The extended symmetry approach for studying the general Korteweg-de Vries-type equation*

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(Received 8 August 2009; revised manuscript received 8 September 2009)

The extended symmetry approach is used to study the general Korteweg-de Vries-type (KdV-type) equation. Several variable-coefficient equations are obtained. The solutions of these resulting equations can be constructed by the solutions of original models if their solutions are well known, such as the standard constant coefficient KdV equation and the standard compound KdV–Burgers equation, and so on. Then any one of these variable-coefficient equations can be considered as an original model to obtain new variable-coefficient equations whose solutions can also be known by means of transformation relations between solutions of the resulting new variable-coefficient equations and the original equation.

Keywords: extended symmetry approach, general Korteweg-de Vries-type (KdV-type) equation, variable-coefficient equation

PACC: 0200, 0420J

1. Introduction

Nonlinear phenomena exist in various scientific fields, such as optical fibres,[1−3] biology,[4,5] fluid mechanics,[6] plasma physics,[7] chemical physics,[8] and so on. How to find exact solutions of nonlinear partial differential equations (PDEs) plays an important role in the research of nonlinear physical phenomena. So it is always an interesting topic to search for meaningful solutions for PDEs. Many effective methods have been found to solve a nonlinear PDE, such as the inverse scattering transformation approach,[9] Bäcklund and Darboux transformations,[10] the bilinear method,[11] the variable separation approach[12] and symmetry reductions.[13] However, most of the activities are only restricted to constant coefficient PDEs. Because many nonlinear PDE in physics have coefficients which also depend on independent time and space variables, it is important to develop effective approaches to solve this kind of variable-coefficient PDE. Recently, this has been attracting more and more attention.[14−17] In Ref. [17], the author proposed a new symmetry approach to solve the space-time-dependent KdV system

$$u_t + u_{xxx} + W(x,t,u)u_x = 0, \tag{1}$$

where W is a function of x, t, and u. The difference between the extended symmetry approach and the standard technique in Lie group analysis of PDEs is that the latter is restricted to the condition that W is a function of x, t, and u, while the former assumes that $W(x,t,u)$ is a new variable dependent on equal conditions to u. With the help of this method, some special types of space-time-dependent equations have been obtained from the original model. These variable-coefficient equations could then be solved from corresponding space-time-independent equations such as the KdV equation and modified KdV equation by choosing appropriate $W(x,t,u)$.

In this paper , we use the extended symmetry approach to study the more general KdV-type equation

$$u_t = F(x,t,u,u_x,u_{xx})u_{xxx} + G(x,t,u,u_x,u_{xx}), \tag{2}$$

where F and G are functions of $x, t, u, u_x,$ and u_{xx} and their actual forms depend on the physics of the problems. Starting from Eq. (2) by using the extended symmetry approach, several new variable-coefficient equations are obtained. Considering the original model as constant coefficient equations by choosing appropriate $F(x,t,u,u_x,u_{xx})$ and $G(x,t,u,u_x,u_{xx})$ whose solutions are well known, the resulting variable-coefficient equations can be solved easily. Then we can also consider each of these resulting variable-coefficient models as an original model

*Project supported by the National Natural Science Foundation of China (Grant No. 10675065) and the Scientific Research Fund of the Education Department of Zhejiang Province of China (Grant No. 20070979).

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to gain new variable-coefficient equations whose solutions can also be obtained by means of the original variable-coefficient model.

The organization of the paper is as follows. In Section 2, the extended symmetry is applied to Eq. (2). The solutions of the resulting variable-coefficient equations are derived from the original model in Section 3. A conclusion is given in the last section.

2. Extended symmetries of the general KdV-type equation

According to the extended symmetry approach, we consider $F(x,t,u,u_x,u_{xx})$ and $G(x,t,u,u_x,u_{xx})$ as two dependent variables which are parallel to the variable u in Eq. (2) and introduce the following first-order differential operator and corresponding transformation group

$$\begin{aligned} K &= \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u \\ &\quad + f(x,t,u,F)\partial_F + g(x,t,u,G)\partial_G, \qquad (3)\\ x_1 &= x + \varepsilon\xi(x,t,u), \quad t_1 = t + \varepsilon\tau(x,t,u),\\ u_1 &= u + \varepsilon\eta(x,t,u), \quad F_1 = F + \varepsilon f(x,t,u,F),\\ G_1 &= G + \varepsilon g(x,t,u,G), \qquad (4) \end{aligned}$$

where ε is the group parameter. Because of the invariance condition of Eq. (2) under the operator (3) with its prolongation, we can obtain the following equation:

$$[\eta_t] - f u_{xxx} - F[\eta_{xxx}] - g = 0, \qquad (5)$$

where $[\eta_t]$ and $[\eta_{xxx}]$ are the infinite similarities of u_t and u_{xxx}, respectively, which can be expressed as follows:

$$\begin{aligned} [\eta_t] &= D_t(\eta - \xi u_x - \tau u_t) + \xi u_{tx} + \tau u_{tt},\\ [\eta_{xxx}] &= D_x^3(\eta - \xi u_x - \tau u_t) + \xi u_{xxxx} + \tau u_{xxxt}, \end{aligned} \qquad (6)$$

where the operator D_i $(i = x,t)$ means total differentiation for corresponding variables and $D_x^3 = D_x \cdot D_x \cdot D_x$. From Eq. (5), we may obtain a set of determining equations for ξ, τ, η, f and g

$$\begin{aligned} &\xi_u = \tau_x = \tau_u = 0, \quad \eta_{uu} = 0,\\ &\eta_{xuu} = 0, \quad \xi_{xx} - \eta_{xu} = 0,\\ &-\tau' F - f + 3F\xi_x = 0, \quad -\xi_t - 3F\eta_{xxu} + F\xi_{xxx} = 0,\\ &\eta_t + G\eta_u - \tau' G - F\eta_{xxx} - g = 0. \qquad (7) \end{aligned}$$

By solving the above determining equations, we can gain the following particular result

$$\begin{aligned} &\xi = \frac{c_1}{2}x^2 + c_3 x + c_4, \quad \tau = \tau(t),\\ &\eta = A + Bu + (c_1 x + c_2)u,\\ &f = (-\tau' + 3c_1 x + 3c_3)F,\\ &g = A_t + B_t u - F A_{xxx} + (c_1 x + c_2 + B - \tau')G. \qquad (8) \end{aligned}$$

Here $A = A(x,t)$, B and τ are arbitrary smooth functions of independent variable t, c_1, c_2, c_3 and c_4 are arbitrary real constants.

Because it is too complicated to obtain the general finite transformation of system (8) directly, here we consider some special cases of Eq. (8). Assuming that $A = A(x), c_1 = \tau = 0$, we can have the general finite transformation as follows:

$$\begin{aligned} &t_1 = t, \quad x_1 = x\mathrm{e}^{\varepsilon c_3} + c_4(\mathrm{e}^{\varepsilon c_3} - 1)/c_3,\\ &u_1(x_1,t_1) = \xi(x_1)^{[B(t_1)+c_2]/c_3}[R(x_1,t_1) - R(x,t) + u(x,t)\xi(x)^{-[B(t)+c_2]/c_3}],\\ &F_1(x_1,t_1,u_1,u_{1x_1},u_{1x_1x_1}) = \frac{F(x,t,u,u_x,u_{xx})\xi(x_1)^3}{\xi(x)^3},\\ &G_1(x_1,t_1,u_1,u_{1x_1},u_{1x_1x_1}) = \xi(x_1)^{(B(t_1)+c_2)/c_3}\{B'(t_1)[q_0(x_1,t_1) + q_2(x,t)\ln(c_3x_1 + c_4)/c_3] - B'(t)\\ &\qquad \times [q_0(x,t) + q_2(x,t)\ln(c_3x + c_4)/c_3] - F(x,t,u,u_x,u_{xx})/\xi(x)^3\\ &\qquad \times [q_1(x_1,t_1) - q_1(x,t)] + G(x,t,u,u_x,u_{xx})\xi(x)^{-(B(t)+c_2)/c_3}\}, \qquad (9) \end{aligned}$$

where

$$\begin{aligned} &R(x_1,t_1) = \int^{x_1} A(\sigma)\xi(\sigma)^{-(c_3+B(t_1)+c_2)}/c_3 \mathrm{d}\sigma, \quad q_0(x_1,t_1) = \int^{x_1}[R(\sigma,t_1)/\xi(\sigma)]\mathrm{d}\sigma,\\ &q_1(x_1,t_1) = \int^{x_1}\xi(\sigma)^{[2c_3-B(t_1)-c_2]/c_3}A'''(\sigma)\mathrm{d}\sigma, \quad q_2(x,t) = u(x,t)\xi(x)^{-(B(t)+c_2)/c_3} - R(x,t). \end{aligned}$$

In the case of $A(x,t)=A(t)$ and $c_1=0$, the general finite transformation of system (8) is

$$\begin{aligned}
&f_0(t_1)=\varepsilon+f_0(t),\quad x_1=x\mathrm{e}^{\varepsilon c_3}+c_4(\mathrm{e}^{\varepsilon c_3}-1)/c_3,\\
&u_1(x_1,t_1)=f_1(t_1)[f_3(t_1)-f_3(t)+u(x,t)/f_1(t)],\\
&F_1(x_1,t_1,u_1,u_{1x_1},u_{1x_1x_1})=F(x,t,u,u_x,u_{xx})\tau(t)/\tau(t_1)\mathrm{e}^{3c_3(f_0(t_1)-f_0(t))},\\
&G_1(x_1,t_1,u_1,u_{1x_1},u_{1x_1x_1})=\frac{f_1(t_1)}{f_1(t)\tau(t_1)}\Bigg[(u(x,t)-f_1(t)f_3(t))(B(t_1)-B(t))+G(x,t,u,u_x,u_{xx})\tau(t)\\
&\qquad\qquad+f_1(t)\Bigg(\int^{t_1}A'(\sigma)/f_1(\sigma)\mathrm{d}\sigma-\int^{t}A'(\sigma)/f_1(\sigma)\mathrm{d}\sigma\\
&\qquad\qquad+\int^{t_1}B'(\sigma)f_3(\sigma)\mathrm{d}\sigma-\int^{t}B'(\sigma)f_3(\sigma)\mathrm{d}\sigma\Bigg)\Bigg],
\end{aligned}\tag{10}$$

where

$$f_0(t_1)=\int^{t_1}\mathrm{d}\sigma/\tau(\sigma),\quad f_1(t_1)=\mathrm{e}^{\int^{t_1}(B(\sigma)+c_2)/\tau(\sigma)\mathrm{d}\sigma},\quad f_3(t_1)=\int^{t_1}[A(\sigma)/f_1(\sigma)/\tau(\sigma)]\mathrm{d}\sigma.$$

Taking $A(x,t)=A(t)$, $c_1=0$ and $\tau=0$, we can obtain the following general finite transformation

$$\begin{aligned}
&t_1=t,\quad x_1=x\mathrm{e}^{\varepsilon c_3}+c_4(\mathrm{e}^{\varepsilon c_3}-1)/c_3,\\
&u_1(x_1,t_1)=-\frac{A(t_1)}{B(t_1)+c_2}+\xi(x_1)^{(B(t_1)+c_2)/c_3}\xi(x)^{(B(t)+c_2)/c_3}\left(u(x,t)+\frac{A(t)}{B(t)+c_2}\right),\\
&F_1(x_1,t_1,u_1,u_{1x_1},u_{1x_1x_1})=F(x,t,u,u_x,u_{xx})\xi(x_1)^3/\xi(x)^3,\\
&G_1(x_1,t_1,u_1,u_{1x_1},u_{1x_1x_1})=\xi(x_1)^{(B(t_1)+c_2)/c_3}\Bigg[f_4(t_1)\int^{x_1}\xi(\sigma)^{-(B(t_1)+c_2+c_3)/c_3}\mathrm{d}\sigma-f_4(t)\\
&\qquad\qquad\times\int^{x}\xi(\sigma)^{-(B(t)+c_2+c_3)/c_3}\mathrm{d}\sigma+(u(x,t)+A(t)/(B(t)+c_2))\\
&\qquad\qquad\times\xi(x)^{(B(t)+c_2)/c_3}\left(B'(t_1)\int^{x_1}\mathrm{d}\sigma/\xi(\sigma)-B'(t)\int^{x}\mathrm{d}\sigma/\xi(\sigma)\right)\\
&\qquad\qquad+G(x,t,u,u_x,u_{xx})/\xi(x)^{(B(t)+c_2)/c_3}\Bigg],
\end{aligned}\tag{11}$$

where

$$f_4(t_1)=A'(t_1)-A(t_1)B'(t_1)/(B(t_1)+c_2).$$

Now we show some determined expressions of the finite transformations (9)–(11) by taking some arbitrary functions as concrete forms.

Choosing $B=1$, $A=x$, $\tau=c_2=c_4=0$, equation (9) can be expressed as follows:

$$x_1=\mathrm{e}^{c_3\varepsilon}x,\quad t_1=t,\quad u_1=x_1/(c_3-1)+\mathrm{e}^{\varepsilon}(u-x/(c_3-1)),\quad F_1=\mathrm{e}^{3c_3\varepsilon}F,\quad G_1=\mathrm{e}^{\varepsilon}G.\tag{12}$$

Taking $B=t$, $A=\mathrm{e}^t$, $\tau=t$, $c_2=c_4=0$ in transformation (10), we can obtain

$$x_1=\mathrm{e}^{c_3\varepsilon}x,\quad t_1=\mathrm{e}^{\varepsilon}t,\quad u_1=\varepsilon\mathrm{e}^{t_1}+\mathrm{e}^{t_1-t}u,\quad F_1=\mathrm{e}^{3c_3-1}F,\quad G_1=\mathrm{e}^{t_1-t}(u+(G-u)/\mathrm{e}^{\varepsilon})+\varepsilon\mathrm{e}^{t_1}.\tag{13}$$

Under the condition $B=0$, $A=1/t$, $\tau=1/t$, $c_2=c_4=0$, the transformation (10) can be written as

$$x_1=\mathrm{e}^{c_3\varepsilon}x,\quad t_1=\sqrt{2\varepsilon+t^2},\quad u_1=u+t_1-t,\quad F_1=\mathrm{e}^{3c_3\varepsilon}t_1/tF,\quad G_1=1+t_1/t(G-1).\tag{14}$$

For $B=1$, $A=\sin t$, $\tau=0$, $c_2=c_4=0$, the transformation (11) can be written as

$$x_1=\mathrm{e}^{c_3\varepsilon}x,\quad t_1=t,\quad u_1=-\sin t_1+\mathrm{e}^{\varepsilon}(u+\sin t),\quad F_1=\mathrm{e}^{3c_3\varepsilon}F,\quad G_1=-\cos t_1+\mathrm{e}^{\varepsilon}(G+\cos t).\tag{15}$$

3. Some variable-coefficient KdV-type equations and their solutions

By using the extended symmetry operator and the corresponding finite transformations (9)–(15), we can obtain several variable-coefficient KdV-type equations. Their solutions can be obtained with the help of the original equation which is a constant coefficient equation or variable-coefficient equation whose solutions are known.

Using the finite transformation (9), we can gain the variable-coefficient equation as follows:

$$\begin{aligned} u_{1t_1} &- \frac{F(x_2,t_2,u_2,u_{2x_2},u_{2x_2x_2})\xi(x_1)^3}{\xi(x_2)^3}u_{1x_1x_1x_1} - \xi(x_1)^{B(t_1)+c_2/c_3}\{B'(t_1)[q_0(x_1,t_1)+q_2(x_2,t_2) \\ &\times \ln(c_3x_1+c_4)/c_3] - B'(t_2)[q_0(x_2,t_2)+q_2(x_2,t_2)\ln(c_3x_2+c_4)/c_3] \\ &- F(x_2,t_2,u_2,u_{2x_2},u_{2x_2x_2})/\xi(x_2)^3[q_1(x_1,t_1)-q_1(x_2,t_2)] \\ &+ G(x_2,t_2,u_2,u_{2x_2},u_{2x_2x_2})\xi(x_2)^{-(B(t_2)+c_2)/c_3}\} = 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} &t_2 = t_1 = t, \quad x_2 = x_1\mathrm{e}^{-\varepsilon c_3} - c_4(1-\mathrm{e}^{-\varepsilon c_3})/c_3, \\ &u(x_2,t_2) = \xi(x_2)^{[B(t_2)+c_2]/c_3}[R(x_2,t_2) - R(x_1,t_1) + u_1(x_1,t_1)\xi(x_1)^{-[B(t_1)+c_2]/c_3}]. \end{aligned}$$

The solution of Eq. (16) can be expressed as follows:

$$u_1(x_1,t_1) = \xi(x_1)^{[B(t_1)+c_2]/c_3}[R(x_1,t_1) - R(x,t) + u(x,t)\xi(x)^{-[B(t)+c_2]/c_3}]\,|_{t=t_2,x=x_2}\ , \tag{17}$$

where $u(x,t)$ is the solution of the original equation (2).

According to the finite transformation (10), the variable-coefficient equation can be written as

$$\begin{aligned} u_{1t_1} &- F(x_2,t_2,u_2,u_{2x_2},u_{2x_2x_2})\tau(t_2)/\tau(t_1)\mathrm{e}^{3c_3(f_0(t_1)-f_0(t_2))}u_{1x_1x_1x_1} \\ &- \frac{f_1(t_1)}{f_1(t)\tau(t_1)}\Bigg[(u(x_2,t_2)-f_1(t_2)f_3(t_2))(B(t_1)-B(t_2)) + G(x_2,t_2,u_2,u_{2x_2},u_{2x_2x_2})\tau(t_2) + f_1(t_2) \\ &\times\left(\int^{t_1}A'(\sigma)/f_1(\sigma)\mathrm{d}\sigma - \int^{t_2}A'(\sigma)/f_1(\sigma)\mathrm{d}\sigma + \int^{t_1}B'(\sigma)f_3(\sigma)\mathrm{d}\sigma - \int^{t_2}B'(\sigma)f_3(\sigma)\mathrm{d}\sigma\right)\Bigg] = 0 \end{aligned} \tag{18}$$

with $t_2 = f_0^{-1}(f_0(t_1)-\varepsilon)$, $x_2 = x_1\mathrm{e}^{-\varepsilon c_3} - c_4(1-\mathrm{e}^{-\varepsilon c_3})/c_3$,

$$u(x_2,t_2) = f_1(t_2)[f_3(t_2) - f_3(t_1) + u_1(x_1,t_1)/f_1(t_1)].$$

The solution of Eq. (18) can be written as

$$u_1(x_1,t_1) = f_1(t_1)[f_3(t_1) - f_3(t) + u(x,t)/f_1(t)]\,|_{t=t_2,x=x_2} \tag{19}$$

with $u(x,t)$ being the solution of original equation (2).

With the help of the finite transformation (11), we can obtain the following variable-coefficient equation:

$$\begin{aligned} &u_{1t_1} - F(x_2,t_2,u_2,u_{2x_2},u_{2x_2x_2})\xi(x_1)^3/\xi(x_2)^3u_{1x_1x_1x_1} - \xi(x_1)^{(B(t_1)+c_2)/c_3}\Bigg[f_4(t_1)\int^{x_1}\xi(\sigma)^{-(B(t_1)+c_2+c_3)/c_3}\mathrm{d}\sigma \\ &- f_4(t_2)\int^{x_2}\xi(\sigma)^{-(B(t)+c_2+c_3)/c_3}\mathrm{d}\sigma + (u(x_2,t_2)+A(t_2)/(B(t_2)+c_2))\xi(x_2)^{(B(t_2)+c_2)/c_3}(B'(t_1)\int^{x_1}\mathrm{d}\sigma/\xi(\sigma) \\ &- B'(t_2)\int^{x_2}\mathrm{d}\sigma/\xi(\sigma)) + G(x_2,t_2,u_2,u_{2x_2},u_{2x_2x_2})/\xi(x_2)^{(B(t_2)+c_2)/c_3}\Bigg] = 0, \end{aligned} \tag{20}$$

where

$$\begin{aligned} &t_2 = t_1 = t, \quad x_2 = x_1\mathrm{e}^{-\varepsilon c_3} - c_4(1-\mathrm{e}^{-\varepsilon c_3})/c_3, \\ &u(x_2,t_2) = -\frac{A(t_2)}{B(t_2)+c_2} + \xi(x_1)^{-(B(t_1)+c_2)/c_3}\xi(x_2)^{-(B(t_2)+c_2)/c_3}\left(u_1(x_1,t_1) + \frac{A(t_1)}{B(t_1)+c_2}\right). \end{aligned}$$

The solution of equation (20) can be constructed by the solution of original equation (2):

$$u_1(x_1,t_1)=-\frac{A(t_1)}{B(t_1)+c_2}+\xi(x_1)^{(B(t_1)+c_2)/c_3}\xi(x)^{(B(t)+c_2)/c_3}\left(u(x,t)+\frac{A(t)}{B(t)+c_2}\right)\Big|_{t=t_2,x=x_2}. \tag{21}$$

According to the transformations (12)–(15), the variable-coefficient equations can be written as follows, respectively:

$$\begin{aligned}&u_{1t_1}-\mathrm{e}^{3c_3\varepsilon}F(x=x_1\mathrm{e}^{-\varepsilon c_3},\quad t=t_1,\quad u=x_1/(c_3-1)(\mathrm{e}^{-c_3\varepsilon}-\mathrm{e}^{-\varepsilon})+\mathrm{e}^{-\varepsilon}u_1,\\&u_x=(u_{1x_1}-1/(c_3-1))\mathrm{e}^{(c_3-1)\varepsilon}+1/(c_3-1),\quad u_{xx}=\mathrm{e}^{(2c_3-1)\varepsilon}u_{1x_1x_1})u_{1x_1x_1x_1}-\mathrm{e}^{\varepsilon}G(x=x_1\mathrm{e}^{-\varepsilon c_3},\quad t=t_1,\\&u=\mathrm{e}^{-\varepsilon}(u_1-x_1/(c_3-1))+x/(c_3-1),\quad u_x=(u_{1x_1}-1/(c_3-1))\mathrm{e}^{(c_3-1)\varepsilon}+1/(c_3-1),\\&u_{xx}=\mathrm{e}^{(2c_3-1)\varepsilon}u_{1x_1x_1})=0,\end{aligned} \tag{22}$$

$$\begin{aligned}&u_{1t_1}-\mathrm{e}^{(3c_3-1)\varepsilon}F(x=\mathrm{e}^{-c_3\varepsilon}x_1,\quad t=\mathrm{e}^{-\varepsilon}t_1,\quad u=(u_1-\varepsilon\mathrm{e}^{t_1})\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1},\quad u_x=\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1+c_3\varepsilon}u_{1x_1},\\&u_{xx}=\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1+2c_3\varepsilon}u_{1x_1x_1})u_{1x_1x_1x_1}-\mathrm{e}^{(1-\mathrm{e}^{-\varepsilon})t_1}[(u_1-\varepsilon\mathrm{e}^{t_1})\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1}(1-\mathrm{e}^{-\varepsilon})+G(x=\mathrm{e}^{-c_3\varepsilon}x_1,\quad t=\mathrm{e}^{-\varepsilon}t_1,\\&u=(u_1-\varepsilon\mathrm{e}^{t_1})\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1},\quad u_x=\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1+c_3\varepsilon}u_{1x_1},\quad u_{xx}=\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1+2c_3\varepsilon}u_{1x_1x_1})/\mathrm{e}^{\varepsilon}]-\varepsilon\mathrm{e}^{t_1}=0,\end{aligned} \tag{23}$$

$$\begin{aligned}&u_{1t_1}-\mathrm{e}^{3c_3\varepsilon}t_1/\sqrt{t_1^2-2\varepsilon}F(x=\mathrm{e}^{-c_3\varepsilon}x_1,\quad t=\sqrt{t_1^2-2\varepsilon},\quad u=u_1-t_1+\sqrt{t_1^2-2\varepsilon},\quad u_x=\mathrm{e}^{c_3\varepsilon}u_{1x_1},\\&u_{xx}=\mathrm{e}^{2c_3\varepsilon}u_{1x_1x_1})u_{1x_1x_1x_1}-1+t_1/\sqrt{t_1^2-2\varepsilon}-t_1/\sqrt{t_1^2-2\varepsilon}G(x=\mathrm{e}^{-c_3\varepsilon}x_1,\quad t=\sqrt{t_1^2-2\varepsilon},\\&u=u_1-t_1+\sqrt{t_1^2-2\varepsilon},\quad u_x=\mathrm{e}^{c_3\varepsilon}u_{1x_1},\quad u_{xx}=\mathrm{e}^{2c_3\varepsilon}u_{1x_1x_1})=0,\end{aligned} \tag{24}$$

and

$$\begin{aligned}&u_{1t_1}-\mathrm{e}^{3c_3\varepsilon}F(x=\mathrm{e}^{-c_3\varepsilon}x_1,\quad t=t_1,\quad u=(u_1+\sin t_1)\mathrm{e}^{-\varepsilon}-\sin t_1,\quad u_x=\mathrm{e}^{(c_3-1)\varepsilon}u_{1x_1},\quad u_{xx}=u_{1x_1x_1}\mathrm{e}^{(2c_3-1)\varepsilon})\\&u_{1x_1x_1x_1}+\cos t_1-\mathrm{e}^{\varepsilon}[\cos t_1+G(x=\mathrm{e}^{-c_3\varepsilon}x_1,\quad t=t_1,\quad u=(u_1+\sin t_1)\mathrm{e}^{-\varepsilon}-\sin t_1,\\&u_x=\mathrm{e}^{(c_3-1)\varepsilon}u_{1x_1},\quad u_{xx}=u_{1x_1x_1}\mathrm{e}^{(2c_3-1)\varepsilon})]=0.\end{aligned} \tag{25}$$

The solutions of equations (22)–(25) read respectively

$$u_1(x_1,t_1)=x_1/(c_3-1)+\mathrm{e}^{\varepsilon}(u(x=x_1\mathrm{e}^{-c_3\varepsilon},t=t_1)-x_1\mathrm{e}^{-c_3\varepsilon}/(c_3-1)), \tag{26}$$

$$u_1(x_1,t_1)=\varepsilon\mathrm{e}^{t_1}+\mathrm{e}^{t_1(1-\mathrm{e}^{-\varepsilon})}u(x=\mathrm{e}^{-c_3\varepsilon}x_1,t=\mathrm{e}^{-\varepsilon}t_1), \tag{27}$$

$$u_1(x_1,t_1)=u\Big(x=\mathrm{e}^{-c_3\varepsilon}x_1,\quad t=\sqrt{t_1^2-2\varepsilon}\Big)+t_1-\sqrt{t_1^2-2\varepsilon}, \tag{28}$$

and

$$u_1(x_1,t_1)=-\sin t_1+\mathrm{e}^{\varepsilon}(u(x=\mathrm{e}^{-c_3\varepsilon}x_1,t=t_1)+\sin t_1), \tag{29}$$

where $u(x,t)$ is the solution of original equation (2).

According to the above discussion, three kinds of space-time-dependent KdV equations (16), (18), and (20), which contain arbitrary functions, are obtained. By choosing concrete forms of the arbitrary functions, we then obtain determined expressions of the variable-coefficient equations (22)–(25). Their solutions can be constructed by the original equation

$$u_t=F(x,t,u,u_x,u_{xx})u_{xxx}+G(x,t,u,u_x,u_{xx}). \tag{30}$$

If considering that the original equation (30) is a constant coefficient equation, for example, taking $F=-1$, and $G=6uu_x$, then the variable-coefficient equations (22)–(25) have the following forms respectively:

$$u_{1t_1}+\mathrm{e}^{3c_3\varepsilon}u_{1x_1x_1x_1}-6[u_1-x_1/(c_3-1)+\mathrm{e}^{(1-c_3)\varepsilon}x_1/(c_3-1)][(u_{1x_1}-1/(c_3-1))\mathrm{e}^{(c_3-1)\varepsilon}+1/(c_3-1)]=0, \tag{31}$$

$$u_{1t_1}+\mathrm{e}^{(3c_3-1)\varepsilon}u_{1x_1x_1x_1}-(u_1-\varepsilon\mathrm{e}^{t_1})(1-\mathrm{e}^{-\varepsilon}+6\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1+(c_3-1)\varepsilon}u_{1x_1})-\varepsilon\mathrm{e}^{t_1}=0, \tag{32}$$

$$u_{1t_1}+\mathrm{e}^{3c_3\varepsilon}t_1/\sqrt{t_1^2-2\varepsilon}u_{1x_1x_1x_1}-1+t_1/\sqrt{t_1^2-2\varepsilon}\Big[1-6\mathrm{e}^{c_3\varepsilon}(u_1-t_1+\sqrt{t_1^2-2\varepsilon})u_{1x_1}\Big]=0, \tag{33}$$

and

$$u_{1t_1} + \mathrm{e}^{3c_3\varepsilon}u_{1x_1x_1x_1} + (1-\mathrm{e}^{\varepsilon})\cos t_1 - 6\mathrm{e}^{(c_3-1)\varepsilon}(u_1 + \sin t_1 - \mathrm{e}^{\varepsilon}\sin t_1)u_{1x_1} = 0, \tag{34}$$

which are some special forms of the following variable coefficient KdV equation with nonuniformity terms[18]

$$u_{1t_1} + a(t_1)u_1u_{1x_1} + d(t_1)u_{1x_1x_1x_1} + c(x_1,t_1)u_1 + b(x_1,t_1)u_{1x_1} + e(x_1,t_1) = 0, \tag{35}$$

when taking $a(t_1), d(t_1), c(x_1,t_1), b(x_1,t_1)$ and $e(x_1,t_1)$ as special cases. In Ref. [18], the author used the WTC method[19] to study Eq. (35) and discussed some explicit solutions in specific cases. For variable-coefficient equations (31)–(34) as special forms of Eq. (35), one can easily gain their solutions which are constructed by the solutions of the standard constant coefficient KdV equation[20]

$$u_t + u_{xxx} - 6uu_x = 0. \tag{36}$$

For a simple example, we know that equation (36) has the following two-soliton solution[21]

$$u(x,t) = -12\frac{3 + 4\cosh(2x-8t) + \cosh(4x-64t)}{[3\cosh(x-28t) + \cosh(3x-36t)]^2}. \tag{37}$$

According to transformations (26)–(29), we can obtain the following solutions for Eqs. (31)–(34), respectively:

$$u_1(x_1,t_1) = x_1/(c_3-1)(1-\mathrm{e}^{(1-c_3)\varepsilon}) - 12\mathrm{e}^{\varepsilon}\frac{3 + 4\cosh(2\mathrm{e}^{-c_3\varepsilon}x_1 - 8t_1) + \cosh(4\mathrm{e}^{-c_3\varepsilon}x_1 - 64t_1)}{[3\cosh(\mathrm{e}^{-c_3\varepsilon}x_1 - 28t_1) + \cosh(3\mathrm{e}^{-c_3\varepsilon}x_1 - 36t_1)]^2}, \tag{38}$$

$$u_1(x_1,t_1) = \varepsilon\mathrm{e}^{t_1} - 12\mathrm{e}^{t_1(1-\mathrm{e}^{-\varepsilon})}\frac{3 + 4\cosh(2\mathrm{e}^{-c_3\varepsilon}x_1 - 8\mathrm{e}^{-\varepsilon}t_1) + \cosh(4\mathrm{e}^{-c_3\varepsilon}x_1 - 64\mathrm{e}^{-\varepsilon}t_1)}{[3\cosh(\mathrm{e}^{-c_3\varepsilon}x_1 - 28\mathrm{e}^{-\varepsilon}t_1) + \cosh(3\mathrm{e}^{-c_3\varepsilon}x_1 - 36\mathrm{e}^{-\varepsilon}t_1)]^2}, \tag{39}$$

$$u_1(x_1,t_1) = -12\frac{3 + 4\cosh(2\mathrm{e}^{-c_3\varepsilon}x_1 - 8\sqrt{t_1^2-2\varepsilon},) + \cosh(4\mathrm{e}^{-c_3\varepsilon}x_1 - 64\sqrt{t_1^2-2\varepsilon})}{[3\cosh(\mathrm{e}^{-c_3\varepsilon}x_1 - 28\sqrt{t_1^2-2\varepsilon}) + \cosh(3\mathrm{e}^{-c_3\varepsilon}x_1 - 36\sqrt{t_1^2-2\varepsilon})]^2} + t_1 - \sqrt{t_1^2-2\varepsilon}, \tag{40}$$

$$u_1(x_1,t_1) = (\mathrm{e}^{\varepsilon}-1)\sin t_1 - 12\mathrm{e}^{\varepsilon}\frac{3 + 4\cosh(2\mathrm{e}^{-c_3\varepsilon}x_1 - 8t_1) + \cosh(4\mathrm{e}^{-c_3\varepsilon}x_1 - 64t_1)}{[3\cosh(\mathrm{e}^{-c_3\varepsilon}x_1 - 28t_1) + \cosh(3\mathrm{e}^{-c_3\varepsilon}x_1 - 36t_1)]^2}. \tag{41}$$

Taking $F = s$ and $G = -\alpha uu_x - \beta u^2u_x - \mu u_{xx}$, where α, β, μ, and s are real constants, the original equation is a standard compound KdV–Burgers (cKdVB) equation

$$u_t - su_{xxx} + \alpha uu_x + \beta u^2u_x + \mu u_{xx} = 0, \quad (\beta s > 0), \tag{42}$$

which has some applications in quantum field theory, plasma physics, and solid-state physics.[22–27] The solutions of Eq. (42) has been studied by many authors.[28–32] For example, according to Ref. [28], the solitary-wave solution to the cKdVB equation can be written as

$$u(x,t) = \pm\sqrt{6s/\beta}(k\tanh k(x-vt-x_0) + \mu/6s) - \alpha/2\beta, \tag{43}$$

where $v = \mu^2/6s + 2sk^2 - \alpha^2/4\beta$, k and x_0 are real constants.

Then equations (22)–(25) can be expressed as follows respectively:

$$\begin{aligned} &u_{1t_1} - s\mathrm{e}^{3c_3\varepsilon}u_{1x_1x_1x_1} + \left(u_1 + \frac{x_1(\mathrm{e}^{(1-c_3)\varepsilon}-1)}{c_3-1}\right)\left(\mathrm{e}^{(c_3-1)\varepsilon}u_{1x_1} + \frac{1-\mathrm{e}^{(c_3-1)\varepsilon}}{c_3-1}\right) \\ &\quad\times\left(\alpha + \beta\frac{x_1(\mathrm{e}^{-c_3\varepsilon}-\mathrm{e}^{-\varepsilon})}{c_3-1} + \beta\mathrm{e}^{-\varepsilon}u_1\right) + \mu\mathrm{e}^{2c_3\varepsilon}u_{1x_1x_1} = 0, \end{aligned} \tag{44}$$

$$\begin{aligned} &u_{1t_1} - s\mathrm{e}^{(3c_3-1)\varepsilon}u_{1x_1x_1x_1} - \varepsilon\mathrm{e}^{t_1} - \{(u_1-\varepsilon\mathrm{e}^{t_1})[1-\mathrm{e}^{-\varepsilon} - \mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1+(c_3-1)\varepsilon}u_{1x_1}(\alpha + \beta(u_1-\varepsilon\mathrm{e}^{t_1})\mathrm{e}^{(\mathrm{e}^{-\varepsilon}-1)t_1})] \\ &\quad - \mu\mathrm{e}^{(2c_3-1)\varepsilon}u_{1x_1x_1}\} = 0, \end{aligned} \tag{45}$$

$$\begin{aligned} &u_{1t_1} - s\mathrm{e}^{3c_3\varepsilon}t_1u_{1x_1x_1x_1}/\sqrt{t_1^2-2\varepsilon} - 1 + t_1/\sqrt{t_1^2-2\varepsilon} + t_1/\sqrt{t_1^2-2\varepsilon}\Big\{\mathrm{e}^{c_3\varepsilon}\Big(u_1 - t_1 + \sqrt{t_1^2-2\varepsilon}\Big) \\ &\quad\times u_{1x_1}\Big[\alpha + \beta\Big(u_1 - t_1 + \sqrt{t_1^2-2\varepsilon}\Big)\Big] + \mu\mathrm{e}^{2c_3\varepsilon}u_{1x_1x_1}\Big\} = 0, \end{aligned} \tag{46}$$

and

$$\begin{aligned} &u_{1t_1} - s\mathrm{e}^{3c_3\varepsilon}u_{1x_1x_1x_1} + (1-\mathrm{e}^{\varepsilon})\cos t_1 + \mathrm{e}^{c_3\varepsilon}u_{1x_1}[u_1\mathrm{e}^{-\varepsilon} + (\mathrm{e}^{-\varepsilon}-1)\sin t_1] \\ &\quad\times[\alpha + \beta(u_1+\sin t_1)\mathrm{e}^{-\varepsilon} - \beta\sin t_1] + \mu\mathrm{e}^{2c_3\varepsilon}u_{1x_1x_1} = 0. \end{aligned} \tag{47}$$

Their solutions can be written as

$$u(x_1,t_1)=(1-\mathrm{e}^{(1-c_3)\varepsilon})\frac{x_1}{c_3-1}+\mathrm{e}^{\varepsilon}\{\pm\sqrt{6s/\beta}(k\tanh k(\mathrm{e}^{-c_3\varepsilon}x_1-vt_1-x_0)+\mu/6s)-\alpha/2\beta\}, \tag{48}$$

$$u(x_1,t_1)=\varepsilon\mathrm{e}^{t_1}+\mathrm{e}^{t_1(1-\mathrm{e}^{-\varepsilon})}\{\pm\sqrt{6s/\beta}(k\tanh k(\mathrm{e}^{-c_3\varepsilon}x_1-\mathrm{e}^{-\varepsilon}vt_1-x_0)+\mu/6s)-\alpha/2\beta\}, \tag{49}$$

$$u(x_1,t_1)=\pm\sqrt{6s/\beta}(k\tanh k(\mathrm{e}^{-c_3\varepsilon}x_1-v\sqrt{t_1^2-2\varepsilon}-x_0)+\mu/6s)-\alpha/2\beta+t_1-\sqrt{t_1^2-2\varepsilon}, \tag{50}$$

and

$$u(x_1,t_1)=\sin t_1(\mathrm{e}^{\varepsilon}-1)+\mathrm{e}^{\varepsilon}\{\pm\sqrt{6s/\beta}(k\tanh k(\mathrm{e}^{-c_3\varepsilon}x_1-vt_1-x_0)+\mu/6s)-\alpha/2\beta\}. \tag{51}$$

Because the solutions of Eqs. (31)–(34) and (44)–(47) are known, we can consider each of them as an original model. For example, taking $F=-\mathrm{e}^{3c_3\varepsilon}t/\sqrt{t^2-2\varepsilon}$ and $G=1-t/\sqrt{t^2-2\varepsilon}[1-6\mathrm{e}^{c_3\varepsilon}(u-t+\sqrt{t^2-2\varepsilon})u_x]$ in the variable-coefficient equation (25), it can be expressed as

$$u_{1t_1}+\mathrm{e}^{6c_3\varepsilon}t_1/\sqrt{t_1^2-2\varepsilon}u_{1x_1x_1x_1}+(1-\mathrm{e}^{\varepsilon})\cos t_1-\mathrm{e}^{\varepsilon}\Big\{1-t_1/\sqrt{t_1^2-2\varepsilon}\Big[1-6\mathrm{e}^{(2c_3-1)\varepsilon}\Big((u_1+\sin t_1)\mathrm{e}^{-\varepsilon}-\sin t_1-t_1+\sqrt{t_1^2-2\varepsilon}\Big)u_{1x_1}\Big]\Big\}=0. \tag{52}$$

The solution of the new resulting variable-coefficient equation can be constructed by the solution of the following original model,

$$u_t+\mathrm{e}^{3c_3\varepsilon}t/\sqrt{t^2-2\varepsilon}u_{xxx}-1+t/\sqrt{t^2-2\varepsilon}\Big[1-6\mathrm{e}^{c_3\varepsilon}\Big(u-t+\sqrt{t^2-2\varepsilon}\Big)u_x\Big]=0, \tag{53}$$

which is the same as Eq. (33) whose solution has been known.

With the help of the transformation (29), the solution of Eq. (52) reads

$$u_1(x_1,t_1)=-\sin t_1+\mathrm{e}^{\varepsilon}\Big[u\Big(x=\mathrm{e}^{-2c_3\varepsilon}x_1,\quad t=\sqrt{t_1^2-2\varepsilon}\Big)+t_1-\sqrt{t_1^2-2\varepsilon}+\sin t_1\Big], \tag{54}$$

where $u(x,t)$ is the solution of Eq. (36).

4. Conclusion

In this paper, we used the extended symmetry method to study the general KdV-type equation. With the help of this method, several new variable-coefficient KdV-type equations which contain arbitrary functions are obtained and their solutions can be constructed by solutions of the original model. In order to better illustrate this problem, we choose some determined forms for arbitrary functions and obtain some concrete space-time-dependent equations (22)–(25). Then considering the original model as constant coefficient equations such as the standard constant coefficient KdV equation and standard compound KdV equation by taking appropriate forms of $F(x,t,u,u_x,u_{xx})$ and $G(x,t,u,u_x,u_{xx})$ whose solutions are well known, we can easily obtain the solutions of these new variable-coefficient equations with the help of the finite transformations (26)–(29). Because the solutions of resulting models are known, we can consider any one of them as an original model by choosing appropriate $F(x,t,u,u_x,u_{xx})$ and $G(x,t,u,u_x,u_{xx})$ to obtain new variable-coefficient KdV-type equations whose solutions can also be obtained with the help of the previous resulting variable-coefficient equations.

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