathrm{i}p(q-q')} = 1. \qquad (36)
\end{aligned}$$

The inverse of Eq. (35) is

$$\iint \frac{\mathrm{d}q\mathrm{d}p}{\pi} \mathrm{e}^{-2\,\mathrm{i}(p-p')(q-q')} G\left(p,q\right) = h\left(p',q'\right). \quad (37)$$

This transformation's Parsval-like theorem is[17]

$$\iint_{-\infty}^{\infty} \frac{\mathrm{d}q\mathrm{d}p}{\pi} |h(p,q)|^2 = \iint \frac{\mathrm{d}q'\mathrm{d}p'}{\pi} |G\left(p',q'\right)|^2. \quad (38)$$

6. *P*–*Q* ordering and *Q*–*P* ordering to Weyl ordering

We now use the above transformation to discuss some operator ordering problems. For instance, from the integration formula we have

$$\begin{aligned}
&\iint_{-\infty}^{\infty} \frac{\mathrm{d}x\mathrm{d}y}{\pi} x^m y^r \exp[2\mathrm{i}\,(y-s)\,(x-t)] \\
&= \left(\frac{1}{\sqrt{2}}\right)^{m+r} (-\mathrm{i})^r\, \mathrm{H}_{m,r}\left(\sqrt{2}t, \mathrm{i}\sqrt{2}s\right), \qquad (39)
\end{aligned}$$

where $\mathrm{H}_{m,r}$ is the two-variable Hermite polynomials,[18,19]

$$\mathrm{H}_{m,r}(t,s) = \sum_{l=0}^{\min(m,r)} \frac{m!r!(-1)^l}{l!(m-l)!(r-l)!} t^{m-l} s^{r-l}. \quad (40)$$

Equation (39) can be proved as follows:

$$\begin{aligned}
&\text{L.H.S. of Eq. (39)} \\
&= \mathrm{e}^{2\mathrm{i}st} \left(\frac{\partial}{\partial t}\right)^r \left(\frac{\partial}{\partial s}\right)^m \iint_{-\infty}^{\infty} \frac{\mathrm{d}x\mathrm{d}y}{\pi} \mathrm{e}^{2\mathrm{i}xy-2\mathrm{i}yt-2\mathrm{i}sx} \\
&= \mathrm{e}^{2\mathrm{i}st} \left(\frac{\partial}{\partial t}\right)^r \left(\frac{\partial}{\partial s}\right)^m \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-2\mathrm{i}sx} \delta\left(x-t\right) \\
&= \mathrm{e}^{2\mathrm{i}st} \left(\frac{\partial}{\partial t}\right)^r \left(\frac{\partial}{\partial s}\right)^m \mathrm{e}^{-2\mathrm{i}st} \\
&= \text{R.H.S. of Eq. (39)}. \qquad (41)
\end{aligned}$$

Using Eqs. (27) and (39) we know

$$\begin{aligned}
Q^m P^r &= \iint_{-\infty}^{\infty} \mathrm{d}p\mathrm{d}q q^m p^r \delta\left(q-Q\right) \delta\left(p-P\right) \\
&= \iint_{-\infty}^{\infty} \frac{\mathrm{d}p\mathrm{d}q}{\pi} q^m p^r \vdots \exp[2\mathrm{i}\,(p-P)\,(q-Q)] \vdots \\
&= \left(\frac{1}{\sqrt{2}}\right)^{m+r} (-\mathrm{i})^r \vdots \mathrm{H}_{m,r}\left(\sqrt{2}Q, \mathrm{i}\sqrt{2}P\right) \vdots, \qquad (42)
\end{aligned}$$

this is a simpler way to put $Q^m P^r$ into its Weyl ordering. Similarly, using Eq. (26) and the complex conjugate of Eq. (39) we see that the Weyl ordered form of $P^r Q^m$ is

$$\begin{aligned}
P^r Q^m &= \iint_{-\infty}^{\infty} \mathrm{d}p\mathrm{d}q p^r q^m \delta\left(p-P\right) \delta\left(q-Q\right) \\
&= \iint_{-\infty}^{\infty} \frac{\mathrm{d}p\mathrm{d}q}{\pi} \vdots \mathrm{e}^{-2\mathrm{i}(q-Q)(p-P)} \vdots q^m p^r \\
&= \left(\frac{1}{\sqrt{2}}\right)^{m+r} (\mathrm{i})^r \vdots \mathrm{H}_{m,r}\left(\sqrt{2}Q, -\mathrm{i}\sqrt{2}P\right) \vdots. \qquad (43)
\end{aligned}$$

7. Weyl ordering to *P*–*Q* ordering and *Q*–*P* ordering

According to Eqs. (35) and (37) we know that the inverse transform of Eq. (39) is

$$\iint \frac{\mathrm{d}s\mathrm{d}t}{\pi} \left(\frac{1}{\sqrt{2}}\right)^{m+r} (-\mathrm{i})^r\, \mathrm{H}_{m,r}\left(\sqrt{2}t, \mathrm{i}\sqrt{2}s\right) \mathrm{e}^{-2\mathrm{i}(y-s)(x-t)} = x^m y^r, \quad (44)$$

which is a new integration formula. Then from Eqs. (26) and (44) we have

$$\left(\frac{1}{\sqrt{2}}\right)^{m+r} (-\mathrm{i})^r\, \mathrm{H}_{m,r}\left(\sqrt{2}Q, \mathrm{i}\sqrt{2}P\right) |_{P \text{ before } Q}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{m+r} (-\mathrm{i})^r \iint \mathrm{d}p\mathrm{d}q\delta\left(p-P\right)\delta\left(q-Q\right) \mathrm{H}_{m,r}\left(\sqrt{2}q, \mathrm{i}\sqrt{2}p\right)$$
$$= \left(\frac{1}{\sqrt{2}}\right)^{m+r} (-\mathrm{i})^r \iint \frac{\mathrm{d}p\mathrm{d}q}{\pi} \mathrm{H}_{m,r}\left(\sqrt{2}q, \mathrm{i}\sqrt{2}p\right) \vdots \mathrm{e}^{-2\mathrm{i}(q-Q)(p-P)} \vdots = \vdots Q^m P^r \vdots. \tag{45}$$

Due to Eq. (40) we see

$$\left(\frac{1}{\sqrt{2}}\right)^{m+r} (-\mathrm{i})^r \mathrm{H}_{m,r}\left(\sqrt{2}Q, \mathrm{i}\sqrt{2}P\right) |_{P \text{ before } Q} = \sum_{l=0} \left(\frac{\mathrm{i}}{2}\right)^l l! \binom{r}{l} \binom{m}{l} P^{r-l} Q^{m-l}, \tag{46}$$

so equations (45) and (46) lead to

$$\vdots Q^m P^r \vdots = \sum_{l=0} \left(\frac{\mathrm{i}}{2}\right)^l l! \binom{r}{l} \binom{m}{l} P^{r-l} Q^{m-l}. \tag{47}$$

Equation (45) or equation (47) is the fundamental formula of converting the Weyl ordered operator to its P–Q ordering.

Similarly, from Eq. (27) and the Hermite conjugate of Eq. (44) we have

$$\left(\frac{1}{\sqrt{2}}\right)^{m+r} (\mathrm{i})^r \mathrm{H}_{m,r}\left(\sqrt{2}Q, -\mathrm{i}\sqrt{2}P\right)\Big|_{Q \text{ before } P}$$
$$= \iint \mathrm{d}p\mathrm{d}q\delta\left(q-Q\right)\delta\left(p-P\right) \left(\frac{1}{\sqrt{2}}\right)^{m+r} (\mathrm{i})^r \mathrm{H}_{m,r}\left(\sqrt{2}q, -\mathrm{i}\sqrt{2}p\right)$$
$$= \iint \frac{\mathrm{d}p\mathrm{d}q}{\pi} \left(\frac{1}{\sqrt{2}}\right)^{m+r} (\mathrm{i})^r \mathrm{H}_{m,r}\left(\sqrt{2}q, -\mathrm{i}\sqrt{2}p\right) \vdots \mathrm{e}^{2\mathrm{i}(q-Q)(p-P)} \vdots$$
$$= \vdots Q^m P^r \vdots = \vdots P^r Q^m \vdots, \tag{48}$$

so

$$\vdots Q^m P^r \vdots = \sum_{l=0} \left(\frac{-\mathrm{i}}{2}\right)^l l! \binom{r}{l} \binom{m}{l} Q^{m-l} P^{r-l}, \tag{49}$$

this is the fundamental formula of converting the Weyl ordered operator to its Q–P ordering, which is in contrast with Eq. (47).

8. *Q–P* ordering to *P–Q* ordering and *vice versa*

Combining Eqs. (42) and (47) together we derive

$$Q^m P^r = \sum_{l=0} \frac{m!r!}{l!(m-l)!(r-l)!} \left(\frac{\mathrm{i}}{2}\right)^l \vdots Q^{m-l} P^{r-l} \vdots$$
$$= \sum_{l=0} \frac{m!r!}{l!(m-l)!(r-l)!} \left(\frac{\mathrm{i}}{2}\right)^l \times \sum_{k=0} \left(\frac{\mathrm{i}}{2}\right)^k k! \binom{r-l}{k} \binom{m-l}{k} P^{r-l-k} Q^{m-l-k}$$
$$= \sum_{l=0} \sum_{k=0} \frac{m!r!}{l!(m-l-k)!(r-l-k)!k!} \left(\frac{\mathrm{i}}{2}\right)^{l+k} P^{r-l-k} Q^{m-l-k}$$
$$= \sum_{k=0} \frac{m!r!}{(m-k)!(r-k)!k!} (\mathrm{i})^k P^{r-k} Q^{m-k}, \tag{50}$$

which puts $Q^m P^r$ to its P–Q ordering. It then follows the commutator

$$[Q^m, P^r] = \sum_{k=1} \frac{m!r!}{(m-k)!(r-k)!k!} (\mathrm{i})^k P^{r-k} Q^{m-k}. \tag{51}$$

On the other hand, from Eqs. (43), (40) and (49) we have

$$\begin{aligned} P^r Q^m &= \left(\frac{1}{\sqrt{2}}\right)^{m+r} (\mathrm{i})^r \vdots \mathrm{H}_{m,r}\left(\sqrt{2}Q, -\mathrm{i}\sqrt{2}P\right) \vdots \\ &= \sum_{l=0} \frac{m!r!}{l!(m-l)!(r-l)!}\left(\frac{-\mathrm{i}}{2}\right)^l \vdots Q^{m-l}P^{r-l} \vdots \\ &= \sum_{l=0} \frac{m!r!}{l!(m-l)!(r-l)!}\left(\frac{-\mathrm{i}}{2}\right)^l \times \sum_{k=0}\left(\frac{-\mathrm{i}}{2}\right)^k k!\binom{r-l}{k}\binom{m-l}{k} Q^{m-l-k}P^{r-l-k} \\ &= \sum_{k=0} \frac{m!r!}{(m-k)!(r-k)!k!}(-\mathrm{i})^k Q^{m-k}P^{r-k}, \end{aligned} \tag{52}$$

which puts P^rQ^m to its Q–P ordering. Thus equation (51) is also equal to

$$[Q^m, P^r] = \sum_{k=1} \frac{m!r!}{(m-k)!(r-k)!k!}(-\mathrm{i})^k Q^{m-k}P^{r-k}. \tag{53}$$

9. P–Q ordering or Q–P ordering expansion of $(P+Q)^n$

Due to

$$(P+Q)^n = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\,\mathrm{e}^{\lambda(P+Q)}\Big|_{\lambda=0} = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\,\vdots \mathrm{e}^{\lambda(P+Q)} \vdots\Big|_{\lambda=0} = \vdots\,(P+Q)^n\,\vdots = \sum_{l=0}^{n}\binom{n}{l}\vdots Q^l P^{n-l}\vdots, \tag{54}$$

substituting Eq. (47) into Eq. (54) we derive

$$(P+Q)^n = \sum_{l=0}^{n}\binom{n}{l}\sum_{k=0}\left(\frac{\mathrm{i}}{2}\right)^k k!\binom{l}{k}\binom{n-l}{k}P^{l-k}Q^{n-l-k}, \tag{55}$$

or using Eq. (49) we have

$$(P+Q)^n = \sum_{l=0}^{n}\binom{n}{l}\sum_{k=0}\left(\frac{-\mathrm{i}}{2}\right)^k k!\binom{l}{k}\binom{n-l}{k}Q^{l-k}P^{n-l-k}. \tag{56}$$

In sum, by virtue of the formula of the Weyl ordering of operators expansion and the technique of integration within the Weyl ordered product of operators we have found a two-fold integration transformation about the Wigner operator $\Delta(q', p')$ in PSQM, which provides us with a new approach for deriving mutual converting formulae among Q–P ordering, P–Q ordering and Weyl ordering of operators. In this way, the contents of PSQM[1−6,19,20,21] can be enriched.

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