Conformal invariance and conserved quantities of a general holonomic system with variable mass*

Xia Li-Li(夏丽莉)[a)†] and Cai Jian-Le(蔡建乐)[b)]

[a)] *Department of Physics, Henan Institute of Education, Zhengzhou* 450014, *China*
[b)] *College of Science, Hangzhou Normal University, Hangzhou* 310018, *China*

(Received 18 August 2008; revised manuscript received 7 September 2009)

Conformal invariance and conserved quantities of a general holonomic system with variable mass are studied. The definition and the determining equation of conformal invariance for a general holonomic system with variable mass are provided. The conformal factor expression is deduced from conformal invariance and Lie symmetry. The relationship between the conformal invariance and the Lie symmetry is discussed, and the necessary and sufficient condition under which the conformal invariance would be the Lie symmetry of the system under an infinitesimal one-parameter transformation group is deduced. The conserved quantities of the system are given. An example is given to illustrate the application of the result.

Keywords: variable mass, conformal invariance, conformal factor, conserved quantity

PACC: 0320

1. Introduction

In 1918, Noether published the Noether theorem which established a potential relationship between symmetries and conserved quantities of dynamical mechanical systems,[1] and gave a simpler method to solve the problems of dynamical mechanical systems. Since then the theory of the symmetries and conserved quantities has developed rapidly and deeply. There are three main kinds of symmetries: Noether symmetry,[1] Lie symmetry[2] and Mei symmetry.[3] Correspondingly one has obtained three kinds of conserved quantities (i.e. the Noether conserved quantity, the Hojman conserved quantity and a new type of conserved quantity) from these symmetries directly or indirectly.[4−17] For a mechanical system, there exists a close relation between its integrability and the variations in its symmetries and conserved quantities; even though differential equations of motion of a system cannot be integrated, the existence of some conserved quantities can enable us to obtain some knowledge about the local physical state. So the symmetries and the conserved quantities of a dynamical mechanical system play an important role in mechanics and physics.

Galiulin *et al.*[18] studied conformal invariance (or conformal symmetry) for a Birkhoff system under special infinitesimal transformations and obtained the Noether conserved quantity in 1997. Cai studied the conformal invariance of Hamilton systems[19] and the conformal invariance and conserved quantities of Mei symmetry for a general holonomic system,[20] Liu *et al.*[21] presented the conformal invariance of non-conserved Lagrange systems with Lie point symmetry, and Fu[22] studied the conserved quantities and the conformal mechanico-electrical systems; in Ref. [23], Cai studied the conformal invariance and the conserved quantities of a general holonomic system, which are more complex than those of the Lagrange system. However, research on conformal invariance for a general holonomic system with variable mass has not been reported yet, thus in the present paper, we study the conformal invariance of a general holonomic system with variable mass and obtain the Noether conserved quantity.

2. Conformal invariance of a general holonomic system with variable mass

We consider a mechanical system composed of N particles, whose configuration is determined by n gen-

*Project supported by the Key Disciplines' Building Foundation of Henan Institute of Education, the Natural Science Foundation of Education Bureau of Henan Province, China (Grant No. 2009A140003) and the Young Core Instructor from Henan Institute of Education.

†Corresponding author. E-mail: xialilixialili@yahoo.com.cn

© 2010 Chinese Physical Society and IOP Publishing Ltd

eralized coordinates q_s $(s = 1, \ldots, n)$. The mass of a representative point i of the system is

$$m_i = m_i(t, \boldsymbol{q}) \ (i = 1, \ldots, N), \tag{1}$$

then its motion can be described by the differential equations as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + P_s \quad (s = 1, \ldots, n), \tag{2}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the nonpotential generalized forces, and P_s are the generalized counter thrusts. We have[16]

$$P_s = (\dot{m}_i \boldsymbol{\mu}_i + m_i \dot{\boldsymbol{r}}) \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_s} - \frac{1}{2}\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \frac{\partial m_i}{\partial q_s}$$
$$(s = 1, \ldots, n; i = 1, \ldots, N), \tag{3}$$

where $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ are the vector radius and the velocity of the particle i, respectively. $\boldsymbol{\mu}_i$ is the relative velocity of the small particle corresponding to the particle i.

From Eq. (2), we can have

$$F_s \equiv A_{sk}(t, \boldsymbol{q})\ddot{q}_k + B_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - Q_s - P_s = 0, \tag{4}$$

where

$$A_{sk} = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k},$$
$$B_s = \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_s \partial t} - \frac{\partial L}{\partial q_s}. \tag{5}$$

Assuming that equation (2) are nonsingular, namely

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0, \tag{6}$$

from Eq. (4), we can obtain

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, \ldots, n). \tag{7}$$

Considering the symmetries to be inherent in Eq. (4), we introduce a one-parameter infinitesimal transformation with respect to time and coordinates

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),$$

and

$$q_s^* = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{8}$$

where ε is a small parameter, ξ_0 and ξ_s are the infinitesimal generators.

It is convenient to express the invariance condition for system (7) in terms of a formula involving vector fields and their prolongations. The vector field itself has the form

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} \tag{9}$$

with ξ_0 and ξ_s being the same as those in Eq. (8). The first prolongation of vector field (9) is

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}, \tag{10}$$

and the second prolongation is

$$X^{(2)} = X^{(1)} + (\ddot{\xi}_s - 2\ddot{q}_s\dot{\xi}_0 - \dot{q}_s\ddot{\xi}_0)\frac{\partial}{\partial \ddot{q}_s}. \tag{11}$$

Definition If there is a nonsingular matrix Γ_s^l that satisfies the condition

$$X^{(2)}(F_s) = \Gamma_s^l(F_l), \tag{12}$$

then this kind of invariance is called the conformal invariance of Eq. (4) under one parameter-infinitesimal transformation (8). Equation (12) is the determining equation of conformal invariance of the general holonomic systems with variable mass, and Γ_s^l is called the conformal factor. If the generators ξ_0 and ξ_s of the infinitesimal transformation (8) satisfy the determining equation (12), then the above transformations are called the conformal transformations.

3. General expressions of conformal invariance

In order to obtain the expressions of the conformal factor of conformal invariance, we calculate the difference of

$$X^{(2)}(F_s) - X^{(2)}(F_s)\,|_{F_s=0}\,. \tag{13}$$

Since

$$\dot{\xi}_k = \frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial q_r}\dot{q}_r + \frac{\partial \xi_k}{\partial \dot{q}_r}\ddot{q}_r, \tag{14}$$
$$\ddot{\xi}_k = \frac{\partial^2 \xi_k}{\partial t^2} + 2\frac{\partial^2 \xi_k}{\partial q_r \partial t}\dot{q}_r + 2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t}\ddot{q}_r + \frac{\partial^2 \xi_k}{\partial q_r \partial q_j}\dot{q}_r\dot{q}_j + 2\frac{\partial^2 \xi_k}{\partial q_r \partial \dot{q}_j}\dot{q}_r\ddot{q}_j + \frac{\partial \xi_k}{\partial q_r}\ddot{q}_r + \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}\ddot{q}_r\ddot{q}_j$$

$$+\frac{\partial\xi_k}{\partial\dot{q}_r}\left(\frac{\partial\alpha_r}{\partial t}+\frac{\partial\alpha_r}{\partial q_j}\dot{q}_j+\frac{\partial\alpha_r}{\partial\dot{q}_j}\ddot{q}_j\right)\quad(k=0,1\ldots,n;r,j=1,\ldots,n),\tag{15}$$

then

$$\begin{aligned}X^{(2)}(F_s) &= A_{sk}(\ddot{\xi}_k-2\ddot{q}_k\dot{\xi}_0-\dot{q}_k\ddot{\xi}_0)+X^{(2)}(A_{sk})\ddot{q}_k+X^{(2)}(B_s-Q_s-P_s)\\ &= A_{sk}(\ddot{\xi}_k-2\ddot{q}_k\dot{\xi}_0-\dot{q}_k\ddot{\xi}_0)+X^{(0)}(A_{sk})\ddot{q}_k+X^{(0)}(B_s-Q_s-P_s)+(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)\frac{\partial(B_s-Q_s-P_s)}{\partial\dot{q}_k}\\ &= A_{sk}\left[2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial t}\ddot{q}_r+2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial q_j}\ddot{q}_r\dot{q}_j+\frac{\partial\xi_k}{\partial q_r}\ddot{q}_r+\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial\dot{q}_j}\ddot{q}_r\alpha_j+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\ddot{q}_r-2\ddot{q}_k\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r+\frac{\partial\xi_0}{\partial\dot{q}_r}\alpha_r\right)\right]\\ &\quad-A_{sk}\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial t}\ddot{q}_r+2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial q_j}\ddot{q}_r\dot{q}_j+\frac{\partial\xi_0}{\partial q_r}\ddot{q}_r+\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial\dot{q}_j}\ddot{q}_r\alpha_j+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\ddot{q}_r\right)+X^{(0)}(A_{sk})\ddot{q}_k\\ &\quad+\left(\frac{\partial\xi_k}{\partial\dot{q}_r}-\frac{\partial\xi_0}{\partial\dot{q}_r}\dot{q}_k\right)\ddot{q}_r\frac{\partial(B_s-Q_s-P_s)}{\partial\dot{q}_k}+C(t,q,\dot{q})\quad(s,k,r,j=1,\ldots,n),\end{aligned}\tag{16}$$

where $C(t,q,\dot{q})$ is the summation of items in which $\ddot{q}_s$ are not included. For the same reason,

$$\begin{aligned}&X^{(2)}(F_s)\,|_{F_s=0}\\ &= A_{sk}\left[2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial t}\alpha_r+2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial q_j}\alpha_r\dot{q}_j+\frac{\partial\xi_k}{\partial q_r}\alpha_r+\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_r\alpha_j+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\alpha_r-2\alpha_k\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r+\frac{\partial\xi_0}{\partial\dot{q}_r}\alpha_r\right)\right]\\ &\quad-A_{sk}\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial t}\alpha_r+2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial q_j}\alpha_r\dot{q}_j+\frac{\partial\xi_0}{\partial q_r}\alpha_r+\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_r\alpha_j+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\alpha_r\right)\\ &\quad+X^{(0)}(A_{sk})\alpha_k+\left(\frac{\partial\xi_k}{\partial\dot{q}_r}-\frac{\partial\xi_0}{\partial\dot{q}_r}\dot{q}_k\right)\alpha_r\frac{\partial(B_s-Q_s-P_s)}{\partial\dot{q}_k}+C(t,q,\dot{q}),\end{aligned}\tag{17}$$

where $\alpha_s=-A^{sk}(B_k-Q_k-P_k)$.

Let Eq. (16) subtract Eq. (17), we can obtain

$$\begin{aligned}&X^{(2)}(F_s)-X^{(2)}(F_s)\,|_{F_s=0}\\ &= A_{sk}\left(2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial t}+2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial q_j}\dot{q}_j+\frac{\partial\xi_k}{\partial q_r}+\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\right)(\ddot{q}_r-\alpha_r)\\ &\quad-2A_{sk}(\ddot{q}_k-\alpha_k)\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r+\frac{\partial\xi_0}{\partial\dot{q}_r}\alpha_r\right)\\ &\quad-A_{sk}\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial t}+2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial q_j}\dot{q}_j+\frac{\partial\xi_0}{\partial q_r}+\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\right)(\ddot{q}_r-\alpha_r)+X^{(0)}(A_{sk})(\ddot{q}_k-\alpha_k)\\ &\quad+\left(\frac{\partial\xi_k}{\partial\dot{q}_r}-\frac{\partial\xi_0}{\partial\dot{q}_r}\dot{q}_k\right)(\ddot{q}_r-\alpha_r)\frac{\partial(B_k-Q_k-P_k)}{\partial\dot{q}_k}\qquad(s,k,j,r=1,\ldots,n).\end{aligned}\tag{18}$$

Since

$$\ddot{q}_k-\alpha_k=\ddot{q}_k+A^{kl}B_l=A^{kl}(A_{lm}\dot{q}_m+B_l-Q_l-P_l)=A^{kl}F_l,\tag{19}$$

we have

$$\begin{aligned}&X^{(2)}(F_s)-X^{(2)}(F_s)\,|_{F_s=0}\\ &= \left[A_{sk}\left(2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial t}+2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial q_j}\dot{q}_j+\frac{\partial\xi_k}{\partial q_r}+\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\right)A^{rl}\right.\\ &\quad-A_{sk}\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial t}+2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial q_j}\dot{q}_j+\frac{\partial\xi_0}{\partial q_r}+\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\right)A^{rl}\\ &\quad\left.-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r+\frac{\partial\xi_0}{\partial\dot{q}_r}\alpha_r\right)+X^{(0)}(A_{sk})A^{kl}+\left(\frac{\partial\xi_k}{\partial\dot{q}_r}-\frac{\partial\xi_0}{\partial\dot{q}_r}\dot{q}_k\right)A^{rl}\frac{\partial(B_k-Q_k-P_k)}{\partial\dot{q}_k}\right]F_l\\ &= \beta_s^lF_l,\quad(s,k,j,r,l=1,\ldots,n),\end{aligned}\tag{20}$$

where $\delta_s^l = A_{sk}A^{kl}$.

If equation (4) is Lie symmetric under the infinitesimal point transformation (8), then

$$X^{(2)}(F_s)\,|_{F_s=0} = 0. \tag{21}$$

From Eqs. (20) and (12), we have $\Gamma_s^l = \beta_s^l$. If equation (4) has conformal invariance under the infinitesimal point transformation (8), from Eq. (20), we have

$$(\Gamma_s^l - \beta_s^l)(F_l) = X^{(2)}(F_s)\,|_{F_s=0}\ ,$$

$$(s, l = 1, \ldots, n). \tag{22}$$

Then the necessary and sufficient condition under which the conformal invariance would be Lie symmetry of the system is

$$\Gamma_s^l = \beta_s^l. \tag{23}$$

Proposition 1 Under the infinitesimal point transformation (8), the necessary and sufficient condition under which the conformal invariance would be Lie symmetric for Eq. (4) is the conformal factor

$$\begin{aligned}\Gamma_s^l =\ & A_{sk}\left(2\frac{\partial^2\xi_k}{\partial\dot q_r\partial t} + 2\frac{\partial^2\xi_k}{\partial\dot q_r\partial q_j}\dot q_j + \frac{\partial\xi_k}{\partial q_r} + \frac{\partial^2\xi_k}{\partial\dot q_r\partial\dot q_j}\alpha_j + \frac{\partial\xi_k}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\right)A^{rl} \\ & - A_{sk}\dot q_k\left(2\frac{\partial^2\xi_0}{\partial\dot q_r\partial t} + 2\frac{\partial^2\xi_0}{\partial\dot q_r\partial q_j}\dot q_j + \frac{\partial\xi_0}{\partial q_r} + \frac{\partial^2\xi_0}{\partial\dot q_r\partial\dot q_j}\alpha_j + \frac{\partial\xi_0}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\right)A^{rl} \\ & - 2\delta_s^l\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial q_r}\dot q_r + \frac{\partial\xi_0}{\partial\dot q_r}\alpha_r\right) + X^{(0)}(A_{sk})A^{kl} + \left(\frac{\partial\xi_k}{\partial\dot q_r} - \frac{\partial\xi_0}{\partial\dot q_r}\dot q_k\right)A^{rl}\frac{\partial(B_k - Q_k - P_k)}{\partial\dot q_k} \\ & (s, k, j, r, l = 1, \ldots, n).\end{aligned} \tag{24}$$

Corollary If equation (4) can be expressed in the standard form

$$F_s \equiv \ddot q_s - \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad (s = 1, \ldots, n), \tag{25}$$

under infinitesimal point transformation (8), the necessary and sufficient condition under which the conformal invariance would be Lie symmetric for Eq. (4) is the conformal factor satisfying

$$\begin{aligned}\Gamma_s^l =\ & 2\frac{\partial^2\xi_s}{\partial\dot q_l\partial t} + 2\frac{\partial^2\xi_s}{\partial\dot q_l\partial q_j}\dot q_j + \frac{\partial\xi_s}{\partial q_l} + \frac{\partial^2\xi_s}{\partial\dot q_l\partial\dot q_j}\alpha_j + \frac{\partial\xi_s}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_l} \\ & - \dot q_s\left(2\frac{\partial^2\xi_0}{\partial\dot q_l\partial t} + 2\frac{\partial^2\xi_0}{\partial\dot q_l\partial q_j}\dot q_j + \frac{\partial\xi_0}{\partial q_l} + \frac{\partial^2\xi_0}{\partial\dot q_l\partial\dot q_j}\alpha_j + \frac{\partial\xi_0}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_l}\right) \\ & - 2\delta_s^l\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial q_r}\dot q_r + \frac{\partial\xi_0}{\partial\dot q_r}\alpha_r\right) - \left(\frac{\partial\xi_k}{\partial\dot q_l} - \frac{\partial\xi_0}{\partial\dot q_l}\dot q_k\right)\frac{\partial\alpha_s}{\partial\dot q_k}.\end{aligned} \tag{26}$$

Comparing Eq. (4) with Eq. (25) leads to $A_{sk} = \delta_{sk}$, $B_s - Q_s - P_s = -\alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$. Using Eq. (24), we can easily obtain Eq. (26).

4. Conformal invariance and conserved quantity of a general holonomic system with variable mass

The conformal invariance of a general holonomic system with variable mass can also lead to a conserved quantity under a certain condition.

Proposition 2 For a general holonomic system with variable mass, if infinitesimal transformations (8) satisfy conformal factor (24) or (26), and there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following structure equation of Lie symmetry

$$L\dot\xi_0 + X^{(1)}(L) + (Q_s + P_s)(\xi_s - \dot q_s\xi_0) + \dot G = 0, \tag{27}$$

then the conformal invariance of the general holonomic system with variable mass possesses the following conserved quantity:

$$I = L\xi_0 + \frac{\partial L}{\partial\dot q_s}(\xi_s - \dot q_s\xi_0) + G = \text{const}. \tag{28}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} =\ & \dot L\xi_0 + L\dot\xi_0 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot q_s}(\xi_s - \dot q_s\xi_0) \\ & + \frac{\partial L}{\partial\dot q_s}(\dot\xi_s - \ddot q_s\xi_0 - \dot q_s\dot\xi_0) - L\dot\xi_0 - \frac{\partial L}{\partial t}\xi_0\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial L}{\partial q_s}\xi_s - \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) \\
&- (Q_s + P_s)(\xi_s - \dot{q}_s\xi_0) \\
&= (\xi_s - \dot{q}_s\xi_0)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s - P_s\right) \\
&= 0.
\end{aligned}
$$

5. Illustrative example

We consider a particle of variable mass in the following form:

$$m = m_0 \exp t \ (m_0 = \text{const.}). \tag{29}$$

The Lagrangian is

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2). \tag{30}$$

The nonpotential generalized forces are

$$Q_1 = 2m\dot{q}_1,$$

and

$$Q_2 = 2m\dot{q}_2. \tag{31}$$

The absolute velocity of the separate corpuscle is zero, i.e.

$$\boldsymbol{u} = -\dot{\boldsymbol{r}}_i = -\dot{q}_1\boldsymbol{i} - \dot{q}_2\boldsymbol{j}. \tag{32}$$

Now we come to study the conformal invariance and the conserved quantities of this system.

From Eqs. (3), (29) and (32), we have

$$P_1 = P_2 = 0. \tag{33}$$

The differential equations of motion are

$$\ddot{q}_1 = \dot{q}_1 = \alpha_1,$$

and

$$\ddot{q}_2 = \dot{q}_2 = \alpha_2. \tag{34}$$

From determining equation (21), we can obtain the following equations:

$$
\begin{aligned}
&\ddot{\xi}_1 - 2\dot{q}_1\dot{\xi}_0 - \dot{q}_1\ddot{\xi}_0 - \dot{\xi}_1 + \dot{q}_1\dot{\xi}_0 = 0, \\
&\ddot{\xi}_2 - 2\dot{q}_2\dot{\xi}_0 - \dot{q}_2\ddot{\xi}_0 - \dot{\xi}_2 + \dot{q}_2\dot{\xi}_0 = 0
\end{aligned} \tag{35}
$$

with solution

$$\xi_0 = 0, \quad \xi_1 = -\dot{q}_1 + q_1, \quad \xi_2 = -\dot{q}_2 + q_2. \tag{36}$$

According to Eq. (26), we have

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{37}$$

So the determining equations of conformal invariance are

$$X^{(2)}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{38}$$

or

$$
\begin{aligned}
X^{(2)}(\ddot{q}_1 - \dot{q}_1) &= (\ddot{q}_1 - \dot{q}_1), \\
X^{(2)}(\ddot{q}_2 - \dot{q}_2) &= (\ddot{q}_2 - \dot{q}_2).
\end{aligned} \tag{39}
$$

Therefore the conformal invariance of the system would be Lie symmetric.

Substituting Eqs. (29), (30), (31) and (33) into structural equation (27) leads to

$$\dot{G} = 2m(\dot{q}_1^2 + \dot{q}_2^2) - 2m(q_1\dot{q}_1 + q_2\dot{q}_2). \tag{40}$$

Thus

$$G = m(\dot{q}_1^2 + \dot{q}_2^2) - m(q_1^2 + q_2^2). \tag{41}$$

From the conserved quantity in Eq. (28), we can conclude that

$$I = m(-q_1^2 - q_2^2 + \dot{q}_1 q_1 + \dot{q}_2 q_2) = \text{const.} \tag{42}$$

6. Conclusions

For a general holonomic system with variable mass, by using the definition of conformal invariance and the Lie symmetrical determining equations of conformal invariance, we can obtain the conformal factor of determining equations of conformal invariance. The conformal factor is also a necessary and sufficient condition under which the conformal invariance would be Lie symmetry of the system. The conformal invariance can also lead to a conserved quantity under a certain condition. The main results are those that are expressed by Eqs. (23), (26), (27) and (28). The results obtained in this paper are of general significance, and they may be applied to a general holonomic system; that is, when $m_i = \text{const.}$, the main results obtained in this paper will be those given in Ref. [23].

References

[1] Noether A E 1918 *Math. Phys.* **KI II** 235
[2] Lutzky M 1979 *J. Phys.* A: *Math. Gen.* **12** 973
[3] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120
[4] Mei F X 2001 *Chin. Phys.* **10** 177
[5] Luo S K and Cai J L 2003 *Chin. Phys.* **12** 357
[6] Lou Z M 2006 *Chin. Phys.* **15** 891
[7] Mei F X 2003 *Acta Phys. Sin.* **52** 1048 (in Chinese)
[8] Luo S K, Chen X W and Guo Y X 2007 *Chin. Phys.* **16** 3176
[9] Mei F X 2000 *Acta Phys. Sin.* **49** 1207 (in Chinese)
[10] Luo S K, Guo Y X and Chen X W 2001 *Acta Phys. Sin.* **50** 2053 (in Chinese)
[11] Fang J H, Ding N and Wang P 2007 *Chin. Phys.* **16** 887
[12] Luo S K 2002 *Acta Phys. Sin.* **51** 712 (in Chinese)
[13] Xia L L and Li Y C 2007 *Commun. Theor. Phys.* (Beijing, China) **48** 23
[14] Xia L L, Li Y C, Wang J and Hou Q B 2006 *Acta Phys. Sin.* **55** 4995 (in Chinese)
[15] Zhao Y Y and Mei F X 1999 *Symmetries and Invariants of Mechanical Systems* (Beijing: Science Press) (in Chinese)
[16] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Techanology Press) (in Chinese)
[17] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[18] Galiullin A S, Gafarov G G, Malaishka R P and Khwan A M 1997 *Analytical Dynamics of Helmholtz, Birkhoff and Nambu Systems* (Moscow: UFN) (in Russian)
[19] Cai J L, Luo S K and Mei F X 2008 *Chin. Phys.* B **17** 3170
[20] Cai J L 2009 *Acta Phys. Sin.* **59** 22 (in Chinese)
[21] Liu C, Mei F X and Guo Y X 2009 *Chin. Phys.* B **18** 395
[22] Fu J L 2008 *Chin. Phys. Lett.* **25** 2413
[23] Cai J L 2008 *Chin. Phys. Lett.* **25** 1523