Bilinear Bäcklund transformation and explicit solutions for a nonlinear evolution equation*

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The bilinear form of two nonlinear evolution equations are derived by using Hirota derivative. The Bäcklund transformation based on the Hirota bilinear method for these two equations are presented, respectively. As an application, the explicit solutions including soliton and stationary rational solutions for these two equations are obtained.

Keywords: Hirota method, Bäcklund transformation, soliton solution

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1. Introduction

The explicit expression of solutions to the nonlinear evolution equation (NEE) plays an important role in nonlinear science. In the past decades, much effort has been made for the construction of exact solutions of the NEEs, and there are some systematic approaches to obtain them, such as the inverse scattering transformation (IST) method, the Bäcklund and Darboux transformation methods, Hirota's bilinear method, the algebra-geometric method, and so on.[1−7] Recently, the nonlinearization method of Lax pairs (or constrained flow),[8−13] the homogeneous balance method,[14−17] the hyperbolic function method,[18−22] and the Jacobian elliptic function expansion method[23,24] have been applied to construct exact solutions of NEEs. Therefore, it is interesting to find a new type of solution to an NEE. In this paper, we consider the bilinear Bäcklund transformation for NEEs resorting to Hirota's bilinear transformation method. The Bäcklund transformation has been proven to be one of the most powerful tools to obtain explicit solutions of NEEs. In order to state how this method works, the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation

$$u_t + u_{xxy} + 4uu_y + 4u_x\partial_x^{-1}u_y = 0, \tag{1}$$

and the (3+1)-dimensional Kadomtsev–Petviashvili equation

$$u_{xt} + 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0, \tag{2}$$

are provided as two illustrative examples.

2. Bilinear Bäcklund transformation and explicit solutions for equation (1)

Equation (1) is a typical "breaking soliton" equation and was studied by Bogoyavlenskii, where overlapping solutions were generated.[25] Radha and Lakshmanan showed that equation (1) possesses the Painlevé property and dromion-like structures.[26] Ikeda and Takasaki presented a Bogoyavlenskii's hierarchy and its breaking soliton solutions.[27] In recent years, many papers have been focusing their topics on various exact solutions of Eq. (1).[28−34] In the following, we shall derive the bilinear Bäcklund transformation and then give the explicit solutions for Eq. (1).

The Hirota bilinear operator (D-operators) is defined as[7]

$$\begin{aligned} & D_x^l D_y^m D_t^n a(x,y,t)\cdot b(x,y,t) \\ \equiv\ & (\partial_x - \partial_{x'})^l(\partial_y - \partial_{y'})^m(\partial_t - \partial_{t'})^n \\ & \times a(x,y,t)b(x',y',t')|_{x'=x,y'=y,t'=t}. \end{aligned} \tag{3}$$

The bilinear form of the (2+1)-dimensional Bogoyavlenskii equation (1) reads

$$F(D_x, D_y, D_t)f\cdot f \equiv (D_tD_x + D_yD_x^3)f\cdot f = 0 \tag{4}$$

with the property of the D-operators and the dependent variable transformation[26,32]

$$u(x,y,t) = \frac{3}{2}\partial_x^2 \ln f(x,y,t). \tag{5}$$

We know that the Bäcklund transformation is a transformation that is related to a pair of solutions of

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NEEs. In order to derive the bilinear Bäcklund transformation for the (2+1)-dimensional Bogoyavlenskii equation (1), we consider the pair of solutions f and f' of the bilinear equation (4), that is

$$(D_tD_x + D_yD_x^3)f\cdot f = 0, \tag{6}$$
$$(D_tD_x + D_yD_x^3)f'\cdot f' = 0, \tag{7}$$

and we start from

$$\begin{aligned} P \equiv\ & [(D_tD_x + D_yD_x^3)f'\cdot f']f^2 \\ & - [(D_tD_x + D_yD_x^3)f\cdot f]f'^2 \\ =\ & 0. \end{aligned} \tag{8}$$

Obviously, if $P = 0$, f satisfies Eq. (6), then f' also satisfies the same equation. Therefore, equation (8) can be regarded as a relation that connects the solutions f and f'. Furthermore, if we can obtain from $P = 0$ the bilinear form of equations

$$\begin{aligned} & F_1(D_t, D_x, D_y)f'\cdot f = 0, \\ & F_2(D_t, D_x, D_y)f'\cdot f = 0, \\ & \ldots \\ & F_k(D_t, D_x, D_y)f'\cdot f = 0, \end{aligned}$$

where $F_1, F_2, \ldots, F_k$ are undetermined functions, then they provide a bilinear Bäcklund transformation for the (2+1)-dimensional Bogoyavlenskii equation (1). In order to gain concrete results, the following four formulae are very useful.

Proposition 1[6,7] For arbitrary derivable functions f and g, we have

$$[D_xD_tf\cdot f]g^2 - [D_xD_tg\cdot g]f^2 = 2D_x(D_tf\cdot g)\cdot gf, \tag{9}$$
$$[D_y^2f\cdot f]g^2 - [D_y^2g\cdot g]f^2 = 2D_y(D_yf\cdot g)\cdot gf, \tag{10}$$
$$[D_x^4f\cdot f]g^2 - [D_x^4g\cdot g]f^2 = 2D_x(D_x^3f\cdot g)\cdot gf + 6D_x(D_x^2f\cdot g)\cdot(D_xg\cdot f), \tag{11}$$
$$\begin{aligned} 2[D_x^3D_yf\cdot f]g^2 - 2[D_x^3D_yg\cdot g]f^2 =\ & D_x[(3D_x^2D_yf\cdot g)\cdot gf + (3D_x^2f\cdot g)\cdot(D_yg\cdot f) \\ & +(6D_xD_yf\cdot g)\cdot(D_xg\cdot f)] + D_y[(D_x^3f\cdot g)\cdot gf + (3D_x^2f\cdot g)\cdot(D_xg\cdot f)]. \end{aligned} \tag{12}$$

Using the properties (9) and (12), equation (8) can be written as

$$\begin{aligned} 2P \equiv\ & D_x[(4D_t + 3D_x^2D_y)f'\cdot f]\cdot ff' + D_x[(3D_x^2)f'\cdot f]\cdot(D_yf\cdot f') \\ & + D_x(6D_xD_yf'\cdot f)\cdot(D_xff') + D_y(D_x^3f'\cdot f)\cdot ff' + D_y[(3D_x^2)f'\cdot f]\cdot(D_xf\cdot f') = 0. \end{aligned} \tag{13}$$

Thus, we have

$$\begin{aligned} 2P \equiv\ & D_x[(4D_t + 3D_x^2D_y + \nu D_y + \alpha)f'\cdot f]\cdot ff' + D_y[(D_x^3 + \zeta D_x + \beta)f'\cdot f]\cdot ff' \\ & + D_y[(3D_x^2 - \gamma D_x - \delta)f'\cdot f]\cdot(D_xf\cdot f') \\ & + D_x[(3D_x^2 + \lambda D_y - \nu - \zeta + \delta)f'\cdot f]\cdot(D_yf\cdot f') \\ & + D_x[(6D_xD_y - 6\mu D_x)f'\cdot f]\cdot(D_xf\cdot f') = 0, \end{aligned} \tag{14}$$

where eight arbitrary parameters $\alpha, \beta, \gamma, \lambda, \mu, \nu, \delta$, and ζ have been introduced. Equation (14) is equivalent to Eq. (13) because the coefficients of α, β, γ, λ, μ, ν, δ, and ζ in Eq. (14) are all zeros, which can be easily proved. Equation (14) is satisfied provided that the following equations hold

$$(4D_t + 3D_x^2D_y + \nu D_y + \alpha)f'\cdot f = 0, \tag{15}$$
$$(D_x^3 + \zeta D_x + \beta)f'\cdot f = 0, \tag{16}$$
$$(3D_x^2 - \gamma D_x - \delta)f'\cdot f = 0, \tag{17}$$
$$(3D_x^2 + \lambda D_y - \nu - \zeta + \delta)f'\cdot f = 0, \tag{18}$$
$$(D_xD_y - \mu D_x)f'\cdot f = 0. \tag{19}$$

These constitute the Bäcklund transformation of the (2+1)-dimensional Bogoyavlenskii equation (1) in bilinear formalism. In what follows, as an application of the bilinear Bäcklund transformations (15)–(19), we shall derive explicit solutions including soliton solutions and rational solution for the (2+1)-dimensional Bogoyavlenskii equation (1). To illustrate the process of generating these solutions starting from a known solution, we shall take the simplest case

$$f = 1, \tag{20}$$

which is transformed into the original variable u as

$$u(x,y,t) = \frac{3}{2}\partial_x^2 \ln f(x,y,t) = 0, \tag{21}$$

so that equation (20) can be regarded as a vacuum solution. Substitution of Eq. (20) into Eqs. (15)–(19) yields the following several cases.

Case 1.1 By setting $\alpha = \beta = \delta = 0, \lambda = -3k, \mu = k, \gamma = 3k, \nu = k^2, \zeta = -k^2$, and choosing the function f' in the bilinear Bäcklund transformation (15)–(19) to be

$$f' = 1 + \mathrm{e}^{kx+ky-k^3t+\xi_0}, \quad \xi_0 = \text{const.}, \tag{22}$$

then a soliton solution is engendered

$$u(x,y,t) = \frac{3k^2}{8}\,\mathrm{sech}^2\,\frac{kx+ky-k^3t+\xi_0}{2}. \tag{23}$$

Case 1.2 By setting $\alpha = \beta = \delta = 0, \lambda = -3, \mu = k^2, \gamma = 3k, \nu = k^2, \zeta = -k^2$, it is easy to see that the function

$$f' = 1 + \mathrm{e}^{kx+k^2y-k^4t+\xi_0}, \quad \xi_0 = \text{const.}, \tag{24}$$

satisfies the compatibility condition of the Bäcklund transformation (15)–(19). With the help of Eq. (5), we obtain the one-soliton solution for Eq. (1),

$$u(x,y,t) = \frac{3k^2}{8}\,\mathrm{sech}^2\,\frac{kx+k^2y-k^4t+\xi_0}{2}. \tag{25}$$

Case 1.3 By considering $\alpha = \beta = \delta = 0, \lambda = -3k, \mu = k^3, \gamma = 3k^2, \nu = k^4, \zeta = -k^4$, it is easy to see that the function f' in the Bäcklund transformation (15)–(19) can be chosen as

$$f' = 1 + \mathrm{e}^{k^2x+k^3y-k^7t+\xi_0}, \quad \xi_0 = \text{const.}, \tag{26}$$

which together with Eq. (5) yields another soliton solution for the (2+1)-dimensional Bogoyavlenskii equation (1)

$$u(x,y,t) = \frac{3k^4}{8}\,\mathrm{sech}^2\,\frac{k^2x+k^3y-k^7t+\xi_0}{2}. \tag{27}$$

Case 1.4 Let $\alpha = -4$, others $= 0$, then the function

$$f' = \mathrm{e}^t(c_1x + c_2y + c_3), \quad c_1, c_2, c_3 = \text{const.}, \tag{28}$$

satisfies the compatibility condition of the Bäcklund transformation (15)–(19). We arrive at a stationary rational solution for the (2+1)-dimensional Bogoyavlenskii equation (1)

$$u(x,y,t) = -\frac{3}{2}\frac{c_1^2}{(c_1x+c_2y+c_3)^2}. \tag{29}$$

3. Bilinear Bäcklund transformation and explicit solutions for equation (2)

Equation (2) is introduced to describe the dynamics of solitons and nonlinear waves in the fields of plasma physics, fluid dynamics, etc.,[35] which has been studied by many authors. Chen *et al.* obtained some new exact solutions for Eq. (2) by using a new generalized transformation in the homogeneous balance method.[36] El-Sayed *et al.* studied Eq. (2) by considering the decomposition scheme and obtained some exact solutions.[37] Hu obtained some travelling wave solutions by applying the algebraic method.[38] Recently, Liu *et al.* derived the similarity solutions by using the compatibility method.[39] In the following, we shall derive the bilinear Bäcklund transformation and give new explicit solutions for Eq. (2).

The bilinear form of the (3+1)-dimensional Kadomtsev–Petviashvili equation (2) reads

$$\begin{aligned} &F(D_x, D_y, D_z, D_t)f\cdot f \\ &\equiv (D_xD_t - D_x^4 - D_y^2 - D_z^2)f\cdot f \\ &= 0 \end{aligned} \tag{30}$$

with the dependent variable transformation

$$u(x,y,z,t) = -2\partial^2 \ln f(x,y,z,t)/\partial x^2. \tag{31}$$

We start from

$$\begin{aligned} P \equiv\ & [(D_xD_t - D_x^4 - D_y^2 - D_z^2)f'\cdot f']f^2 \\ & - [(D_xD_t - D_x^4 - D_y^2 - D_z^2)f\cdot f]f'^2 = 0, \end{aligned} \tag{32}$$

and use the formulae (9)–(11). Adopting a similar procedure of obtaining the bilinear Bäcklund transformation for Eq. (1), we arrive at

$$\begin{aligned} &\frac{P}{2} \\ \equiv\ & D_x[(D_x^3 - \alpha D_x)f\cdot f']\cdot f'f \\ & + 3D_x[(D_x^2 - \beta D_x)f\cdot f']\cdot (D_xf'\cdot f) \\ & - D_x[(D_t - \alpha D_x - \gamma D_y - \delta D_z - \lambda)f\cdot f']\cdot f'f \\ & + D_y[(D_y - \gamma D_x - \varepsilon D_z - \mu)f\cdot f']\cdot f'f \\ & + D_z[(D_z - \delta D_x + \varepsilon D_y - \nu)f\cdot f']\cdot f'f = 0. \end{aligned} \tag{33}$$

Thus, the Bäcklund transformation for Eq. (2) in bilinear form is as follows:

$$(D_x^3 - \alpha D_x)f\cdot f' = 0, \tag{34}$$

$$(D_x^2 - \beta D_x)f\cdot f' = 0, \tag{35}$$

$$(D_t - \alpha D_x - \gamma D_y - \delta D_z - \lambda)f\cdot f' = 0, \tag{36}$$

$$(D_y - \gamma D_x - \varepsilon D_z - \mu) f \cdot f' = 0, \tag{37}$$

$$(D_z - \delta D_x + \varepsilon D_y - \nu) f \cdot f' = 0, \tag{38}$$

where $\alpha, \beta, \gamma, \lambda, \mu, \nu, \varepsilon$, and δ are eight arbitrary parameters. Similarly, substituting $f = 1$ into Eqs. (34)–(38) yields the solutions of Eq. (2).

Case 2.1 By setting $\lambda = \mu = \nu = 0, \alpha = k^2, \beta = k, \gamma = 1, \varepsilon = 0, \delta = 1$, and choosing the function f' in the bilinear Bäcklund transformation (34)–(38) to be

$$f' = 1 + \mathrm{e}^{kx+ky+kz+(k^3+2k)t+\xi_0}, \quad \xi_0 = \text{const.}, \tag{39}$$

then a soliton solution is derived

$$u(x, y, z, t) = -\frac{k^2}{2} \operatorname{sech}^2 \frac{kx + ky + kz + (k^3 + 2k)t + \xi_0}{2}. \tag{40}$$

Case 2.2 By setting $\lambda = \mu = \nu = 0, \alpha = k^2, \beta = k, \gamma = 0, \varepsilon = 1, \delta = 2k$, it is easy to see that the function

$$f' = 1 + \mathrm{e}^{kx+k^2y+k^2z+3k^3t+\xi_0}, \quad \xi_0 = \text{const.}, \tag{41}$$

satisfies the compatibility condition of the Bäcklund transformation (34)–(38). With the help of Eq. (31), we obtain a one-soliton solution for Eq. (2)

$$u(x, y, z, t) = -\frac{k^2}{2} \operatorname{sech}^2 \frac{kx + k^2y + k^2z + 3k^3t + \xi_0}{2}. \tag{42}$$

Case 2.3 By considering $\lambda = \mu = \nu = 0, \alpha = k^2, \beta = k, \gamma = 1, \varepsilon = 1, \delta = 3$, it is easy to see that the function f' in the Bäcklund transformation (34)–(38) can be chosen as

$$f' = 1 + \mathrm{e}^{kx+2ky+kz+(k^3+5k)t+\xi_0}, \quad \xi_0 = \text{const.}, \tag{43}$$

which together with Eq. (31) yields another soliton solution for the (3+1)-dimensional Kadomtsev–Petviashvili equation (2)

$$u(x, y, z, t) = -\frac{k^2}{2} \operatorname{sech}^2 \frac{kx + 2ky + kz + (k^3 + 5k)t + \xi_0}{2}. \tag{44}$$

Case 2.4 Assuming that $\lambda = \mu = \nu = 0, \alpha = k^2, \beta = k, \gamma = k^2, \varepsilon = 1, \delta = 3k^2$, we can verify that the function f' in the Bäcklund transformation (34)–(38) can be chosen as

$$f' = 1 + \mathrm{e}^{kx+2k^3y+k^3z+(k^3+5k^5)t+\xi_0}, \quad \xi_0 = \text{const.}, \tag{45}$$

thus, another soliton solution for the (3+1)-dimensional Kadomtsev–Petviashvili equation (2) is obtained

$$u(x, y, z, t) = -\frac{k^2}{2} \operatorname{sech}^2 \frac{kx + 2k^3y + k^3z + (k^3 + 5k^5)t + \xi_0}{2}. \tag{46}$$

Case 2.5 By setting λ, μ, ν arbitrary, others $= 0$, then the function

$$f' = \mathrm{e}^{\lambda t+\mu y+\nu z}(c_1 x + c_2), \quad c_1, c_2, = \text{const.}, \tag{47}$$

satisfies the compatibility condition of the Bäcklund transformation (34)–(38), thus we arrive at a stationary rational solution for the (3+1)-dimensional Kadomtsev–Petviashvili equation (2)

$$u(x, y, z, t) = \frac{2c_1^2}{(c_1 x + c_2)^2}. \tag{48}$$

4. Summary

In summary, we have obtained several explicit solutions for the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation (1) and the (3+1)-dimensional Kadomtsev–Petviashvili equation (2) by the bilinear Bäcklund transformation. This method is valid for quite a few nonlinear evolution equations, and generally speaking, an infinite number of soliton solutions can be derived in this way. Furthermore, if the solution derived is taken as a new seed solution, we can do the Bäcklund transformation again and a new explicit solution can be obtained. This process can be carried out continuously and will usually yield a series of explicit solutions. Further discussion will be given in other work.

References

[1] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

[2] Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons, The Inverse Scattering Methods* (New York: Consultants Bureau)

[3] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)

[4] Belokolos E D, Bobenko A I, Enolskii V Z, Its A R and

Matveev V B 1994 *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Berlin: Springer)
[5] Cherednik I 1996 *Basic Methods of Soliton Theory* (Singapore: World Scientific)
[6] Matsuno Y 1984 *Bilinear Transformation Method* (London: Academic Press, Inc.)
[7] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
[8] Konopelchenko B, Sidorenko J and Strampp W 1991 *Phys. Lett.* A **157** 17
[9] Chen Y and Li Y S 1991 *Phys. Lett.* A **157** 22
[10] Cao C W, Wu Y T and Geng X G 1999 *J. Math. Phys.* **40** 3948
[11] Cao C W, Geng X G and Wu Y T 1999 *J. Phys.* A **32** 8059
[12] Geng X G, Wu Y T and Cao C W 1999 *J. Phys.* A **32** 3733
[13] Zhou R G 1997 *J. Math. Phys.* **38** 2535
[14] Wang M L 1995 *Phys. Lett.* A **199** 169
[15] Lei Y 1999 *Phys. Lett.* A **260** 55
[16] Fan E G and Zhang H Q 1998 *Acta Phys. Sin.* **47** 353 (in Chinese)
[17] Fan E G 2000 *Acta Phys. Sin.* **49** 1409 (in Chinese)
[18] Parkes E J and Duffy B R 1997 *Phys. Lett.* A **229** 217
[19] Zhang G X, Li Z B and Duan Y S 2000 *Sci. Chin.* A **30** 1103 (in Chinese)
[20] Fan E G 2000 *Phys. Lett.* A **277** 212
[21] Shi Y R, Lü K P, Duan W S and Zhao J B 2001 *Acta Phys. Sin.* **50** 2074 (in Chinese)
[22] Guo G P and Zhang J F 2002 *Acta Phys. Sin.* **51** 1159 (in Chinese)
[23] Liu S K, Fu Z T, Liu S D and Zhao Q 2001 *Phys. Lett.* A **289** 69
[24] Fu Z T, Liu S K, Liu S D and Zhao Q 2001 *Phys. Lett.* A **290** 72
[25] Bogoyavlenskii O I 1990 *Russ. Math. Surveys* **45** 1
[26] Radha R and Lakshmanan M 1995 *Phys. Lett.* A **197** 7
[27] Ikeda T and Takasaki K 2001 *Int. Math. Res. Notices* **7** 329
[28] Lou S Y and Ruan H Y 2001 *J. Phys.* A **34** 305
[29] Zhang J F and Meng J P 2004 *Phys. Lett.* A **321** 173
[30] Xie Z and Zhang H Q 2005 *Commun. Theor. Phys.* **43** 401
[31] Wang D S and Li H B 2007 *Appl. Math. Comput.* **188** 762
[32] Huang W H, Liu Y L and Zhang J F 2008 *Commun. Theor. Phys.* **49** 268
[33] Chen L X and Zhang J X 2008 *Appl. Math. Comput.* **198** 184
[34] Fan E G and Hon Y C 2008 *Phys. Rev.* E **78** 036607
[35] Swnatorski A and Infeld E 1996 *Phys. Rev. Lett.* **77** 2855
[36] Chen Y, Yan Z Y and Zhang H Q 2003 *Phys. Lett.* A **307** 107
[37] El-Sayed S M and Kaya D 2004 *Appl. Math. Comput.* **157** 523
[38] Hu J Q 2005 *Chaos, Solitons and Fractals* **23** 391
[39] Liu N and Liu X Q 2008 *Chin. Phys. Lett.* **25** 3527