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A fuzzy crisis in a Duffing–van der Pol system*

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A crisis in a Duffing–van del Pol system with fuzzy uncertainties is studied by means of the fuzzy generalised cell mapping (FGCM) method. A crisis happens when two fuzzy attractors collide simultaneously with a fuzzy saddle on the basin boundary as the intensity of fuzzy noise reaches a critical point. The two fuzzy attractors merge discontinuously to form one large fuzzy attractor after a crisis. A fuzzy attractor is characterized by its global topology and membership function. A fuzzy saddle with a complicated pattern of several disjoint segments is observed in phase space. It leads to a discontinuous merging crisis of fuzzy attractors. We illustrate this crisis event by considering a fixed point under additive and multiplicative fuzzy noise. Such a crisis is fuzzy noise-induced effects which cannot be seen in deterministic systems.

Keywords: fuzzy dynamical systems, fuzzy noise, fuzzy bifurcation, cell mapping methods
PACC: 0545, 0547

1. Introduction

Noise is ubiquitous in real-life physical systems and can usually be modeled as a random variable or a fuzzy set dependent on the available information about the noise.[1−4] Noise acting on nonlinear dynamical systems can be a source of new phenomena. It may qualitatively change the system behaviour and induce bifurcations. This paper presents a method to analyse the response and bifurcation of nonlinear dynamical systems with fuzzy noise. We are interested in a nonlinear dynamical system whose response is a fuzzy process, and study how the fuzzy response changes as the fuzzy noise intensity varies. Specifically, our attention is focused on the analysis of crisis in fuzzy attractors by fuzzy noise. It should be noted that few works dealing with this problem have been published to date.

In the theory of deterministic dissipative systems, sudden changes in chaotic attractors with parameter variation have been called crisis.[5−8] Three types of crises can be distinguished according to the nature of the discontinuous change that the crisis induces in the chaotic attractor: in the first type a chaotic attractor is suddenly destroyed as the parameter passes through its critical value; in the second type the size of the attractor in phase space suddenly increases; in the third type two or more chaotic attractors merge to form one chaotic attractor.

The bifurcation analysis of noisy (stochastic and fuzzy) nonlinear dynamical systems is in general a difficult subject, partly because even the definition of bifurcation is open to discussion. Take the stochastic system as an example. The commonly accepted definition of the bifurcation is the 'qualitative change' of the system response as a control parameter varies. Meunier and Verga studied the pitchfork bifurcation of a stochastic dynamical system.[9] They examined the quantities such as invariant measures, Lyapunov exponents, correlation functions, and exit times. It turns out that the behaviour of all these quantities near the deterministic bifurcation point changes for different values of the bifurcation parameter, making them a poor indicator of bifurcation in some cases. They proposed an effective potential function of the invariant probability density function of the system response to describe the bifurcation and concluded that corresponding to the bifurcation point of the deterministic system, there is a bifurcation transition region for the stochastic system. Doi, Inoue, and Kumagai[10] have also found that the invariant probability density of the system response is not indicative of bifurcation in some cases, and proposed to examine the qualitative changes of the spectrum of a Markov operator. The spectrum of the Markov operator reveals the bifurcation of the stochastic phase lockings of a van der Pol oscillator that does not show up in the invariant probability density of the system response. Other

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studies phenomenologically define stochastic bifurcation based on the observation of collision of stable attractors with saddle nodes.[11−13]

For fuzzy nonlinear dynamical systems, the subject is even more difficult because the evolution of the membership function of the fuzzy response process is not readily obtained analytically. There is little study in the literature on the bifurcation of fuzzy nonlinear dynamical systems. There are studies of bifurcations of fuzzy control systems where the fuzzy control law leads to a nonlinear and deterministic dynamical system. The bifurcation studies are practically the same as that of deterministic systems.[14,15] The work in Ref. [16] deals with the bifurcation of fuzzy dynamical systems having a fuzzy response. Numerical simulations are used to simulate the system response with a given parameter and fuzzy membership grade. The eigenvalues and the membership distribution are both used to describe the bifurcation. For a given membership grade, the bifurcation of the system is defined in the same manner as that of the deterministic system. The authors have recently proposed a fuzzy generalised cell mapping (FGCM) method for the bifurcation analysis of fuzzy nonlinear dynamical systems and considered several very interesting scenarios of fuzzy bifurcations.[17,18] The current paper studies a sudden change in a fuzzy attractor which is characterized by its global topology and membership function. Such a change is called a fuzzy crisis following Grebogi's definition of crisis in deterministic chaotic systems.[5,6] We shall study a discontinuously merging crisis involving the collision of two fuzzy attractors with a fuzzy saddle on the basin boundary. The origin and evolution of the fuzzy attractor and saddle under fuzzy noise are also investigated. A fuzzy saddle develops from a saddle point into a complicated saddle pattern of several disjoint segments and plays an extremely important role in a fuzzy crisis. It leads to a discontinuously merging crisis of fuzzy attractors. We illustrate this crisis event by considering a fixed point under additive and multiplicative fuzzy noise.

The remainder of the paper is organised as follows. In Section 2, we describe the FGCM method, and discuss its properties. In Section 3, we study a discontinuously merging crisis of two fuzzy attractors in the cases of additive and multiplicative fuzzy noises. The paper concludes in Section 4.

2. The FGCM method

2.1. The FGCM

Consider a dynamical system with a fuzzy parameter

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t, S), \quad \boldsymbol{x} \in \boldsymbol{D}, \tag{1}$$

where $\boldsymbol{x}$ is the state vector, t the time variable, S a fuzzy set with a membership function $\mu_S(s) \in (0, 1]$ where $s \in S$, and $\boldsymbol{f}$ is a vector-valued nonlinear function of its arguments. It is assumed to be periodic in t with period T for all $s \in S$ and to satisfy the Lipschitz condition for all $s \in S$, and $\boldsymbol{D}$ is a bounded domain of interest in the state space. A fuzzy Poincaré map can be obtained from Eq. (1) as

$$\boldsymbol{x}(n+1) = \boldsymbol{G}(\boldsymbol{x}(n), \quad S), \ n = 0, 1, 2, \ldots . \tag{2}$$

The cell mapping method proposes to further discretize the state space in searching for the global solution of the system.[19] In order to apply the cell mapping method, we also need to discretize the fuzzy set S. Suppose that S is a finite interval in R. We divide S into M segments of appropriate length and sample a value $s_k \in S$ $(k = 1, \ldots, M)$ in the middle of each segment. The division of S is such that there is at least one s_k with membership grade equal to one. The domain $\boldsymbol{D}$ is then discretized into N small cells. Each cell is identified by an integer ranging from 1 to N. For a cell, say cell j, N_p points are uniformly sampled from cell j, $M \times N_p$ fuzzy sample trajectories are computed for one period T, or one mapping step. Each trajectory carries a membership grade determined by that of s_k's. We then find the cells in which the end points of the trajectories fall. Assume that cell i is one of the image cells of cell j, and that there are m $(0 < m \leq MN_p)$ trajectories falling in cell i. Define a quantity

$$p_{ij} = \max_{i_k}[\mu_S(s_{i_k})], \quad 0 < p_{ij} \leq 1, \tag{3}$$

where i_k $(k = 1, 2, \ldots, m)$ are referred to the trajectories falling in cell i, and $\mu_S(s_{i_k})$ are the membership grades of the corresponding trajectories. This procedure for computing p_{ij} is known as the sampling point method in the context of generalised cell mapping.[19]

Now, assume that the membership grade of the system being in cell j at the nth mapping step is $p_j(n)$ $(0 < p_j(n) \leq 1)$. Cell j is mapped in one step to cell

i with the membership grade given by

$$\begin{aligned}&\max\{\min\left[\mu_S\left(s_{i_1}\right), p_j\left(n\right)\right],\\&\min\left[\mu_S\left(s_{i_2}\right), p_j\left(n\right)\right], \ldots,\\&\min\left[\mu_S\left(s_{i_m}\right), p_j\left(n\right)\right]\}\\=&\min[\max_{i_k}\left(\mu_S\left(s_{i_k}\right)\right), p_j\left(n\right)]\\=&\min[p_{ij}, p_j\left(n\right)]. \end{aligned} \tag{4}$$

Considering all possible pre-images of cell i, we have the membership grade of the system being in cell i at the $(n+1)$th step as

$$p_i\left(n+1\right) = \max_j \min\left[p_{ij}, p_j\left(n\right)\right]. \tag{5}$$

Let $\boldsymbol{p}\left(n\right)$ be a vector with components $p_i\left(n\right)$, and $\boldsymbol{P}$ a matrix with components p_{ij}. Equation (5) can be written in a compact matrix notation

$$\boldsymbol{p}\left(n+1\right) = \boldsymbol{P} \circ \boldsymbol{p}\left(n\right), \quad \boldsymbol{p}\left(n\right) = \boldsymbol{P}^n \circ \boldsymbol{p}\left(0\right), \tag{6}$$

where $\boldsymbol{P}^{n+1} = \boldsymbol{P} \circ \boldsymbol{P}^n$ and $\boldsymbol{P}^0 = \boldsymbol{I}$. The matrix $\boldsymbol{P}$ is called the one-step transition membership matrix. The vector $\boldsymbol{p}\left(n\right)$ is called the n-step membership distribution vector, and $\boldsymbol{p}\left(0\right)$ the initial membership distribution vector. The (i,j)th element p_{ij} of the matrix $\boldsymbol{P}$ is called the one-step transition membership from cell j to cell i.

Equation (6) is called the FGCM system, which describes the evolution of the fuzzy solution process $\boldsymbol{x}\left(n\right)$ and its membership function, and is a finite approximation to the mapping (2) in $\boldsymbol{D}$.

Consider the master equation for the possibility transition of continuous fuzzy processes,[20−22]

$$p(\boldsymbol{x}, t) = \sup_{\boldsymbol{x}_0 \in \boldsymbol{D}} \left[\min\{p(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0), p(\boldsymbol{x}_0, t_0)\}\right], \quad \boldsymbol{x} \in \boldsymbol{D}, \tag{7}$$

where $\boldsymbol{x}$ is a fuzzy process, $p(\boldsymbol{x}, t)$ is the membership function of $\boldsymbol{x}$, and $p(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0)$ is the transition possibility function, also known as a fuzzy relation.[20] Equation (5) of the FGCM can be viewed as a discrete representation of Eq. (7). Friedman and Sandler have derived a partial differential equation from Eq. (7) for continuous time processes.[21,22] This equation is analogous to the Fokker–Planck–Kolmogorov equation for the probability density function of stochastic processes.[23] The solution to this equation is in general very difficult to obtain analytically. Numerically, the FGCM offers a very effective method for solutions to this equation, particularly, for fuzzy nonlinear dynamical systems.

2.2. Properties of FGCM

(A) Recall that the min–max operation in Eq. (5) really represents the intersection (product) and union (summation) of fuzzy sets in the form of cells in $\boldsymbol{D}$. Hence, $\boldsymbol{P} \circ \boldsymbol{p}$ is an inner product of fuzzy sets. The topological matrix of $\boldsymbol{P}$, denoted by $[\bar{p}_{ij}]$, and the topological vector of $\boldsymbol{p}\left(n\right)$, denoted by $\{\bar{p}_i(n)\}$, are defined as

$$\bar{p}_{ij} = \begin{cases} 1, & p_{ij} > 0 \\ 0, & p_{ij} = 0 \end{cases};$$

$$\bar{p}_i(n) = \begin{cases} 1, & p_i(n) > 0, \\ 0, & p_i(n) = 0, \end{cases} \quad n \geq 0. \tag{8}$$

Topologically, Eq. (6) becomes

$$\bar{\boldsymbol{p}}\left(n+1\right) = \bar{\boldsymbol{P}} \circ \bar{\boldsymbol{p}}\left(n\right), \quad \bar{\boldsymbol{p}}\left(n\right) = \bar{\boldsymbol{P}}^n \circ \bar{\boldsymbol{p}}\left(0\right), \tag{9}$$

where $\bar{\boldsymbol{P}}^{n+1} = \bar{\boldsymbol{P}} \circ \bar{\boldsymbol{P}}^n$ and $\bar{\boldsymbol{P}}^0 = \boldsymbol{I}$. Note that the min–max operation is equivalent to the logic operations of multiplication $\wedge$ and addition $\vee$ of binary numbers: $0 \wedge 1 = 0$, $1 \wedge 0 = 0$, $0 \wedge 0 = 0$, $1 \wedge 1 = 1$, $0 \vee 1 = 1$, $1 \vee 0 = 1$, $0 \vee 0 = 0$, and $1 \vee 1 = 1$. Hence, the min–max operation in Eq. (9) leads to the identical result to that of the topological matrix of the Markov chains. Hence, $\bar{\boldsymbol{P}}$ forms a digraph and can be partitioned to identify persistent groups of cells representing stable solutions and the transient cells including the unstable solutions.[24]

(B) When $\bar{p}_{ij} = 1$, this signifies that cell j is mapped to cell i in one step with a certain possibility. In other words, if the system is found to be in cell i with a possibility, it must be in cell j one step before with another possibility. We introduce a notation $\bar{b}_{ji}$ to represent this backward mapping relationship. When $\bar{b}_{ji} = 1$, cell j is a pre-image of cell i. When $\bar{b}_{ji} = 0$, it is not a pre-image. The matrix defined by these elements such that $m\bar{\boldsymbol{B}} = [\bar{b}_{ij}]$ is called the backward one-step transition topological matrix of the FGCM system. By definition, we have

$$\bar{\boldsymbol{B}} = [\bar{b}_{ij}] = [\bar{p}_{ji}] = \bar{\boldsymbol{P}}^{\mathrm{T}}. \tag{10}$$

Hence, it results in

$$\bar{\boldsymbol{p}}\left(n-1\right) = \bar{\boldsymbol{B}} \circ \bar{\boldsymbol{p}}\left(n\right). \tag{11}$$

Since the matrix $\bar{\boldsymbol{B}}$ describes the dynamics of the original system backward in time topologically, the stable solutions of the original system appear to be unstable and the unstable solutions appear to be stable in Eq. (11). $\bar{\boldsymbol{B}}$ also forms a digraph and can be used to identify unstable solutions of the system.[24]

(C) Under the min–max operation in Eq. (5), the membership function of the cells is non-increasing in the sense that

$$||\boldsymbol{p}(n+1)|| \leq ||\boldsymbol{p}(n)||, \ \forall n \geq 0, \tag{12}$$

where the norm of the fuzzy membership vector $||\boldsymbol{p}(n)|| = \max_i\{p_i(n)\}$. Furthermore, the min–max operator does not introduce any new numbers that are not the entries of the matrix $\boldsymbol{P}$ or the vector $\boldsymbol{p}(0)$. This implies that $\boldsymbol{p}(n)$ can assume only finite number of possible values as $n \to \infty$. In the steady state, $\boldsymbol{p}(n)$ will either converge to a constant vector, or to a set of vectors that form a periodic group and repeat themselves as the iteration continues. In either case, we consider the system to have converged to the steady state. Because there are only a finite number of possible values for $\boldsymbol{p}(n)$, Eq. (5) will converge in finite number of iterations. This is in sharp contrast to the Markov chains, which theoretically converge to steady state in infinite iterations.

3. A Fuzzy crisis in a Duffing–van der Pol system

3.1. A Duffing–van der Pol system with fuzzy additive noise

A Duffing–van der Pol (DVP) oscillator expressed as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \mu_1 x_1 + \mu_2 x_2 - x_1^3 - x_1^2 x_2 \end{aligned} \tag{13}$$

is one of the most extensively studied systems in nonlinear dynamics. Its local and global bifurcation behaviour has been thoroughly studied.[25,26] The influence of stochastic noise on the Duffing–van der Pol oscillator exhibiting codimension one and two bifurcations has also been studied.[27−29] For the case of fuzzy noise, to our knowledge, no attempt has been made regarding this problem.

In the present work, we choose $\mu_1 = 1.0$, $\mu_2 = 0.2$ located in the upper right quadrant of Fig. 7.3.7 of Ref. [25]. When $\mu_1 = 1.0$, $\mu_2 = 0.2$, the DVP equation has two coexistent fixed point attractors and a saddle point on their basin boundary as shown in Fig. 1.

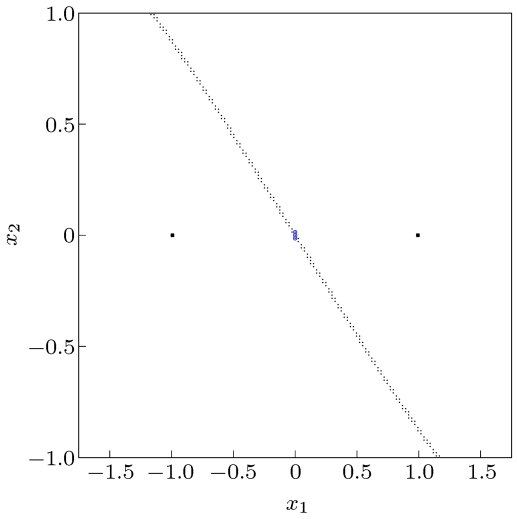


Fig. 1. Global phase portrait of the deterministic Duffing–van der Pol Eq. (13) with $\mu_1 = 1.0$ and $\mu_2 = 0.2$.

We consider the DVP equation driven by additive fuzzy noise,

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= 1.0x_1 + 0.2x_2 - x_1^3 - x_1^2 x_2 + S, \end{aligned} \tag{14}$$

where S is a fuzzy parameter with a triangular membership function,

$$\mu_S(s) = \begin{cases} [s - (s_0 - \varepsilon)]/\varepsilon, & s_0 - \varepsilon \leq s < s_0, \\ -[s - (s_0 + \varepsilon)]/\varepsilon, & s_0 \leq s < s_0 + \varepsilon, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

and $\varepsilon > 0$ is a parameter characterising the intensity of fuzziness of S, and s_0 is the nominal value of S with membership grade $\mu_S(s_0) = 1$.

The domain

$$D = \{-1.75 \leq x_1 \leq 1.75, \ -1.0 \leq x_2 \leq 1.0\}$$

is discretized into 141×141 cells when applying the FGCM method, and 5×5 sampling points are used within each cell. The membership function is discretized into 201 segments ($M = 201$), hence, out of each cell, there are 5025 trajectories with varying membership grades. These trajectories are then used to compute the transition membership matrix.

We fix $s_0 = 0$ and allow the fuzzy noise intensity ε to vary. As ε increases, two coexisting fuzzy fixed point attractors become simultaneously bigger, and a fuzzy saddle changes from a saddle point to a saddle pattern with complicated structure. The global phase portrait is shown in Figs. 2 and 3 when $\varepsilon = 0.2$ and

$\varepsilon = 0.284$, respectively.

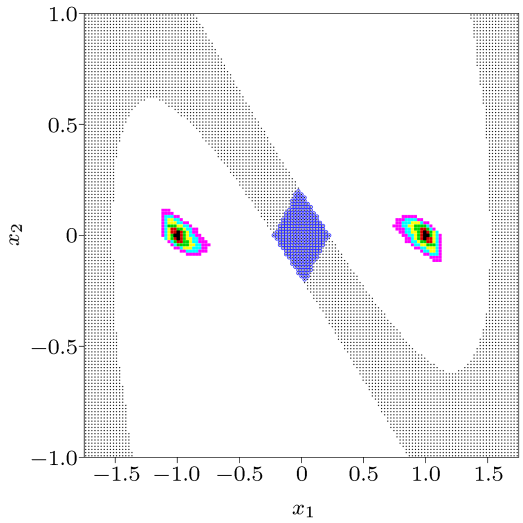


Fig. 2. Global phase portrait of the noisy Duffing–van der Pol Eq. (14) with $\varepsilon = 0.2$. In the figure, the fuzzy attractors are marked by the symbol ■. The membership distribution of fuzzy attractors is colour-coded with black=1.0, 0.8<red<1.0, 0.6<green≤0.8, 0.4<yellow≤0.6, 0.2<cyan≤0.4, and 0.0<purple≤0.2. The saddle is marked by the blue symbol ∘. The basin boundary is marked by the grey symbol '·'.

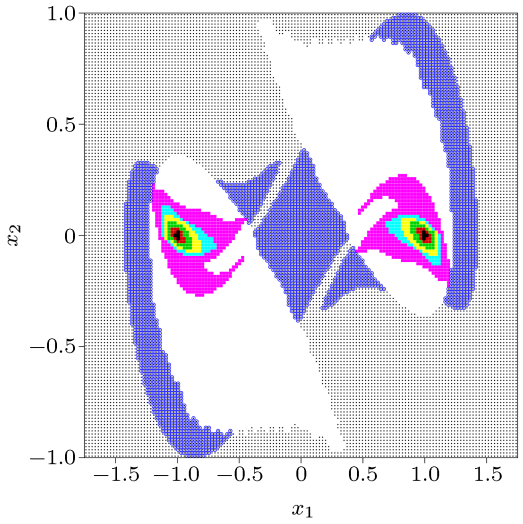


Fig. 3. Global phase portrait of the noisy Duffing–van der Pol Eq. (14) at a crisis value with $s_0 = 0$ and $\varepsilon = 0.284$.

A merging crisis occurs when $\varepsilon = 0.284$ and $\varepsilon = 0.285$. In such a case, the fuzzy attractors collide simultaniously with a fuzzy saddle on the basin boundary, and suddenly merge to form one large fuzzy attractor in the phase space after the crisis. The global phase portraits are shown in Figs. 4 and 5.

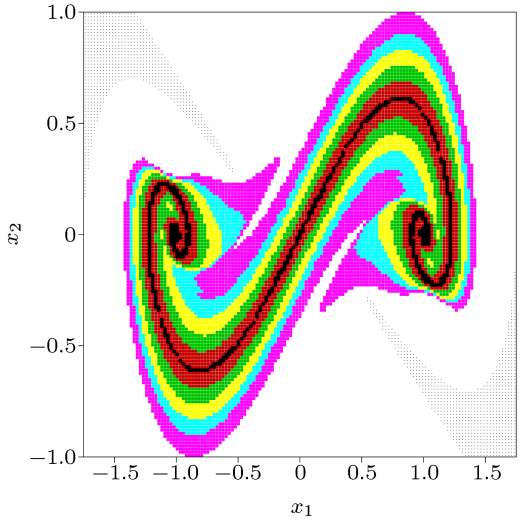


Fig. 4. Large fuzzy attractor of the noisy Duffing–van der Pol Eq. (14) just after a crisis at $\varepsilon = 0.285$.

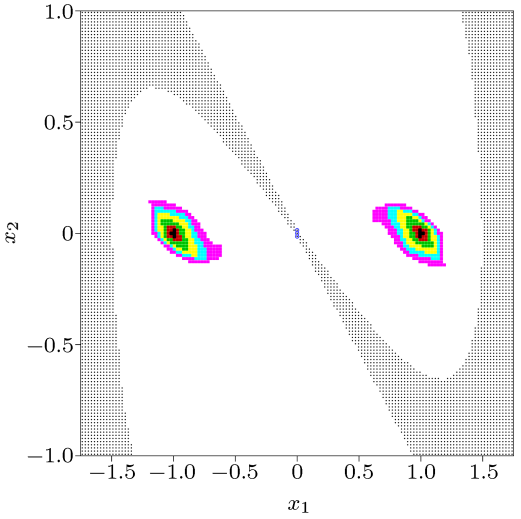


Fig. 5. Global phase portrait of the noisy Duffing–van der Pol Eq. (16) with $\varepsilon = 0.3$.

3.2. A Duffing–van der Pol system with fuzzy multiplicative noise

For the same DVP equation with multiplicative fuzzy noise, we have

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = 1.0x_1 + 0.2x_2 - x_1^3 - x_1^2 x_2 + Sx_1. \qquad (16)$$

The global phase portraits are shown in Figs. 5 and 6. The crisis value is found at $\varepsilon = (0.361, 0.362)$ as shown in Figs. 6 and 7.

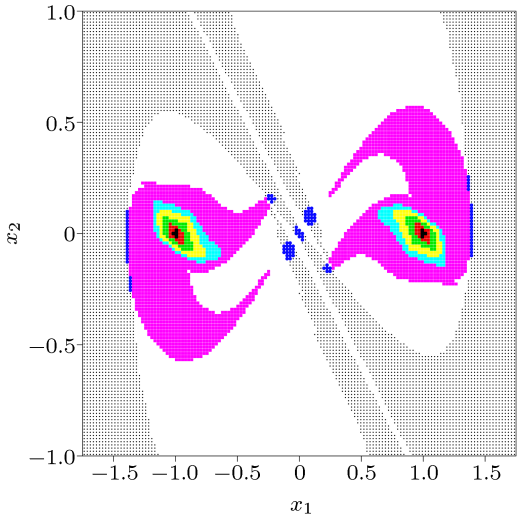


Fig. 6. Global phase portrait of the noisy Duffing–van der Pol Eq. (16) at a crisis value with $\varepsilon = 0.361$.

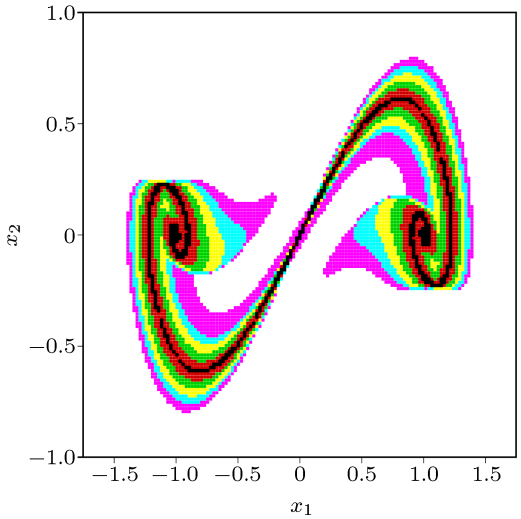


Fig. 7. Large fuzzy attractor of the noisy Duffing–van der Pol Eq. (16) just after a crisis at $\varepsilon = 0.362$.

4. Concluding remarks

In this paper, we have introduced the FGCM method and investigated fuzzy crisis driven by fuzzy noise where a fuzzy saddle in a complicated structure leads two fuzzy attractors discontinuously to merge into a large one. A collision with a fuzzy saddle is the typical mechanism by which two fuzzy attractors can discontinuously merge. These fuzzy crises are difficult to analyse with direct numerical simulations or analytical methods. The FGCM method is at present the only effective tool for the bifurcation analysis of fuzzy nonlinear dynamical systems.

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