Upper bound for the time derivative of entropy for a stochastic dynamical system with double singularities driven by non-Gaussian noise*

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A stochastic dynamical system with double singularities driven by non-Gaussian noise is investigated. The Fokker–Plank equation of the system is obtained through the path-integral approach and the method of transformation. Based on the definition of Shannon's information entropy and the Schwartz inequality principle, the upper bound for the time derivative of entropy is calculated both in the absence and in the presence of non-equilibrium constraint. The present calculations can be used to interpret the effects of the system dissipative parameter, the system singularity strength parameter, the noise correlation time and the noise deviation parameter on the upper bound.

Keywords: non-Gaussian noise, stochastic dynamical system with double singularities, information entropy, upper bound for the time derivative of entropy

PACC: 0570L, 0520, 6550

1. Introduction

In recent years, studying the information entropy and related quantities has become important for understanding the noise driven dynamical systems.[1−8] Daems and Nicolis,[1,2] aiming at the dissipative dynamic system driven by random perturbation, established the non-equilibrium thermodynamics theory which was based on the balanced equation of information entropy. Their research showed that the entropy flux and the entropy production depended on the characteristics of the dynamics in phase space, especially on the rate of phase space volume contraction. Bag *et al.*[3−5] studied the entropy flux, the entropy production and the upper bound for the time derivative of entropy for a dissipative dynamic system driven by Gaussian noises. They also analysed the effects of the dissipative constant and the Gaussian noises on the entropy flux, the entropy production and the upper bound for the time derivative of entropy. Xie *et al.*[6,7] investigated the upper bound for the time derivative of entropy and the time dependence of entropy flux and entropy production for a dynamic system driven by colored cross-correlated white noises. Guo *et al.*[8] researched the time dependence of entropy flux and entropy production of a dissipative dynamical system driven by non-Gaussian noise.

In most of the previous studies, the random perturbations were assumed to be Gaussian noises. However, Gaussian noises are not applicable in some practical cases. Many experimental evidences, particularly in sensory and biological systems,[9] indicate that the study of non-Gaussian noise is necessary.[9−15] At present, the research of the evolution for information entropy in various systems driven by non-Gaussian noise has seldom reported. In this paper, we investigate a stochastic dynamical system with double singularities driven by non-Gaussian noise on the basis of the previous researches. Through the path-integral approach and the method of transformation, we obtain the Fokker–Plank equation corresponding to the system in Section 2. According to the definition of Shannon's information entropy and the Schwartz inequality principle, we give the expression of the upper bound for the time derivative of entropy both in the absence and in the presence of non-equilibrium constraint. Furthermore, we discuss the effects of the system dissipative parameter, the system singularity strength parameter, the noise correlation time and the noise deviation parameter on the upper bound in Sections 3 and 4. In Section 5 we give our conclusions to complete this paper.

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2. Model and the corresponding Fokker–Planck equation

We consider a stochastic dynamical system with double singularities driven by non-Gaussian noise, the relevant Langevin equation is as follows:

$$\dot{X} = -\gamma X + |X|^{\beta}\eta(t), \quad (\gamma > 0), \tag{1}$$

where γ is the system dissipative parameter, β is the system singularity strength parameter, and $\eta(t)$ is the non-Gaussian noise. It is called the stochastic dynamical system with double singularities because $X = 0$ is the singularity of both the deterministic force and the stochastic force. The non-Gaussian noise $\eta(t)$ can be generated from the solution of the following Langevin equation:

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = -\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta) + \frac{1}{\tau}\xi(t) \tag{2}$$

and

$$V_q(\eta) = \frac{D}{\tau(q-1)}\ln\left[1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}\right], \tag{3}$$

where q is the noise deviation parameter, and it denotes the departure from the Gaussian noise. In the limit of $q \to 1$, $\eta(t)$ reduces to the Gaussian colored noise with the noise correlation time τ and the noise intensity D. For $q \neq 1$, it is a non-Gaussian noise process. The variable $\xi(t)$ in Eq. (2) is Gaussian white noise, and the statistical property of $\xi(t)$ is

$$\begin{aligned} &\langle\xi(t)\rangle = 0, \\ &\langle\xi(t)\xi(t')\rangle = 2D\delta(t-t'). \end{aligned} \tag{4}$$

In Eq. (4), D is the intensity of the Gaussian white noise $\xi(t)$. According to Ref. [11], the property of non-Gaussian noise $\eta(t)$ is

$$\begin{aligned} &\langle\eta(t)\rangle = 0, \\ &\langle\eta^2(t)\rangle = \begin{cases} \dfrac{2D}{\tau(5-3q)}, & q \in \left(-\infty, \dfrac{5}{3}\right), \\ \infty, & q \in \left(\dfrac{5}{3}, 3\right). \end{cases} \end{aligned} \tag{5}$$

When $|q-1| \ll 1$, by applying the path-integral approach,[9−15] one has

$$\begin{aligned} \frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta) &= \frac{\eta}{\tau}\left[1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}\right]^{-1} \\ &\approx \frac{\eta}{\tau}\left[1 + \frac{\tau}{D}(q-1)\frac{\langle\eta^2\rangle}{2}\right]^{-1} = \frac{\eta}{\tau_1}. \end{aligned} \tag{6}$$

Here

$$\tau_1 = \frac{2(2-q)}{5-3q}\tau. \tag{7}$$

The associated noise intensity is

$$D_1 = \left[\frac{2(2-q)}{5-3q}\right]^2 D. \tag{8}$$

It is clear that Eq. (2) can be reduced to a renormalised Ornstein–Uhlenbeck process with the noise correlation time τ_1 and the noise intensity D_1. So Eq. (2) can be written as

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = -\frac{1}{\tau_1}\eta + \frac{1}{\tau_1}\xi_1(t), \tag{9}$$

where $\xi_1(t)$ in Eq. (9) satisfies

$$\begin{aligned} &\langle\xi_1(t)\rangle = 0, \\ &\langle\xi_1(t)\xi_1(t')\rangle = 2D_1\delta(t-t'). \end{aligned} \tag{10}$$

Let $Y_1 = X$ and $Y_2 = \eta$, then Eqs. (1) and (9) can be rewritten as

$$\dot{Y}_1 = -\gamma Y_1 + |Y_1|^{\beta}Y_2, \tag{11}$$

$$\dot{Y}_2 = -\frac{1}{\tau_1}Y_2 + \frac{1}{\tau_1}\xi_1(t). \tag{12}$$

In this paper we study mainly the nontrivial situation with double singularities, i.e. $\beta < 1$. Through the following transformation:

$$X_1 = \begin{cases} |Y_1|^{1-\beta}, & (Y_1 > 0), \\ -|Y_1|^{1-\beta}, & (Y_1 < 0), \end{cases} \tag{13}$$

$$X_2 = Y_2, \tag{14}$$

we can transform Eqs. (11) and (12) into

$$\dot{X}_1 = -\gamma(1-\beta)X_1 + (1-\beta)X_2, \tag{15}$$

$$\dot{X}_2 = -\frac{1}{\tau_1}X_2 + \frac{1}{\tau_1}\xi_1(t). \tag{16}$$

Applying the functional analysis, the corresponding Fokker–Planck equation of Eqs. (15) and (16) can be written as

$$\begin{aligned} \frac{\partial P(X_1, X_2, t)}{\partial t} &= \gamma(1-\beta)\frac{\partial X_1 P}{\partial X_1} - (1-\beta)X_2\frac{\partial P}{\partial X_1} \\ &\quad + \frac{1}{\tau_1}\frac{\partial X_2 P}{\partial X_2} + \frac{D_1}{\tau_1^2}\frac{\partial^2 P}{\partial X_2^2}. \end{aligned} \tag{17}$$

Where $P(X_1,\ X_2,\ t)$ is the extended phase space probability distribution function based on the transformation of Eqs. (13) and (14).

3. Upper bound for the time derivative of entropy

Now we use the Fokker–Plank equation (17) to give the expression of the upper bound for the time derivative of entropy for the system (1). Making use of the transformation[16]

$$T = aX_1 + X_2, \tag{18}$$

we can rewrite the Fokker–Plank equation (17) as

$$\frac{\partial P(T,t)}{\partial t} = -\frac{\partial FP}{\partial T} + D_{\rm s}\frac{\partial^2 P}{\partial T^2}, \tag{19}$$

where

$$F = -\lambda T, \tag{20}$$

$$D_{\rm s} = \frac{D_1}{\tau_1^2}, \tag{21}$$

$$\lambda T = a\gamma(1-\beta)X_1 - a(1-\beta)X_2 + \frac{1}{\tau_1}X_2, \tag{22}$$

where a and λ are constants to be determined. Substituting Eq. (18) into Eq. (22) and comparing the coefficient of X_1 and X_2, one has

$$a = \frac{1}{\tau_1(1-\beta)} - \gamma, \tag{23}$$

$$\lambda = \gamma(1-\beta). \tag{24}$$

The Fokker–Plank equation (19) can be rearranged into a general form of continuity equation

$$\frac{\partial P(T,t)}{\partial t} = -\nabla_T \cdot J, \tag{25}$$

where ∇_T denotes the phase space divergence and J is

$$J = FP - D_{\rm s}\frac{\partial P}{\partial T}. \tag{26}$$

The definition of Shannon's information entropy with continuous probability distribution is[17]

$$S_{\rm I} = -\int P(T,t)\ln P(T,t){\rm d}T. \tag{27}$$

Combining Eqs. (25), (26) and (27), the time evolution equation for entropy can then be written as

$$\frac{{\rm d}S_{\rm I}}{{\rm d}t} = \int \nabla_T \cdot J \ln P {\rm d}T. \tag{28}$$

Integrating Eq. (28) by parts and using the following boundary conditions:[18]

$$\begin{aligned} &J|_{\rm boundary} = 0, \\ &J\ln P|_{\rm boundary} = 0, \end{aligned} \tag{29}$$

we can have

$$\frac{{\rm d}S_{\rm I}}{{\rm d}t} = -\int \frac{1}{P} J \cdot \nabla_T \ln P {\rm d}T. \tag{30}$$

Applying the Schwartz inequality to Eq. (30), one obtains

$$\frac{{\rm d}S_{\rm I}}{{\rm d}t} \le U_{\rm B}(t), \tag{31}$$

where

$$U_{\rm B}(t) = \left(\int \frac{J^2}{P}{\rm d}T\right)^{\frac{1}{2}} \left(\int \frac{(\nabla_T P)^2}{P}{\rm d}T\right)^{\frac{1}{2}}. \tag{32}$$

In the way similar to the derivation of Ref. [4], through Eqs. (19)–(32) we can finally obtain the upper bound for the time derivative of entropy for the system (1) as

$$U_{\rm B}(t) = \frac{\left[2\lambda^2\alpha^2(t)\sigma(t) + \lambda^2\sigma^2(t) - 4\lambda D_{\rm s}\sigma(t) + 4D_{\rm s}^2\right]^{1/2}}{\sigma(t)}, \tag{33}$$

where

$$\alpha(t) = \alpha(0)\exp(-\lambda t), \tag{34}$$

$$\sigma(t) = \frac{2D_{\rm s}}{\lambda}\left[1 - \exp\left(-2\lambda t\right)\right] + \sigma(0)\exp(-2\lambda t), \tag{35}$$

where $D_{\rm s}$ and λ are given by Eqs. (21) and (24), and the initial conditions $\alpha(0)$ and $\sigma(0)$ are constants. At steady state ($t \to \infty$), Eqs. (34) and (35) reduce to $\alpha(\infty) = 0$, and $\sigma(\infty) = 2D_{\rm s}/\lambda$. The numerator on the right-hand side of Eq. (33) vanishes. Therefore we obtain ${\rm d}S_{\rm I}/{\rm d}t = 0$. That is to say, the upper bound for the time derivative of entropy for the system (1) is zero at the steady state. Now we will use Eq. (33) to analyse the effects of the system dissipative parameter γ, the system singularity strength parameter β, the noise correlation time τ and the noise deviation parameter q on the upper bound $U_{\rm B}(t)$.

By combining the results of Eqs. (7), (8) and (21), we can obtain $D_{\rm s} = D_1/\tau_1^2 = D/\tau^2$. So Eq. (33) shows that the upper bound $U_{\rm B}(t)$ does not depend

on the noise deviation parameter q. Thus, we need to discuss only the effects of the system dissipative parameter γ, the system singularity strength parameter β and the noise correlation time τ on the upper bound $U_{\rm B}(t)$. In Figs. 1–3, we choose $D = 0.5$, $\sigma(0) = 0.2$ and $\alpha(0) = 0.5$(all units are arbitrary).

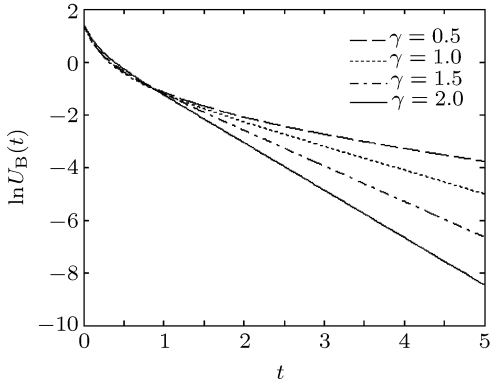


Fig. 1. Upper bound for the time derivative of entropy, $\ln U_{\rm B}(t)$, versus time t at $\beta = 0.1$ and $\tau = 1.0$ for Eq. (33).

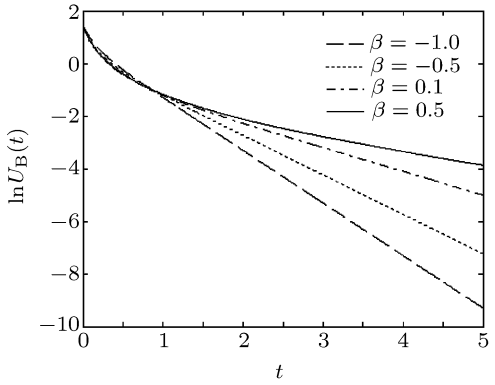


Fig. 2. Upper bound for the time derivative of entropy, $\ln U_{\rm B}(t)$, versus time t at $\gamma = 1.0$ and $\tau = 1.0$ for Eq. (33).

Figures 1, 2 and 3 show the changes of the upper bound for the time derivative of entropy $U_{\rm B}(t)$ with the time t for different values of the system dissipative parameter γ, the system singularity strength parameter β and the noise correlation timeτ, respectively. As shown in Figs. 1, 2 and 3, $U_{\rm B}(t)$ decreases monotonically with the increase of time t. It is clearly seen that the system finally reaches the steady state when time t keeps on increasing. By now $U_{\rm B}(t) = 0(t \to \infty)$, that is, ${\rm d}S_{\rm I}/{\rm d}t = 0$. For larger γ, smaller β or smaller τ, the relaxation time is shorter. For a given time t, when it is larger, the upper bound $U_{\rm B}(t)$ corresponding to the larger γ is smaller, but that corresponding to the larger β or the larger τ is larger.

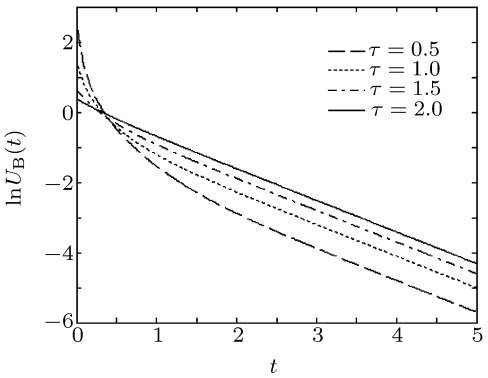


Fig. 3. Upper bound for the time derivative of entropy, $\ln U_{\rm B}(t)$, versus time t at $\gamma = 1.0$ and $\beta = 0.1$ for Eq. (33).

4. Upper bound in the presence of non-equilibrium constraint

We now consider the upper bound for the time derivative of entropy for the system (1) in the presence of non-equilibrium constraint. To this end, we add a constant force $f_{\rm e}$ to Eq. (1), owing to external force, so that the total drift in Eq. (19) now becomes

$$F = F_0 + F_1, \tag{36}$$

where $F_0 = -\lambda T$ and $F_1 = af_{\rm e}$. Also in the way similar to the derivation of Ref. [4], we can obtain the upper bound for the time derivative of entropy for the system (1) in the presence of non-equilibrium constraint as

$$\Delta U_{\rm B}(t) = \frac{\left[2\lambda^2\alpha^2(t)\sigma(t) - 4\lambda F_1\alpha(t)\sigma(t) + 2F_1^2\sigma(t) + \lambda^2\sigma^2(t) - 4\lambda D_{\rm s}\sigma(t) + 4D_{\rm s}^2\right]^{1/2}}{\sigma(t)}, \tag{37}$$

where

$$\alpha(t) = \frac{F_1}{\lambda}\left[1 - \exp\left(-\lambda t\right)\right] + \alpha(0)\exp(-\lambda t), \tag{38}$$

$$\sigma(t) = \frac{2D_{\rm s}}{\lambda}\left[1 - \exp\left(-2\lambda t\right)\right] + \sigma(0)\exp(-2\lambda t), \tag{39}$$

and the meanings of F_1, D_s and λ here are the same as the above. The initial conditions $\alpha(0)$ and $\sigma(0)$ are constants. At steady state ($t \to \infty$), Eqs. (38) and (39) reduce to $\alpha(\infty) = F_1/\lambda$, and $\sigma(\infty) = 2D_s/\lambda$respectively. The numerator on the right-hand side of Eq. (37) also vanishes. Therefore we also obtain $\mathrm{d}S_I/\mathrm{d}t = 0$. That is to say, the upper bound for the time derivative of entropy for the system (1) in the presence of non-equilibrium constraint is also zero in the steady state. Now we use Eq. (37) to analyse the effects of the system dissipative parameter γ, the system singularity strength parameter β, the noise correlation time τ and the noise deviation parameter q on the upper bound $\Delta U_B(t)$ in the presence of non-equilibrium constraint. In Figs. 4–10, we choose the non-equilibrium constraint force $f_e = 1.0$ and $D = 0.5$, $\sigma(0) = 0.2$, $\alpha(0) = 0.5$ (all units are arbitrary).

Figures 4–7 show the changes of the upper bound for the time derivative of entropy $\Delta U_B(t)$ with the time t for different values of the system dissipative parameter γ, the system singularity strength parameter β, the noise correlation time τ and the noise deviation parameter q, respectively. As seen from Figs. 4–7, $\Delta U_B(t)$ also decreases monotonically with the increase of time t. It is also clearly seen that the system finally reaches a steady state when time t keeps on increasing. By now $\Delta U_B(t) = 0 (t \to \infty)$, that is, $\mathrm{d}S_I/\mathrm{d}t = 0$. For larger γ, smaller β or smaller q, the relaxation time is shorter. But for τ, the relaxation time decreases and then increases with the increase of τ. For a given time t, when it is larger, the upper bound $\Delta U_B(t)$ corresponding to the larger γ is smaller, that corresponding to the larger β or the larger q is larger, and that corresponding to the increase of τ is decreasing and then increasing.

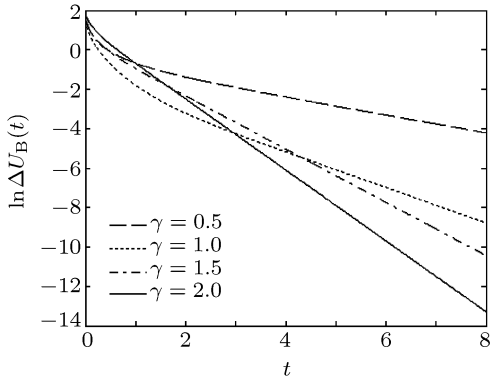


Fig. 4. Upper bound for the time derivative of entropy, $\ln \Delta U_B(t)$, versus time t at $\beta = 0.1$, $\tau = 1.0$ and $q = 0.5$ for Eq. (37).

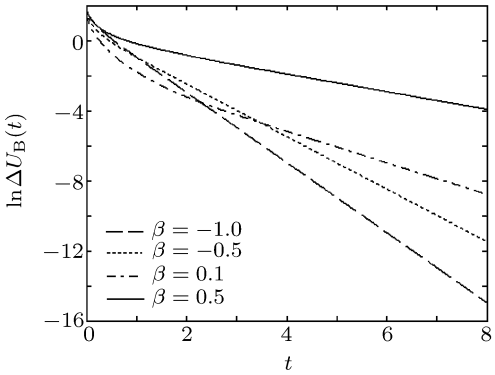


Fig. 5. Upper bound for the time derivative of entropy, $\ln \Delta U_B(t)$, versus time t at $\gamma = 1.0$, $\tau = 1.0$ and $q = 0.5$ for Eq. (37).

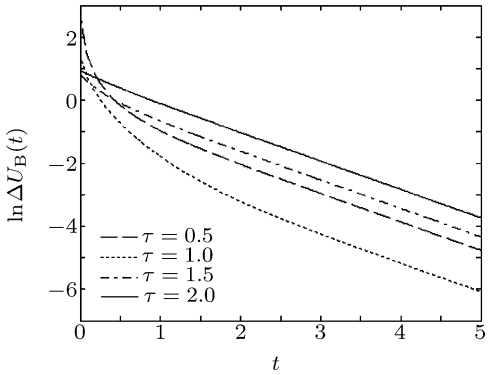


Fig. 6. Upper bound for the time derivative of entropy, $\ln \Delta U_B(t)$, versus time t at $\gamma = 1.0$, $\beta = 0.1$ and $q = 0.5$ for Eq. (37).

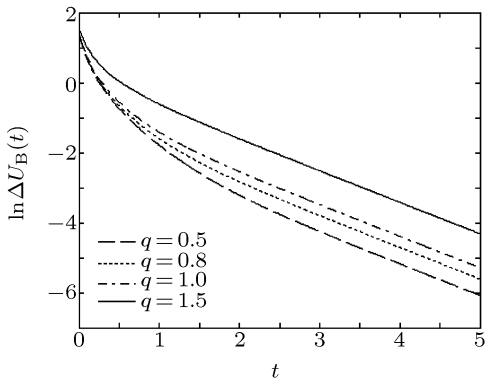


Fig. 7. Upper bound for the time derivative of entropy, $\ln \Delta U_B(t)$, versus time t at $\gamma = 1.0$, $\beta = 0.1$ and $\tau = 1.0$ for Eq. (37).

Figures 8–10 show the changes of the upper bound for the time derivative of entropy $\Delta U_B(t)$ with the system dissipative parameter γ, the system singularity strength parameter β and the noise correlation time τ for different values of the noise deviation parameter q, respectively. In Fig. 8, as the system dissipative parameter γ increases, $\Delta U_B(t)$ decreases down to a minimum and then increases up to a maximum, and finally decreases monotonically. For larger q, the corresponding minimum or maximum is also larger, the

position of the extreme point is closer to the left. In Fig. 9, as the system singularity strength parameter β increases, $\Delta U_{\rm B}(t)$ increases up to a maximum, then decreases down to a minimum, and finally increases monotonically. For larger q, the corresponding maximum or minimum is also larger, but the position of the extreme point is closer to the right. In Fig. 10, as the noise correlation time τ increases, $\Delta U_{\rm B}(t)$ decreases down to a minimum and then increases monotonically. For larger q, the corresponding minimum is also larger, the position of the extreme point is closer to the left.

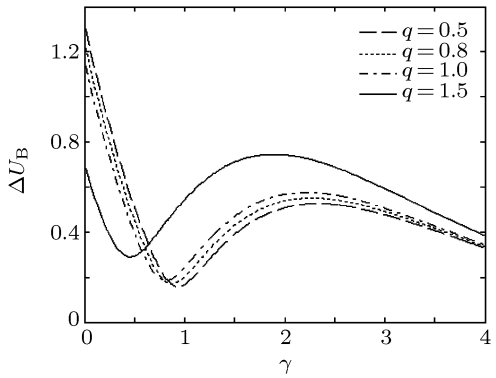


Fig. 8. $\Delta U_{\rm B}$ versus system dissipative parameter γ at $\beta = 0.1$, $\tau = 1.0$ and $t = 1.0$ for Eq. (37).

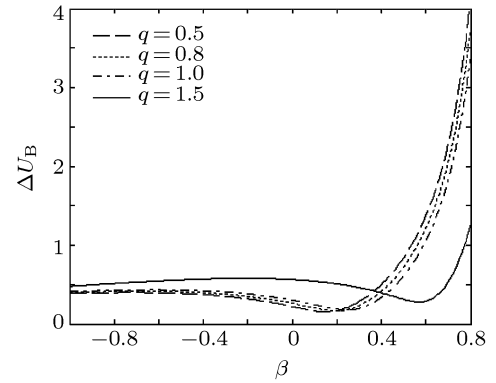


Fig. 9. $\Delta U_{\rm B}$ versus system singularity strength parameter β at $\gamma = 1.0$, $\tau = 1.0$ and $t = 1.0$ for Eq. (37).

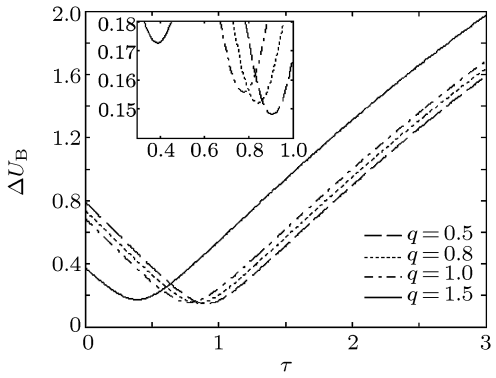


Fig. 10. $\Delta U_{\rm B}$ versus noise correlation time τ at $\gamma = 1.0$, $\beta = 0.1$ and $t = 1.0$ for Eq. (37).

5. Conclusions

In this paper, we study a stochastic dynamical system with double singularities driven by non-Gaussian noise through the path-integral approach and the method of transformation. According to the definition of Shannon's information entropy and the Schwartz inequality principle, we obtain the upper bound for the time derivative of entropy both in the absence and in the presence of non-equilibrium constraint. Moreover, we analyse the effects of the system dissipative parameter, the system singularity strength parameter, the noise correlation time and the noise deviation parameter on the upper bound. In particular, the upper bound for the time derivative of entropy in the presence of non-equilibrium constraint shows extreme properties with the system dissipative parameter, the system singularity strength parameter or the noise correlation time. Since non-Gaussian noise occurs in many circumstances in biological systems which are dissipative and irreversible, we expect that the observations made in this paper can bring new ways of understanding some issues relevant to the dissipative and irreversible systems.

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