A new type of conserved quantity of Lie symmetry for the Lagrange system

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This paper studies a new type of conserved quantity which is directly induced by Lie symmetry of the Lagrange system. Firstly, the criterion of Lie symmetry for the Lagrange system is given. Secondly, the conditions of existence of the new conserved quantity as well as its forms are proposed. Lastly, an example is given to illustrate the application of the result.

Keywords: Lagrange system, Lie symmetry, new conserved quantity

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1. Introduction

The research on symmetries and conserved quantities of a mechanical system has important theoretical and practical significance, and plays a very important role in modern mathematics, mechanics and physics. The modern methods of finding conserved quantities by using symmetries are mainly Noether symmetry,[1,2] Lie symmetry[2,3] and Mei symmetry.[4−6] Noether symmetry is an invariance of a Hamilton action under the infinitesimal transformations of a group. Lie symmetry is an invariance of differential equations under the infinitesimal transformations of a group. Mei symmetry means that the dynamical functions (such as Lagrangians, Hamiltonians, and non-potential generalized forces, etc.) in the equation of motion are replaced by transformed dynamical functions, whereas the form of the equation of motion is invariant. The conserved quantities induced from the three symmetries mentioned above are mainly Noether conserved quantity, Hojman conserved quantity and Mei conserved quantity.[2,7]

In recent years, the research on the three symmetries and corresponding conserved quantities has achieved lots of important results.[8−22] Lie symmetry can induce the Hojman conserved quantity directly. Meanwhile, it can induce the Noether and Mei conserved quantities indirectly.[2] In this paper, we study a new type of conserved quantity which is directly induced by Lie symmetry of the Lagrange system, and obtain conditions of existence of the new conserved quantity as well as its forms. Finally, an example is given to illustrate the application of the result.

2. Lie symmetry of the system

The differential equations of motion for the Lagrange system are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=0, \quad (s=1,2,\ldots,n), \tag{1}$$

where $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the Lagrangian function of the system, and q_s are generalized coordinates. Equation (1) can be written succinctly as

$$E_s(L)=0, \tag{2}$$

where

$$E_s=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}-\frac{\partial}{\partial q_s}. \tag{3}$$

We introduce the infinitesimal transformations

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \quad q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{4}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are single-parameter infinitesimal generators.

Based on the Lie symmetry theory of the Lagrange system, we have the following criterion.[2]

Criterion For Lagrange system (2), if the infinitesimal generators ξ_0 and ξ_s satisfy the determining equation

$$X^{(2)}[E_s(L)]=0, \tag{5}$$

the corresponding transformations are called Lie symmetrical transformations, and the corresponding symmetry is called the Lie symmetry of the system, where

$$X^{(2)}=X^{1)}+[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^{\cdot}-\ddot{q}_s\dot{\xi}_0]\frac{\partial}{\partial \ddot{q}_s}, \tag{6}$$

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$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}, \tag{7}$$

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s}. \tag{8}$$

3. The new conserved quantity of Lie symmetry

The Lie symmetry of the Lagrange system can induce the Hojman conserved quantity directly. Can it induce other new types of conserved quantities directly? The following theorem gives the conditions of existence of a new conserved quantity as well as its forms, which is induced directly by Lie symmetry of the Lagrange system.

Theorem 1 For Lagrange system (2), if the infinitesimal generators ξ_0, ξ_s of Lie symmetry and a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy the condition

$$-\dot{\xi}_0\frac{\partial L}{\partial t} - X^{(1)}\left(\frac{\partial L}{\partial t}\right) + \dot{G} = 0, \tag{9}$$

then the Lie symmetry of the system can lead to the following new conserved quantity

$$I = \dot{q}_s X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - X^{(0)}(L) + G = \text{const.} \tag{10}$$

Proof Taking the derivative of I with respect to t, we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\left[X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right] + \ddot{q}_s X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\mathrm{d}}{\mathrm{d}t}[X^{(0)}(L)] + \dot{G}. \tag{11}$$

Because of

$$X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right) = \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} - \left[\frac{\partial \xi_0}{\partial \dot{q}_s}\frac{\partial L}{\partial t} + \frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial L}{\partial q_k} + \frac{\partial(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)}{\partial \dot{q}_s}\frac{\partial L}{\partial \dot{q}_k}\right], \tag{12}$$

substituting Eq. (12) into the first term of the right-hand side of Eq. (11), we have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} = {} & \dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right) - \dot{q}_s\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_0}{\partial \dot{q}_s}\right)\frac{\partial L}{\partial t} + \frac{\partial \xi_0}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial t}\right)^{\cdot} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_k}{\partial \dot{q}_s}\right)\frac{\partial L}{\partial q_k} + \frac{\partial \xi_k}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial q_k}\right)^{\cdot}\right. \\ & \left. + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)}{\partial \dot{q}_s}\right]\frac{\partial L}{\partial \dot{q}_k} + \frac{\partial(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial \dot{q}_k}\right)^{\cdot}\right\} + \ddot{q}_s X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\mathrm{d}}{\mathrm{d}t}[X^{(0)}(L)] + \dot{G}.\end{aligned} \tag{13}$$

Adding and subtracting the term $\dot{q}_s\partial X^{(1)}(L)/\partial q_s$ in the above equation at the same time, and noticing

$$\frac{\partial X^{(1)}(L)}{\partial q_s} = X^{(1)}\left(\frac{\partial L}{\partial q_s}\right) + \left[\frac{\partial \xi_0}{\partial q_s}\frac{\partial L}{\partial t} + \frac{\partial \xi_k}{\partial q_s}\frac{\partial L}{\partial q_k} + \frac{\partial(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)}{\partial q_s}\frac{\partial L}{\partial \dot{q}_k}\right], \tag{14}$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} = {} & \dot{q}_s\left\{E_s[X^{(1)}(L)] - E_s(\xi_0)\frac{\partial L}{\partial t} - E_s(\xi_k)\frac{\partial L}{\partial q_k} - E_s(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial \xi_0}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial t}\right)^{\cdot} - \frac{\partial \xi_k}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial q_k}\right)^{\cdot}\right. \\ & \left. - \left(\frac{\partial \dot{\xi}_k}{\partial \dot{q}_s} - \dot{q}_k\frac{\partial \dot{\xi}_0}{\partial \dot{q}_s}\right)\left(\frac{\partial L}{\partial \dot{q}_k}\right)^{\cdot}\right\} + \dot{\xi}_0\dot{q}_s\left(\frac{\partial L}{\partial \dot{q}_s}\right)^{\cdot} + \dot{q}_s X^{(1)}\left(\frac{\partial L}{\partial q_s}\right) + \ddot{q}_s X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - (X^{(0)}(L))^{\cdot} + \dot{G}.\end{aligned} \tag{15}$$

Considering the following deductions

$$\begin{aligned}E_s[X^{(1)}(L)] = {} & E_s(\xi_0)\frac{\partial L}{\partial t} + \xi_0 E_s\left(\frac{\partial L}{\partial t}\right) + \dot{\xi}_0\frac{\partial^2 L}{\partial t\partial \dot{q}_s} + \frac{\partial \xi_0}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial t}\right)^{\cdot} + E_s(\xi_k)\frac{\partial L}{\partial q_k} + \xi_k E_s\left(\frac{\partial L}{\partial q_k}\right) + \dot{\xi}_k\frac{\partial^2 L}{\partial q_k\partial \dot{q}_s} \\ & + \frac{\partial \xi_k}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial q_k}\right)^{\cdot} + E_s(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial L}{\partial \dot{q}_k} + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)E_s\left(\frac{\partial L}{\partial \dot{q}_k}\right) + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)^{\cdot}\frac{\partial^2 L}{\partial \dot{q}_k\partial \dot{q}_s} \\ & + \left(\frac{\partial \dot{\xi}_k}{\partial \dot{q}_s} - \dot{q}_k\frac{\partial \dot{\xi}_0}{\partial \dot{q}_s}\right)\left(\frac{\partial L}{\partial \dot{q}_k}\right)^{\cdot} - \dot{\xi}_0\left(\frac{\partial L}{\partial \dot{q}_s}\right)^{\cdot},\end{aligned} \tag{16}$$

$$\begin{aligned}X^{(2)}[E_s(L)] = {} & \xi_0 E_s\left(\frac{\partial L}{\partial t}\right) + \xi_k E_s\left(\frac{\partial L}{\partial q_k}\right) + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)E_s\left(\frac{\partial L}{\partial \dot{q}_k}\right) + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)^{\cdot}\frac{\partial^2 L}{\partial \dot{q}_k\partial q_k} \\ & + [(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)^{\cdot} - \ddot{q}_k\dot{\xi}_0]\frac{\partial^2 L}{\partial \dot{q}_k\partial \dot{q}_s},\end{aligned} \tag{17}$$

we have

$$X^{(2)}[E_s(L)] = E_s[X^{(1)}(L)] - E_s(\xi_0)\frac{\partial L}{\partial t} - E_s(\xi_k)\frac{\partial L}{\partial q_k} - E_s(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial \xi_0}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial t}\right)^{\cdot} - \frac{\partial \xi_k}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial q_k}\right)^{\cdot} - \left(\frac{\partial \dot{\xi}_k}{\partial \dot{q}_s} - \dot{q}_k\frac{\partial \dot{\xi}_0}{\partial \dot{q}_s}\right)\left(\frac{\partial L}{\partial \dot{q}_k}\right)^{\cdot}. \tag{18}$$

Substituting Eq. (18) into Eq. (15), we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \dot{q}_s X^{(2)}[E_s(L)] + \dot{\xi}_0\dot{q}_s\left(\frac{\partial L}{\partial \dot{q}_s}\right)^{\cdot} + \dot{q}_s X^{(1)}\left(\frac{\partial L}{\partial q_s}\right) + \ddot{q}_s X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - (X^{(0)}(L))^{\cdot} + \dot{G}. \tag{19}$$

After further collation, we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \dot{q}_s X^{(2)}[E_s(L)] + \dot{\xi}_s E_s(L) - X^{(1)}\left(\frac{\partial L}{\partial t}\right) - \dot{\xi}_0\frac{\partial L}{\partial t} + \dot{G}. \tag{20}$$

Substituting Eqs. (5) and (9) into Eq. (20), and using Eq. (2), we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0. \tag{21}$$

The conserved quantity Eq. (10) is a new type of conserved quantity, which is directly induced by Lie symmetry of the Lagrange system and is different from the Hojman conserved quantity, Noether conserved quantity and Mei conserved quantity.

Because Nother symmetry of the Lagrange system must be Lie-symmetric,[2] according to Theorem 1, we have the following theorem.

Theorem 2 For Lagrange system (2), if the infinitesimal generators ξ_0, ξ_s of Noether symmetry and a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy the condition (9), Noether symmetry of the system can lead to the new conserved quantity (10).

4. Example

The Lagrangian of a mechanical system is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + q_1^2, \tag{22}$$

we try to study the new type of conserved quantity, which is directly induced by Lie symmetry of the system.

The equations of motion (2) give

$$\ddot{q}_1 - 2q_1 = 0,\ \ddot{q}_2 = 0. \tag{23}$$

According to determining equation (5), we have

$$(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)^{\cdot} - \ddot{q}_1\dot{\xi}_0 - 2\xi_1 = 0,\quad \ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 = 0. \tag{24}$$

Equations (24) have the following solutions:

$$\xi_0 = 1,\ \xi_1 = q_1,\ \xi_2 = q_2 - \dot{q}_2 t + t, \tag{25}$$

i.e. the infinitesimal generators (25) are Lie symmetrical, the corresponding symmetry is Lie symmetry.

For generators (25), the condition equation (9) gives

$$G = 0, \tag{26}$$

from Eq. (10), we can obtain the following new conserved quantity

$$I = \dot{q}_1^2 - 2q_1^2 + \dot{q}_2 = \text{const}. \tag{27}$$

5. Conclusion

A new type of conserved quantity induced directly by Lie symmetry of the Lagrange system is given in this paper, which is different from the Noether conserved quantity, the Hojman conserved quantity and the Mei conserved quantity. The Lie symmetry of the Lagrange system can induce the Hojman conserved quantity directly and can induce the Noether and Mei conserved quantities indirectly. The research in this paper has shown that Lie symmetry of the Lagrange system can also induce the new conserved quantity Eq. (10) directly. The result of this paper can be generalized to non-conservative holonomic mechanical systems and non-holonomic mechanical systems.

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