Quantum fluctuation in excited states of mesoscopic *LC* circuits at finite temperature*

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We discuss quantum fluctuation in excited states (named thermo number states) of mesoscopic *LC* circuits at a finite temperature. By introducing the coherent thermo state into the thermo field dynamics pioneered by Umezawa and using the natural representation of thermo squeezing operator we can concisely derive the fluctuation. The result shows that the noise becomes larger when either temperature or the excitation number increases.

Keywords: quantization of *LC* circuits at finite temperature, coherent thermo state representation, thermo number state, thermo field dynamics

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1. Introduction

Recently much attention has been focused on mesoscopic physics because of the growth of nanometer techniques and the rapid development of microelectronics.[1,2] The miniaturization of integrated circuits tends to the atomic-scale dimension gradually.[3] Thus the quantum effects or quantum fluctuation must be taken into account in mesoscopic circuits, and a correct quantum theory is of necessity to the device physics in integrated circuits of nanoelectronics.[4] It is Louisell[5] who first quantized a single *LC* (*L* stands for the inductance, and *C*, the capacitance) circuit with the Hamiltonian,

$$H_0 = \frac{1}{2L}p^2 + \frac{1}{2C}q^2, \quad p = L\frac{\mathrm{d}q}{\mathrm{d}t}, \tag{1}$$

as a harmonic oscillator by imposing the quantizing condition $[\hat{q}, \hat{p}/\hbar] = \mathrm{i}$, with the electric charge (current) being quantized as a continuous variable— coordinate ('momentum').

$$\begin{aligned} q &= \sqrt{\frac{\hbar\omega C}{2}}\left(a + a^\dagger\right), \\ p &= \mathrm{i}\sqrt{\frac{\hbar\omega L}{2}}\left(a^\dagger - a\right), \\ \left[a, a^\dagger\right] &= 1, \end{aligned} \tag{2}$$

where $\omega = 1/\sqrt{LC}$ is the frequency of the *LC* oscillator. Then the vacuum state of this quantized system is $|0\rangle$, obeying $a|0\rangle = 0$.

As any electric circuit produces Joule heat except superconductors. Thus we should also consider the case at finite temperature $T \neq 0$, in this case the average of any observables should be calculated by ensemble average. In Ref. [6] Fan and Liang have calculated the noise in the ground state of quantized *LC* circuit at $T \neq 0$. The aim of this work is to calculate quantum fluctuation in excited states $|n\rangle = (a^{\dagger n}/\sqrt{n!})|0\rangle$ of this system. By introducing the coherent thermo state into thermo field dynamics (TFD), which was pioneered by Takahashi and Umezawa,[7] and using the natural representation of thermo squeezing operator we can concisely derive the quantum fluctuation in number states at finite temperature (named thermo number states), which concerns about how the noise relies on the excitation number. This paper organized as follows: In Section 2 we briefly review the coherent thermo state representation (CTSR) and its main properties. In Section 3 we express the thermo squeezing operator in CTSR and introduce another kind of squeezing. In Section 4 we calculate the quantum fluctuation by virtue of the CTSR.

2. The coherent thermo state

To begin with, we briefly review the TFD theory which is used to describe the thermal communication between systems and reservoirs in simpler way, i.e., the ensemble average of a operator A is equal to the

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average of operator A in thermal vacuum state, which can convert the calculations of ensemble averages at finite temperature

$$\langle A\rangle = Z^{-1}(\beta)\,\mathrm{Tr}\left(A\,\mathrm{e}^{-\beta H}\right) \tag{3}$$

into the pure state average[8]

$$\langle A\rangle = \langle 0(\beta)|\,A\,|0(\beta)\rangle\,, \tag{4}$$

where $Z(\beta)$ is the partition function, H is the system's Hamiltonian. The worthwhile convenience is at the expense of introducing a fictitious field (or so-called a tilde-conjugate field, denoted as operator $\tilde{a}^\dagger$) in the extending Hilbert space, thus the original optical field state $|n\rangle$ in the Hilbert space H is accompanied by a tilde state $|\tilde{n}\rangle$ in $\tilde{H}$. A similar rule holds for operators: every annihilation operator a acting on H has an image $\tilde{a}$ acting on $\tilde{H}$, the number state $|n\rangle$ is replaced by $|n,\tilde{n}\rangle$. For a harmonic oscillator system, the thermo vacuum state is

$$\begin{aligned}|0(\beta)\rangle &= \mathrm{sech}\,\theta\exp\left[a^\dagger\tilde{a}^\dagger\tanh\theta\right]\left|0,\tilde{0}\right\rangle\\ &= S(\theta)\left|0,\tilde{0}\right\rangle,\end{aligned} \tag{5}$$

where

$$S(\theta)\equiv\exp\left[\theta\left(a^\dagger\tilde{a}^\dagger - a\tilde{a}\right)\right] \tag{6}$$

is the thermo squeezing operator which transforms the zero-temperature vacuum state $\left|0,\tilde{0}\right\rangle$ into the thermo vacuum state $|0(\beta)\rangle$, and $\beta = 1/kT$, k is the Boltzmann constant; θ is related to the Bose distribution by

$$\tanh\theta = \exp\left(-\frac{\hbar\omega}{2kT}\right). \tag{7}$$

By developing TFD to coherent thermo state representation we can express the thermo squeezing operator in a neat way, which brings much convenience for calculating the quantum fluctuation of excited state $S(\theta)\,|n,\tilde{n}\rangle$ at finite temperature. Though the fluctuation in the state $|0(\beta)\rangle$ has been calculated in the literature before, to our knowledge, the quantum fluctuation in thermo excited states has not been discussed yet.

In Ref. [9] we have introduced the coherent thermo state representation

$$|\tau\rangle = \exp\left[-\frac{1}{2}|\tau|^2 + \tau a^\dagger - \tau^*\tilde{a}^\dagger + a^\dagger\tilde{a}^\dagger\right]\left|0,\tilde{0}\right\rangle. \tag{8}$$

$|\tau\rangle$ obeys the eigenvector equations

$$\left(a-\tilde{a}^\dagger\right)|\tau\rangle = \tau\,|\tau\rangle\,,\quad \left(\tilde{a}-a^\dagger\right)|\tau\rangle = \tau^*\,|\tau\rangle\,. \tag{9}$$

Using the technique of integration within an ordered product (IWOP) of operators, and the vacuum projector

$$|00\rangle\langle 00| =: \exp\left(-a^\dagger a - \tilde{a}^\dagger\tilde{a}\right):, \tag{10}$$

we can prove the completeness of $|\tau\rangle$,

$$\int\frac{\mathrm{d}^2\tau}{\pi}\,|\tau\rangle\langle\tau| = 1. \tag{11}$$

While the orthogonality relation is

$$\langle\tau'|\,\tau\rangle = \pi\delta(\tau-\tau')\delta(\tau^*-\tau'^*). \tag{12}$$

Accompanied the charge and current operators we introduce their 'tilde' counterparts

$$\tilde{q} = \sqrt{\frac{\hbar\omega C}{2}}\left(\tilde{a}+\tilde{a}^\dagger\right),\quad \tilde{p} = \mathrm{i}\sqrt{\frac{\hbar\omega L}{2}}\left(\tilde{a}^\dagger-\tilde{a}\right). \tag{13}$$

$|\tau\rangle$ also obeys the eigenvector equations

$$\begin{aligned}&(q-\tilde{q})\,|\tau\rangle = \sqrt{2}\tau_1\,|\tau\rangle\,,\\ &(p+\tilde{p})\,|\tau\rangle = \sqrt{2}\tau_2\,|\tau\rangle\,,\\ &\tau = \tau_1 + \mathrm{i}\tau_2.\end{aligned} \tag{14}$$

Since $|\tau\rangle$ is the common eigenvector of $(q-\tilde{q})$ and $(p+\tilde{p})$, we can also name it the thermo entangled state. Based on $|\tau\rangle$ we can introduce the thermo Wigner operator

$$\Delta_{\mathrm{T}}(\sigma,\gamma) = \int\frac{\mathrm{d}^2\tau}{\pi^3}\,|\sigma-\tau\rangle\langle\sigma+\tau|\,\mathrm{e}^{\tau\gamma^*-\gamma\tau^*}. \tag{15}$$

After setting $\gamma=\alpha+\varepsilon^*$, $\sigma=\alpha-\varepsilon^*$, and performing the integration in Eq. (15) with the IWOP technique we obtain

$$\begin{aligned}\Delta_{\mathrm{T}}(\sigma,\gamma) = \pi^{-2}:\,&\exp\left[-2(a^\dagger-\alpha^*)(a-\alpha)\right.\\ &\left.-2(\tilde{a}^\dagger-\varepsilon^*)(\tilde{a}-\varepsilon)\right]:\,.\end{aligned} \tag{16}$$

$\Delta_{\mathrm{T}}(\sigma,\gamma)$ includes both the real-mode part and the fictitious-mode part of the Wigner operator.

3. Coherent thermo state representation for thermo squeezing operators

By virtue of $|\tau\rangle$ the thermo squeezing operator can be neatly expressed as[9,10]

$$S(\theta) = \int\frac{\mathrm{d}^2\tau}{\pi\mu}\,|\tau/\mu\rangle\langle\tau|\,,\ \mu^2 = \frac{1+\tanh\theta}{1-\tanh\theta}\,. \tag{17}$$

By using the IWOP technique we have

$$\int\frac{\mathrm{d}^2\tau}{\pi\mu}\,|\tau/\mu\rangle\langle\tau|$$

$$
\begin{aligned}
&= \exp\left(a^{\dagger}\tilde{a}^{\dagger}\tanh\theta\right)\exp[\left(a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}+1\right)\ln\operatorname{sech}\theta] \\
&\quad\times\exp\left(-a\tilde{a}\tanh\theta\right) \\
&= \exp\left[\theta\left(a^{\dagger}\tilde{a}^{\dagger}-a\tilde{a}\right)\right], \\
\theta &= \ln\mu, \qquad (18)
\end{aligned}
$$

which leads naturally to

$$
S\left(\theta\right)|\tau\rangle = \frac{1}{\mu}\left|\tau/\mu\right\rangle. \qquad (19)
$$

Thus the thermo vacuum state $|0(\beta)\rangle$ is expressed as

$$
\begin{aligned}
|0(\beta)\rangle &= \int\frac{\mathrm{d}^2\tau}{\pi\mu}\left|\tau/\mu\right\rangle\left\langle\tau\right|0,\tilde{0}\rangle \\
&= \int\frac{\mathrm{d}^2\tau}{\pi\mu}\left|\tau/\mu\right\rangle\,\mathrm{e}^{-\frac{1}{2}|\tau|^2}. \qquad (20)
\end{aligned}
$$

Equation (18) brings convenience for calculating the Wigner function of the thermo number state $S\left(\theta\right)|n,\tilde{n}\rangle$. At this point we mention that thermo squeezing operator is not unique. Writing

$$
|\tau\rangle \equiv |\tau_1,\tau_2\rangle, \qquad (21)
$$

we can introduce another kind of thermo squeezing operator,

$$
S'\left(\phi\right) \equiv \mathrm{e}^{-\phi/2}\int\frac{\mathrm{d}^2\tau}{\pi}\left|\tau\right\rangle\left\langle\tau_1,\,\mathrm{e}^{-\phi}\tau_2\right|. \qquad (22)
$$

Using the IWOP technique we perform this integration and obtain

$$
\begin{aligned}
&S'\left(\phi\right) \\
&= \operatorname{sech}^{1/2}\phi:\exp\left\{-\frac{\tanh\phi}{4}\left[\left(a^{\dagger}+\tilde{a}^{\dagger}\right)^2-\left(a+\tilde{a}\right)^2\right]\right. \\
&\quad\left.-\frac{1}{2}\left(a^{\dagger}+\tilde{a}^{\dagger}\right)\left(a+\tilde{a}\right)\left(\operatorname{sech}\phi-1\right)\right\}:. \qquad (23)
\end{aligned}
$$

With the aid of operator identity

$$
\begin{aligned}
&\exp\left[f\left(a^{\dagger}+\tilde{a}^{\dagger}\right)\left(a+\tilde{a}\right)\right] \\
&=: \exp\left[\frac{1}{2}\left(\mathrm{e}^{2f}-1\right)\left(a^{\dagger}+\tilde{a}^{\dagger}\right)\left(a+\tilde{a}\right)\right]: \qquad (24)
\end{aligned}
$$

we can decompose Eq. (23) as

$$
\begin{aligned}
S'\left(\phi\right) &= \operatorname{sech}^{1/2}\phi\exp\left[-\frac{\tanh\phi}{4}\left(a^{\dagger}+\tilde{a}^{\dagger}\right)^2\right] \\
&\quad\times\exp\left[\frac{1}{2}\left(a^{\dagger}+\tilde{a}^{\dagger}\right)\left(a+\tilde{a}\right)\ln\operatorname{sech}\phi\right] \\
&\quad\times\exp\left[\frac{\tanh\phi}{4}\left(a+\tilde{a}\right)^2\right] \qquad (25)
\end{aligned}
$$

Operating Eq. (25) on $|00\rangle$ yields

$$
\begin{aligned}
&S'\left(\phi\right)|00\rangle \\
&= \operatorname{sech}^{1/2}\phi\exp\left[-\frac{\tanh\phi}{4}\left(a^{\dagger}+\tilde{a}^{\dagger}\right)^2\right]|00\rangle, \qquad (26)
\end{aligned}
$$

which is also a thermo state. Demanding that the number average in this state is equal to the Bose distribution

$$
\left(\mathrm{e}^{\beta\omega\hbar}-1\right)^{-1} = \langle 00|\,S'^{\dagger}\left(\phi\right)a^{\dagger}aS'\left(\phi\right)|00\rangle, \qquad (27)
$$

then we can determine the value of ϕ. However, we still use $S\left(\theta\right)$ for later discussions.

4. Quantum fluctuation of electric charge and current in $S\left|n,\tilde{n}\right\rangle$

The coherent thermo state representation helps lot in calculating various averages. From Eqs. (14) and (18) we see that

$$
\begin{aligned}
&S^{\dagger}\left(q-\tilde{q}\right)S = \frac{1}{\mu}\left(q-\tilde{q}\right), \\
&S^{\dagger}\left(p+\tilde{p}\right)S = \frac{1}{\mu}\left(p+\tilde{p}\right), \\
&\mu = \sqrt{\frac{1+\tanh\theta}{1-\tanh\theta}}. \qquad (28)
\end{aligned}
$$

Thus the fluctuations of $(q-\tilde{q})$ and $(p+\tilde{p})$ in the thermo excited number state $S\left|n,\tilde{n}\right\rangle$ are

$$
\begin{aligned}
&\left[\Delta\left(q-\tilde{q}\right)\right]^2 \\
&= \langle n,\tilde{n}|\,S^{\dagger}\left(q-\tilde{q}\right)^2S\left|n,\tilde{n}\right\rangle \\
&= \frac{1}{\mu^2}\left\langle n,\tilde{n}\right|\left(q-\tilde{q}\right)^2\left|n,\tilde{n}\right\rangle \\
&= \left(2n+1\right)\frac{\hbar\omega C}{2}\frac{1-\,\mathrm{e}^{-\hbar\omega/2kT}}{1+\,\mathrm{e}^{-\hbar\omega/2kT}}, \qquad (29)
\end{aligned}
$$

and

$$
\left[\Delta\left(p+\tilde{p}\right)\right]^2 = \left(2n+1\right)\frac{\hbar\omega L}{2}\frac{1-\,\mathrm{e}^{-\hbar\omega/2kT}}{1+\,\mathrm{e}^{-\hbar\omega/2kT}}. \qquad (30)
$$

Further, under the thermo squeezing transformation

$$
\begin{aligned}
S^{\dagger}aS\left(\theta\right) &= a\cosh\theta+\tilde{a}^{\dagger}\sinh\theta, \\
S^{\dagger}\tilde{a}S\left(\theta\right) &= \tilde{a}\cosh\theta+a^{\dagger}\sinh\theta, \qquad (31)
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left\langle n,\tilde{n}\right|S^{\dagger}q\tilde{q}S\left|n,\tilde{n}\right\rangle \\
&= \frac{\hbar\omega C}{2}\left\langle n,\tilde{n}\right|S^{\dagger}\left(a+a^{\dagger}\right)\left(\tilde{a}+\tilde{a}^{\dagger}\right)S\left|n,\tilde{n}\right\rangle \\
&= \frac{\hbar\omega C}{4}\left\langle n,\tilde{n}\right|\left(\tilde{a}^{\dagger}\tilde{a}+\tilde{a}\tilde{a}^{\dagger}+aa^{\dagger}+a^{\dagger}a\right)\left|n,\tilde{n}\right\rangle\sinh(2\theta) \\
&= \left(2n+1\right)\hbar\omega C\frac{\mathrm{e}^{-\hbar\omega/2kT}}{1-\,\mathrm{e}^{-\hbar\omega/kT}}, \qquad (32)
\end{aligned}
$$

and

$$\begin{aligned}&\langle n,\tilde{n}|\, S^{\dagger}p\tilde{p}S\,|n,\tilde{n}\rangle\\ =&-(2n+1)\,\frac{\hbar\omega L}{2}\frac{\mathrm{e}^{-\hbar\omega/2kT}}{1-\mathrm{e}^{-\hbar\omega/kT}}.\end{aligned}\tag{33}$$

It then follows Eqs. (28)–(29) and (32)–(33) to have:

$$\begin{aligned}&\langle n,\tilde{n}|\, S^{\dagger}q^2S\,|n,\tilde{n}\rangle\\ =&\langle n,\tilde{n}|\, S^{\dagger}\tilde{q}^2S\,|n,\tilde{n}\rangle\\ =&\frac{2n+1}{2}\hbar\omega C\frac{1+\mathrm{e}^{-\omega\hbar/Tk}}{1-\mathrm{e}^{-\omega\hbar/Tk}},\end{aligned}\tag{34}$$

and

$$\langle n,\tilde{n}|\, S^{\dagger}p^2S\,|n,\tilde{n}\rangle=\frac{2n+1}{2}\hbar\omega L\frac{1+\mathrm{e}^{-\omega\hbar/Tk}}{1-\mathrm{e}^{-\omega\hbar/Tk}}.\tag{35}$$

Thus the fluctuation of thermo number state is

$$\begin{aligned}\Delta p\Delta q=&\frac{(2n+1)\,\hbar}{2}\frac{1+\mathrm{e}^{-\omega\hbar/Tk}}{1-\mathrm{e}^{-\omega\hbar/Tk}}\\ =&\frac{(2n+1)\,\hbar}{2}\coth\frac{\beta\omega\hbar}{2}.\end{aligned}\tag{36}$$

Equation (36) gives the fluctuation of thermo number state, and the noise arises with the increase of either n or T. The higher the excited state is, the more noise produces. In particularly, when $|n,\tilde{n}\rangle=\left|0,\tilde{0}\right\rangle$, then noise becomes to $(\hbar/2)\coth(\beta\omega\hbar/2)$.

On the other hand, using Eq. (31) we have

$$\begin{aligned}&\langle n,\tilde{n}|\, S^{\dagger}a^{\dagger}aS\,|n,\tilde{n}\rangle\\ =&\,n\cosh^2\theta+(1+n)\sinh^2\theta,\end{aligned}\tag{37}$$

and

$$\begin{aligned}&\langle n,\tilde{n}|\, S^{\dagger}\left(a^{\dagger}a\right)^2S\,|n,\tilde{n}\rangle\\ =&\,n^2\cosh^4\theta+(1+n)^2\sinh^4\theta\\ &+(2n+1)^2\sinh^2\theta\cosh^2\theta,\end{aligned}\tag{38}$$

then

$$\begin{aligned}\Delta\left(a^{\dagger}a\right)=&\sqrt{2n^2+2n+1}\sinh\theta\cosh\theta\\ =&\frac{\mathrm{e}^{-\omega\hbar/2kT}}{1-\mathrm{e}^{-\omega\hbar/kT}}\sqrt{2n^2+2n+1}.\end{aligned}\tag{39}$$

Since $\left(a^{\dagger}a+1/2\right)\omega\hbar=p^2/2L+q^2/2C$,[11−14] so the whole energy fluctuation of the mesoscopic LC circuit is given by Eq. (39).

In summary, in the context of TFD, by using the neat expression of the thermo Wigner operator in coherent thermo state representation we have derived the quantum fluctuation in the excited state of the mesoscopic LC circuit. The result indicates that the quantum noise becomes larger accompanying the augmentations of T (θ) and n. Our approach could be the easiest method in thermo effect treatment.

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