# Hopf bifurcation analysis of Chen circuit with direct time delay feedback*

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(Received 2 September 2008; revised manuscript received 30 August 2009)

Direct time delay feedback can make non-chaotic Chen circuit chaotic. The chaotic Chen circuit with direct time delay feedback possesses rich and complex dynamical behaviours. To reach a deep and clear understanding of the dynamics of such circuits described by delay differential equations, Hopf bifurcation in the circuit is analysed using the Hopf bifurcation theory and the central manifold theorem in this paper. Bifurcation points and bifurcation directions are derived in detail, which prove to be consistent with the previous bifurcation diagram. Numerical simulations and experimental results are given to verify the theoretical analysis. Hopf bifurcation analysis can explain and predict the periodical orbit (oscillation) in Chen circuit with direct time delay feedback. Bifurcation boundaries are derived using the Hopf bifurcation analysis, which will be helpful for determining the parameters in the stabilisation of the originally chaotic circuit.

**Keywords:** direct time delay feedback, bifurcation diagram, Hopf bifurcation, bifurcation boundary
**PACC:** 0545, 0547

## 1. Introduction

In 1963, Lorenz found the first chaotic attractor in a simple three dimensional autonomous system.[1] So far, tremendous efforts have been devoted to the understanding of chaos,[2−6] eliminating harmful chaos,[7−17] generating helpful chaos[18−32] and applying chaos to various fields including high performance circuits and devices(e.g. delta-sigma modulators and power inverters,[33]) secure communications,[34] liquid mixing,[35,36] chemical reactions,[37] encryption,[38,39] chaotic road roller and abrasive machine,[40−42] high resolution radar imaging,[43,44] chaotic neural networks[45,46] etc. Among these efforts, understanding the local and the global characteristics of the chaotic dynamical systems are of great importance, because this effort often gives hints for the methods of generating/eliminating chaos, explains how these methods work, and indicates the potential applications.

The local and global characteristics of the typical chaotic systems, such as the Lorenz system, Rössler system, Chua system, have been studied in detail.[2−6] It was very helpful for understanding the essence of chaos. When the great potential applications of chaos were recognized,[18] generating chaos from an originally non-chaotic system became a theoretically attractive and yet technically very challenging task.[6] Generating chaos in the non-chaotic Lorenz system using linear state feedback has led to the discovery of the Chen system, which is topologically not equivalent to the Lorenz system. The analysis of the local and global characteristics of the Chen system[47,48] and its difference from the Lorenz family system have led to the discovery of a new chaotic attractor.[49,50] Following the idea of Chen and Lü, many other new chaotic or hyper-chaotic systems were formulated. Their local and global characteristics were analysed.[51−53] The practical applications of the chaotic dynamics were also investigated intensively.[31−42,54−56]

The direct time delay feedback (DTDF) method is proposed for generating chaos.[29] Some simulation results have been given to illustrate that chaos is indeed generated. Chaos generated by the proposed method possesses theoretically infinite dimensions.[57] But the local bifurcation of such a system remains unclear. In this paper, the Hopf bifurcation theory and the centre manifold theorem are used for the bifurcation analysis of Chen circuit with DTDF. The theoretical analysis results are consistent with the previous bifurcation diagrams, the simulation and experimental results. The bifurcation boundaries are also derived using the Hopf bifurcation analysis and the simulation, which is helpful for selecting parameters to stabilise the originally chaotic state at the fixed point or the periodical orbits.

*Project supported in part by the National Natural Science Foundation of China (Grant No. 60804040) and Fok Ying-Tong Education Foundation for Young Teacher (Grant No. 111065).
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This paper is organized as follows: a brief review of the DTDF method and the chaotic attractor generated by this method in the non-chaotic Chen circuit are given in Section 2. The Hopf bifurcation analysis of Chen circuit with DTDF is given in Section 3. In Section 4, the bifurcation points and the bifurcation directions derived by the theoretical analysis are compared with the bifurcation diagrams. Simulation and experimental results are given to show the correctness of the bifurcation analysis. In Section 5, as a direct application of the foregoing bifurcation analysis, not only the periodic orbits derived by DTDF can be explained, but also the reason why chaos can be stabilised at the equilibrium points or the periodic orbits using DTDF can be explained according to the bifurcation boundaries of the originally chaotic system. In Section 6, some conclusion remarks are given. Some formulations are given in the Appendix, which are useful to determine the bifurcation directions and the stability of the bifurcation.

## 2. Chen circuit with DTDF

Chen circuit can be written as[6]

$$\begin{cases} \dot{x} = a\,(y - x)\,, \\ \dot{y} = (c - a)\,x - xz + cy, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

which is non-chaotic with $a = 35$, $b = 3$, $c = 18.5$. Three equilibriums points are given as

$$\begin{cases} \boldsymbol{O} = (0, 0, 0)\,, \\ \boldsymbol{C}^{-} = \left(-\sqrt{6}, -\sqrt{6}, 2\right), \\ \boldsymbol{C}^{+} = \left(\sqrt{6}, \sqrt{6}, 2\right). \end{cases}$$

Chen circuit with DTDF can be given as[29]

$$\begin{cases} \dot{x} = a\,(y - x)\,, \\ \dot{y} = (c - a)\,x - xz + cy, \\ \dot{z} = xy - bz + k_{33}\left(z\,(t) - z\,(t - \tau_3)\right), \end{cases} \tag{2}$$

where $\tau_3$ is the time delay, $k_{33}$ is the gain of the time delay feedback.

For a non-chaotic Chen circuit with the aforementioned parameters, DTDF can be used to generate chaos. The chaotic attractors in three two-dimensional phase plots, the corresponding time sequence and the power spectra of the state $x$ are given in Figs. 1 and 2. Figure 1 shows the simulation results when the parameter pairs ($k_{33}$, $\tau_3$) are (2.85, 0.3). Figure 2 shows the corresponding experimental results. The largest Lyapunov exponent is 0.1214, which can be calculated from the time sequence by the method proposed in Ref. [58].

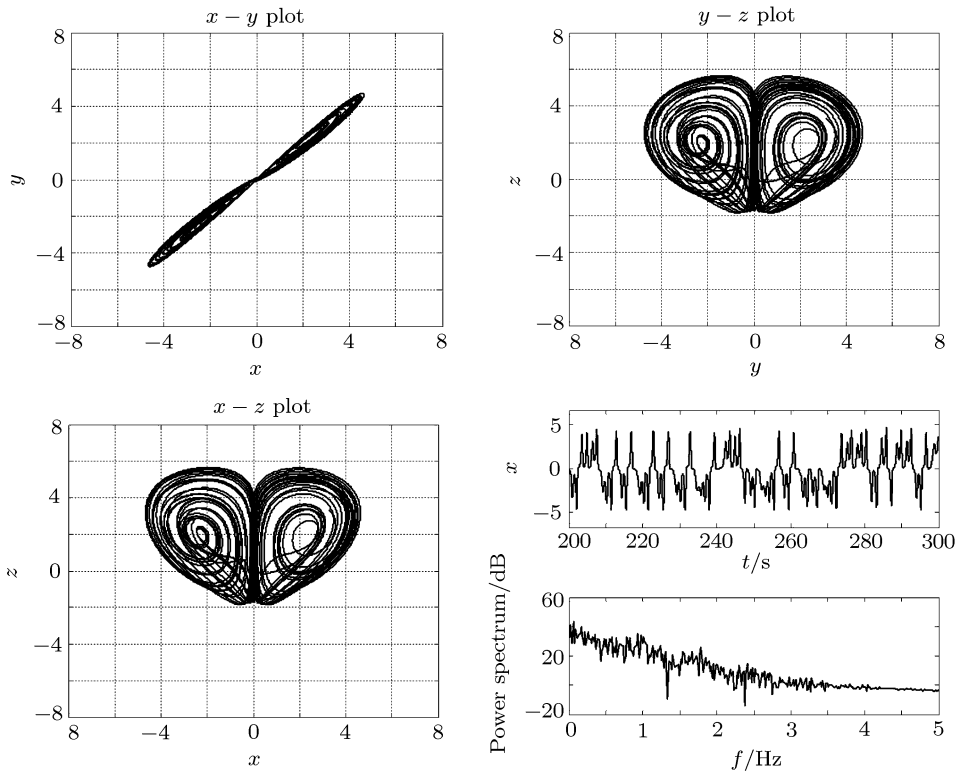



**Fig. 1.** Chaotic attractor in Chen system with DTDF, time sequence, and power spectrum of $x$ at $(k_{33}, \tau_3) = (2.85, 0.3)$.

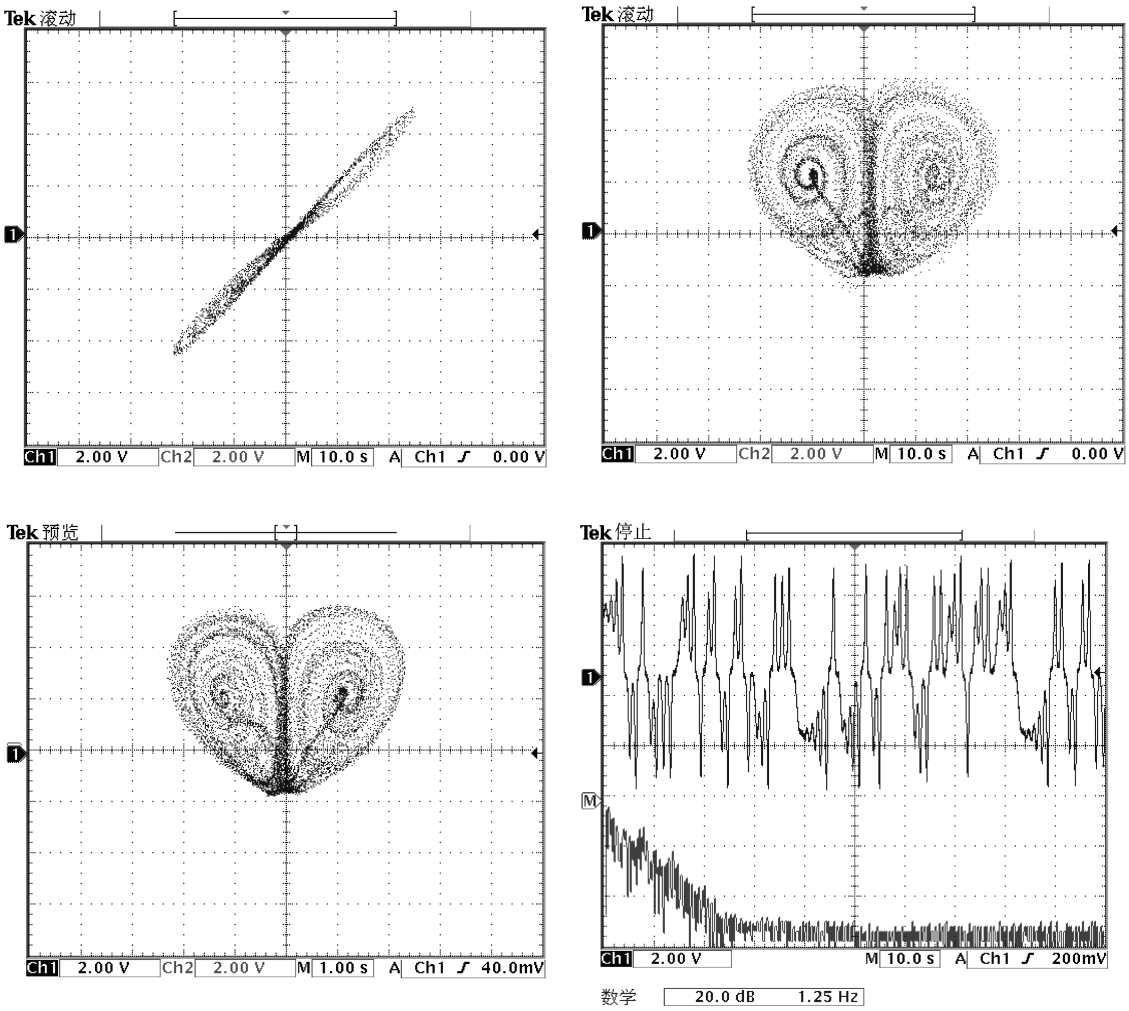



**Fig. 2.** Experimental results corresponding to Fig.1.

Some of the parameter bifurcation diagrams are given in Figs. 3 and 4. One can refer to Ref. [29] for more information about the method. Something to be noticed is that the system undergoes a complex bifurcation procedure that can be seen from the bifurcation diagrams. This is consistent with the conclusion derived for Rössler system in Ref. [59]. The systematic analysis of the Hopf bifurcation will be given in the next section.

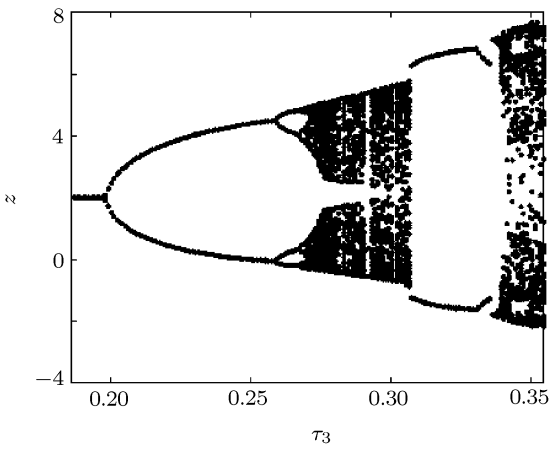



**Fig. 3.** Bifurcation diagram of $\tau_3$ with $k_{33}$ fixed at 2.85.

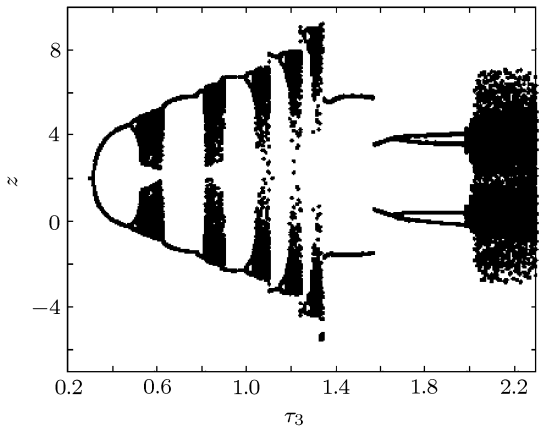



**Fig. 4.** Bifurcation diagram of $\tau_3$ with $k_{33}$ fixed at 1.6.

## 3. Hopf Bifurcation analysis of DTDF-controlled Chen system

From the bifurcation diagrams, we can also find that the equilibrium point $C^+$ will bifurcate at certain parameters, and periodic motion will appear. Local bifurcation analysis will help us understand these behaviours. The local bifurcation will be analysed according to the Hopf bifurcation theory[60,61] in this section.

It is clear that system (2) has the same equilibriums as system (1). Let

$$\begin{cases} x_1 = x - x_0, \\ y_1 = y - y_0, \\ z_1 = z - z_0, \end{cases}$$

where $(x_0, y_0, z_0)$ is the equilibrium point of system (2). The linearisation of system (2) at $C^+$ can be given as

$$\begin{cases} \dot{x}_1(t) = a(y_1(t) - x_1(t)), \\ \dot{y}_1(t) = (c-a)x_1(t) + cy_1(t) - x_0 z_1(t) - z_0 x_1(t), \\ \dot{z}_1(t) = y_0 x_1(t) + x_0 y_1(t) - bz_1(t) + k_{33}[z_1(t) - z_1(t-\tau_3)]. \end{cases} \tag{3}$$

The corresponding characteristic equation is

$$\lambda^3 + (a+b-c-k_{33})\lambda^2 + (k_{33}c + bc - ak_{33})\lambda + 4abc - 2a^2 b + \left[\lambda^2 + (a-c)\lambda\right] k_{33} \mathrm{e}^{-\lambda\tau_3} = 0.$$

We can rewrite the above equation as

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + \left(b_2\lambda^2 + b_1\lambda\right)\mathrm{e}^{-\lambda\tau_3} = 0, \tag{4}$$

where $b_2 = k_{33}$, $b_1 = k_{33}(a-c)$, $a_0 = 4abc - 2a^2 b$, $a_1 = k_{33}c + bc - ak_{33}$, $a_2 = a + b - c - k_{33}$.

According to the Hopf bifurcation theory, we know that the system will undergo a Hopf bifurcation if the corresponding characteristic equation has a pair of purely imaginary roots. Suppose that the imaginary root is $\mathrm{i}\omega$. Substituting $\mathrm{i}\omega$ into Eq. (4), we have

$$-\omega^3 \mathrm{i} - a_2\omega^2 + a_1\omega\mathrm{i} + a_0 + \left(-b_2\omega^2 + b_1\omega\mathrm{i}\right)\cdot(\cos\omega\tau_3 - \mathrm{i}\sin\omega\tau_3) = 0.$$

Separating the real and the imaginary parts, we have

$$\begin{cases} a_2\omega^2 - a_0 = b_1\omega\sin(\omega\tau) - b_2\omega^2\cos(\omega\tau), \\ \omega^3 - a_1\omega = b_2\omega^2\sin(\omega\tau) + b_1\omega\cos(\omega\tau). \end{cases} \tag{5}$$

Taking square of the two sides of the two equations in Eq. (5), and their summation, we can obtain the following equation

$$\omega^6 + \left(a_2^2 - b_2^2 - 2a_1\right)\omega^4 + \left[a_1^2 - 2a_0a_2 - b_1^2\right]\omega^2 + a_0^2 = 0. \tag{6}$$

Let $\omega = \sqrt{u}$, $p = a_2^2 - b_2^2 - 2a_1$, $q = a_1^2 - 2a_0a_2 - b_1^2$, and $r = a_0^2$; Equation (6) becomes

$$u^3 + pu^2 + qu + r = 0. \tag{7}$$

Now, we try to find the conditions under which Eq. (7) has at least one positive root. Let

$$f(u) = u^3 + pu^2 + qu + r. \tag{8}$$

Since $\lim\limits_{u\to\infty} f(u) = \infty$, Eq. (7) has at least one positive root if $r < 0$.

From Eq. (8), we have

$$\mathrm{d}f(u)/\mathrm{d}u = 3u^2 + 2pu + q. \tag{9}$$

Denote $\Delta = p^2 - 3q$. Then $f(u)$ is a monotonically increasing function in $u \in (0, \infty)$ if $\Delta \le 0$. So, if $r \ge 0$ and $\Delta \le 0$, Eq. (7) does not have positive real roots. On the other hand, if $r \ge 0$ and $\Delta > 0$, the equation

$$3u^2 + 2pu + q = 0 \tag{10}$$

has two real roots, i.e.

$$\tilde{u}_1 = \left(-p + \sqrt{\Delta}\right)/3 \quad \text{and} \quad \tilde{u}_2 = \left(-p - \sqrt{\Delta}\right)/3. \tag{11}$$

Since $f''(\tilde{u}_1) = 2\sqrt{\Delta} > 0$ and $f''(\tilde{u}_2) = -2\sqrt{\Delta} < 0$, $\tilde{u}_1$ is the local minimum and $\tilde{u}_2$ is the local maximum. Therefore, it is not difficult to have the following conclusion:

For $r \geq 0$ and $\Delta > 0$, Eq. (7) has positive real roots if and only if $\tilde{u}_1 > 0$ and $f(\tilde{u}_1) \leq 0$. Summarizing the above results, we can obtain the following results:

(i) Equation (7) has at least one positive root if $r < 0$;

(ii) It has no positive root if $r \geq 0$ and $\Delta \leq 0$;

(iii) For $r \geq 0$ and $\Delta > 0$, Eq. (7) has positive roots, if and only if $\tilde{u}_1 > 0$ and $f(\tilde{u}_1) \leq 0$.

Thus we have found the conditions under which Eq. (7) has at least one positive real root. Without loss of generality, we suppose that Eq. (7) has three positive roots: $u_1$, $u_2$ and $u_3$ . So the roots of Eq. (6) are $\omega_1 = \sqrt{u_1}$, $\omega_2 = \sqrt{u_2}$ and $\omega_3 = \sqrt{u_3}$.

From Eq. (5), we have

$$\tau_{3j}^{(k)} = \frac{1}{\omega_j}\left\{\cos^{-1}\left(\frac{b_1\omega_j^2 - a_1b_1 - a_2b_2\omega_j^2 + a_0b_2}{b_2^2\omega_j^2 + b_1^2}\right) + 2k\pi\right\}, \tag{12}$$

where $j = 1, 2, 3$, $k = 0, 1, \ldots$, and $\pm \mathrm{i}\omega_j$ are roots of Eq. (3).

Thus, we have the following conclusion about the bifurcation points of the parameters:

(i) Since $r = a_0^2 = \left(4abc - 2a^2b\right)^2 > 0$, if $\Delta > 0$, $\tilde{u}_1 > 0$ and $f(\tilde{u}_1) \leq 0$, then Eq. (4) has two purely imaginary roots for $\tau_3 = \tau_{3j}^{(k)}$.

(ii) System (2) will undergo Hopf bifurcation at $C^+$ (or $C^-$) for $\tau_3 = \tau_{3j}^{(k)}$ when condition (i) is satisfied and $f'(\tilde{u}_1) \neq 0$, where $\tau_{3j}^{(k)}$ is defined by Eq. (12).

By a similar process, it is easy to obtain the bifurcation condition for the equilibrium point $O$. The procedure is omitted for the limited length.

Following the basic idea of Ref. [60] and the method in Ref. [61], one can draw the conclusion about the bifurcation direction and the stability of the Hopf bifurcation, which are determined by the following parameters:

$$c_1(0) = \frac{\mathrm{i}}{2w_j\tau_j}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \tag{13}$$

$$\mu_2 = -\mathrm{Re}\{c_1(0)\}/\mathrm{Re}\{\lambda'(\tau_j)\}, \tag{14}$$

$$\beta_2 = 2\mathrm{Re}(c_1(0)), \tag{15}$$

$$T_2 = -\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tau_j)\}/\tau_j\omega_j. \tag{16}$$

The notations in Eqs. (13)–(16) can be found in the Appendix. Then the Hopf bifurcation property in the central manifold at the critical time delay value $\tau_i$ can be determined from the above parameters. The parameter $\mu_2$ determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then Hopf bifurcation is supercritical (subcritical); $\beta_2$ determines the stability of the bifurcation periodic solutions: the bifurcation periodic solutions are unstable (stable) if $\beta_2 > 0$ ($\beta_2 < 0$); $T_2$ determines the period of bifurcation periodic solutions.

## 4. Verification of bifurcation analysis

In this section, we will calculate the bifurcation points according to the above discussion and compare the results with the bifurcation diagrams. In addition, simulation results and experimental results of the DTDF-controlled Chen circuit near the bifurcation points are given to show that the bifurcation does occur.

First, we check the bifurcation points in the bifurcation diagrams. For example, in Fig. 3, for $k_{33} = 2.85$, the bifurcation delay time can be calculated to be $\tau_{31}^{(0)} = 0.1985$ and $\tau_{32}^{(0)} = 0.3231$ by using Eq. (12), with $\tau_{31}^{(0)}$ being a stable supercritical bifurcation point, and $\tau_{32}^{(0)}$ an unstable subcritical bifurcation point. In Fig. 4, for $k_{33} = 1.6$, it is easy to derive that $\tau_{31}^{(0)} = 0.3149$ is a stable supercritical bifurcation point. We can derive that $\tau_{31}^{(1)} = 1.55$ is also a stable supercritical bifurcation point. These theoretical results are in good agreement with the bifurcation diagrams.

Secondly, after the equilibrium points undergo stable bifurcations, the system will demonstrate periodic oscillation. Simulations and circuit experiments are performed to verify the oscillation. For example, for $k_{33} = 1.6$, with increasing $\tau_3$, the equilibrium point $C^+$ will bifurcate at $\tau_{31}^{(0)} = 0.3149$, and periodic oscillation will appear after the parameter crosses the bifurcation point, which can be seen from the simulation results in Fig. 5 and the corresponding experimental results in Fig. 6. Behaviours of the system after the bifurcation for other cases (for example, the case in Fig. 3) are easy to obtain by both the simulation and the experiment, which are omitted for the limited length.

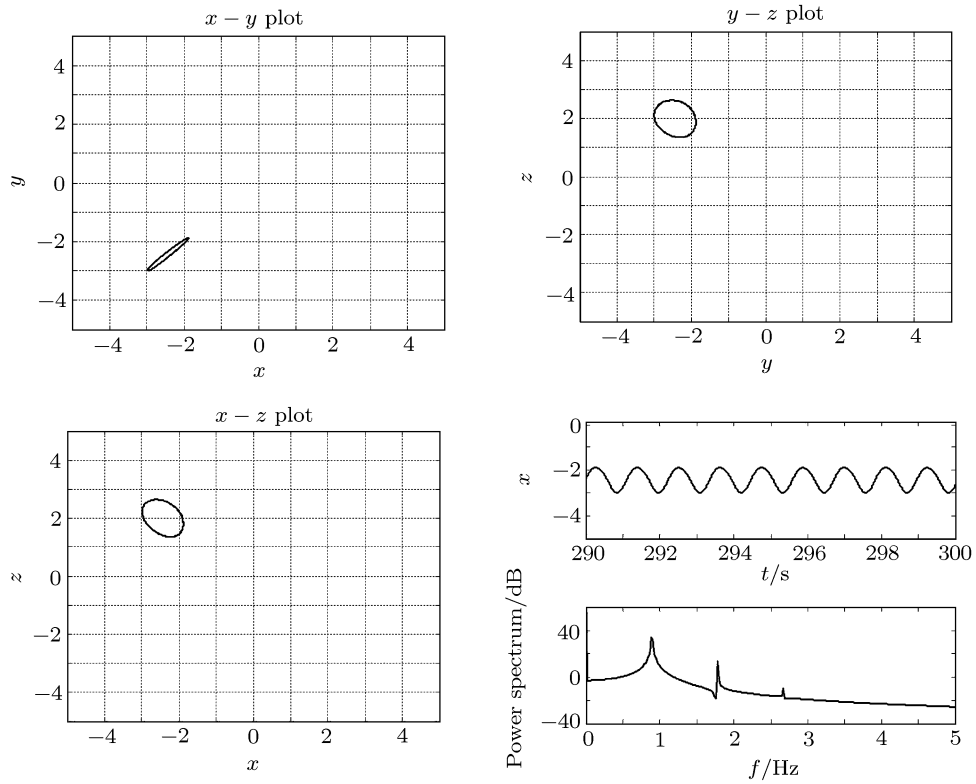



**Fig. 5.** Stable period oscillation after bifurcation occurs with $(k_{33}, \tau_3) = (1.6, 0.32)$.

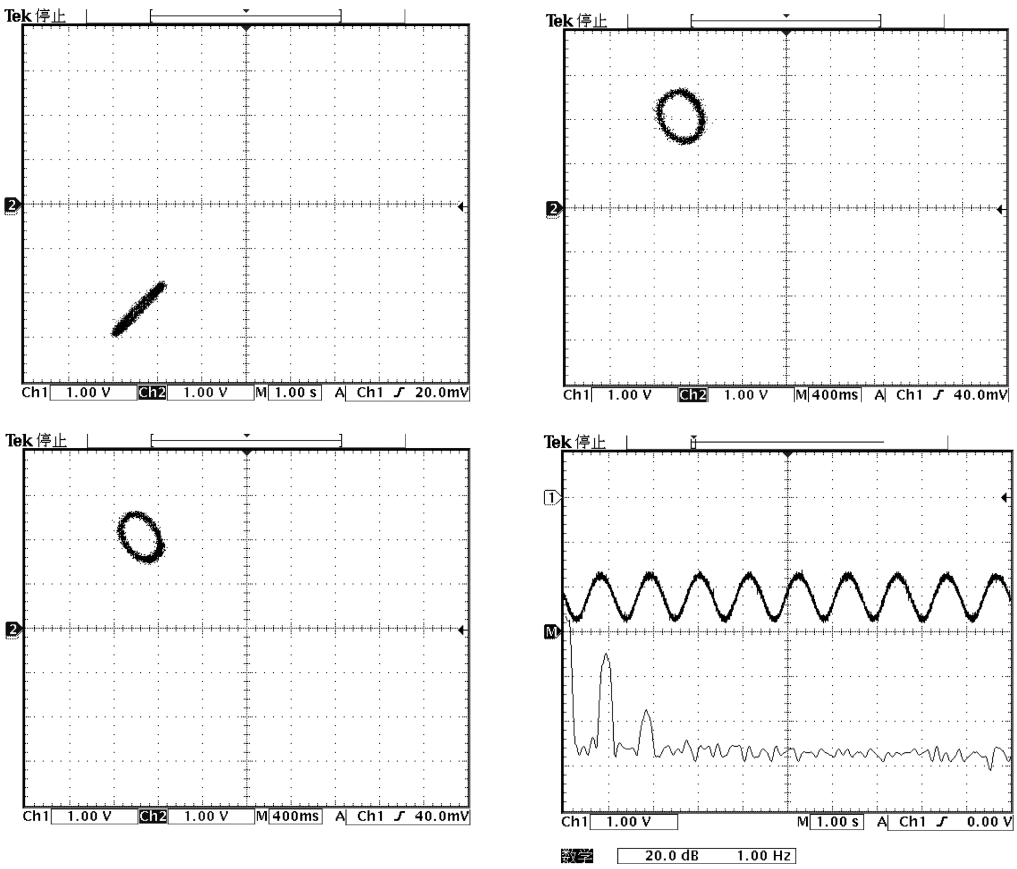



**Fig. 6.** Experimental results corresponding to Fig.5.

## 5. Applications of local bifurcation analysis

In Ref. [29], Ren *et al.* have shown some periodic orbits generated by DTDF, but the origin is unclear. From the bifurcation analysis, we can find and explain what kind of parameters can lead to bifurcation, and which one is the stable bifurcation point. Therefore, some of the periodic orbits in Ref. [29] can be predicted theoretically.

From the Hopf bifurcation analysis, we can also obtain the boundary of bifurcation in an originally chaotic system. This is helpful for selecting parameters to eliminate chaos in the chaotic Chen circuit if the chaos is harmful. The stable area of the control parameters for the chaotic Chen circuit with parameters $a = 35$, $b = 3$, $c = 28$ (these parameters belong to the chaotic region) is given in Fig. 7, from which we can choose the parameters within the marked area so that the chaotic Chen circuit can be stabilised at the equilibriums point. In fact, according to the bifurcation analysis, it is easy to calculate the parameters that drive the chaotic system to a periodic orbit by slightly moving the bifurcation parameters across the bifurcation point and towards the stable direction.

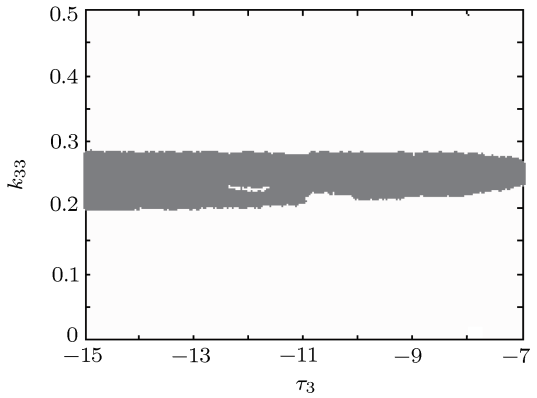



**Fig. 7.** Hopf bifurcation boundary of DTDF-controlled chaotic Chen system.

## 6. Conclusions

Bifurcation analyses of the Chen circuit with DTDF have been conducted in this paper. The bifurcation points and the bifurcation directions are derived by using Hopf bifurcation theory and centre manifold theorem. Theoretical results are consistent with the bifurcation diagrams, the simulation results, and the experimental data. From the results in this paper, the local bifurcation characteristics of the Chen circuits with DTDF are made clear, which can be used to explain the periodical orbits generated by DTDF. The results are also helpful for selecting the DTDF parameters to stabilise the originally chaotic circuit at their equilibrium points or at the low order periodical orbits.

From the viewpoint of understanding the essence of the complex dynamics caused by DTDF (in other words, the complex dynamics in DDE), just the local bifurcation behaviours are not sufficient. The global bifurcation characteristics of such complex dynamics are still open to further research.

## Appendix: Direction and stability of Hopf bifurcation

In this appendix, the bifurcation direction and the stability of the bifurcations are analysed using the central manifold theorem. We assume that system (2) always undergoes Hopf bifurcation at the equilibrium point $(\tilde{x}, \tilde{y}, \tilde{z})$ for $\tau_3 = \tau_j$.

Let $x_1 = x - \tilde{x}$, $y_1 = y - \tilde{y}$, $z_1 = z - \tilde{z}$ and $\tau_3 = \tau_j + \mu$. So system (2) can be transformed into FDE in $\boldsymbol{C} = \boldsymbol{C}\left([-1, 0], R^3\right)$ as

$$\dot{\boldsymbol{x}}_t = \boldsymbol{L}_\mu\left(\boldsymbol{x}_t\right) + \boldsymbol{f}\left(\mu, \boldsymbol{x}_t\right), \tag{A1}$$

where $\boldsymbol{x}_t = \left(x_{1t}, y_{1t}, z_{1t}\right)^{\mathrm{T}} \in R^3$, $\boldsymbol{f} : R \times \boldsymbol{C} \to R$, $\boldsymbol{L}_\mu : \boldsymbol{C} \to R$ are given as

$$\boldsymbol{L}_\mu\left(\phi\right) = \left(\tau_j + \mu\right)\begin{pmatrix} -a & a & 0 \\ c - a - \tilde{z} & c & -\tilde{x} \\ \tilde{x} & \tilde{x} & -b + k_{33} \end{pmatrix}\begin{pmatrix} \phi_1\left(0\right) \\ \phi_2\left(0\right) \\ \phi_3\left(0\right) \end{pmatrix} + \left(\tau_j + \mu\right)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -k_{33} \end{pmatrix}\begin{pmatrix} \phi_1\left(-1\right) \\ \phi_2\left(-1\right) \\ \phi_3\left(-1\right) \end{pmatrix}, \tag{A2}$$

$$\boldsymbol{f}(\tau_3,\phi)=(\tau_j+\mu)\begin{pmatrix}0\\-\phi_1(0)\,\phi_3(0)\\\phi_1(0)\,\phi_2(0)\end{pmatrix}. \tag{A3}$$

Based on the Riesz representation theorem, there is a bounded variation function $\boldsymbol{\eta}(\theta,\mu)$ in $\theta\in[-1,0]$, such that

$$\boldsymbol{L}_\mu(\phi)=\int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\theta,0)\,\phi(\theta),\quad \phi\in C. \tag{A4}$$

And $\boldsymbol{\eta}(\theta,\mu)$ can be chosen as

$$\boldsymbol{\eta}(\theta,\mu)=(\tau_j+\mu)\begin{pmatrix}-a & a & 0\\ c-a-\tilde{z} & c & -\tilde{x}\\ \tilde{x} & \tilde{x} & -b+k_{33}\end{pmatrix}\delta(\theta)-(\tau_j+\mu)\begin{pmatrix}0&0&0\\0&0&0\\0&0&-k_{33}\end{pmatrix}\delta(\theta+1),$$

where $\delta(\cdot)$ is Dirac function.

For $\boldsymbol{\phi}\in C^1\left([-1,0],R^3\right)$, define

$$\boldsymbol{A}(\mu)\,\boldsymbol{\phi}=\begin{cases}\mathrm{d}\boldsymbol{\phi}(\theta)/\mathrm{d}\theta, & \theta\in[-1,0);\\ \int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\mu,s)\,\phi(s), & \theta=0,\end{cases}$$

and

$$\boldsymbol{R}(\mu)\,\phi=\begin{cases}0, & \theta\in[-1,0);\\ \boldsymbol{f}(\mu,\phi), & \theta=0.\end{cases}$$

(A1) can be rewritten as

$$\dot{\boldsymbol{x}}_t=\boldsymbol{A}(\mu)\,\boldsymbol{x}_t+\boldsymbol{R}(\mu)\,\boldsymbol{x}_t, \tag{A5}$$

where $\boldsymbol{x}_t(\theta)=\boldsymbol{x}(t+\theta)$ for $\theta\in[-1,0]$.
For $\boldsymbol{\psi}\in C^1\left([0,1],\left(R^3\right)^*\right)$, define

$$\boldsymbol{A}^*\boldsymbol{\psi}(s)=\begin{cases}-\mathrm{d}\boldsymbol{\psi}(s)/\mathrm{d}s, & s\in(0,1];\\ \int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}^{\mathrm{T}}(t,0)\,\boldsymbol{\psi}(-t), & s=0.\end{cases}$$

Here, we define an inner product by

$$\langle\boldsymbol{\psi},\boldsymbol{\phi}\rangle=\bar{\boldsymbol{\psi}}(0)\,\boldsymbol{\phi}(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\boldsymbol{\psi}}(\xi-\theta)\mathrm{d}\boldsymbol{\eta}(\theta)\,\boldsymbol{\phi}(\xi)\,\mathrm{d}\xi, \tag{A6}$$

where $\boldsymbol{\eta}(\theta)=\boldsymbol{\eta}(\theta,0)$. Then $\boldsymbol{A}(0)$ and $\boldsymbol{A}^*$ are adjoint operators. From the foregoing discussion, we know that $\pm\mathrm{i}\,\omega_j\tau_j$ are eigenvalues of $\boldsymbol{A}(0)$ and thus $\boldsymbol{A}^*$. We need to calculate the eigenvectors of $\boldsymbol{A}(0)$ and $\boldsymbol{A}^*$ corresponding to $\mathrm{i}\,w_j\tau_j$ and $-\mathrm{i}\,w_j\tau_j$, respectively. Suppose that $\boldsymbol{q}(\theta)=(1,\alpha,\beta)^{\mathrm{T}}\,\mathrm{e}^{\mathrm{i}\,\theta w_j\tau_j}$ is the eigenvector of $\boldsymbol{A}(0)$. i.e. $\boldsymbol{A}(0)\,\boldsymbol{q}(\theta)=\mathrm{i}\,\tau_j\omega_j\boldsymbol{q}(\theta)$, then we have

$$\begin{pmatrix}\mathrm{i}\,w_j+a & -a & 0\\ a+\tilde{z}-c & \mathrm{i}\,w_j-c & \tilde{x}\\ -\tilde{y} & -\tilde{x} & \mathrm{i}\,w_j+b-K+Ke^{-i\tau_j w_j}\end{pmatrix}\boldsymbol{q}(0)=\begin{pmatrix}0\\0\\0\end{pmatrix}. \tag{A7}$$

It is easy to obtain

$$\boldsymbol{q}(0)=(1,\alpha,\beta)^{\mathrm{T}}=\left(1,\frac{a+\mathrm{i}\,w_j}{a},\frac{-a\,(a+\tilde{z}-c)-(-c+\mathrm{i}\,w_j)\,(a+\mathrm{i}\,w_j)}{a\tilde{x}}\right)^{\mathrm{T}}.$$

Similarly, we can suppose that $\boldsymbol{q}^*(s)=D\,(1,\alpha^*,\beta^*)\,\mathrm{e}^{\mathrm{i}\,sw_j\tau_j}$ is the eigenvector of $\boldsymbol{A}^*$ corresponding to $-\mathrm{i}\,\tau_j\omega_j$. From the definition of $\boldsymbol{A}^*$, we have

$$\boldsymbol{q}^*(s)=D\left(1,\frac{\tilde{x}\,(a-\mathrm{i}\,w_j)+a\tilde{y}}{-\tilde{x}\,(a+\tilde{z}-c)-\tilde{y}\,(c+\mathrm{i}\,w_j)},\mathrm{tv3}\right)\mathrm{e}^{\mathrm{i}\,sw_j\tau_j}$$

$$\mathrm{tv3}=\frac{a\,[-\tilde{x}\,(a+\tilde{z}-c)-\tilde{y}\,(c+\mathrm{i}\,w_j)]}{\tilde{x}\,[\tilde{x}\,(a+\tilde{z}-c)+\tilde{y}\,(c+\mathrm{i}\,w_j)]}+\frac{(c+\mathrm{i}\,w_j)\,[\tilde{x}\,(a-\mathrm{i}\,w_j)+a\tilde{y}]}{\tilde{x}\,[\tilde{x}\,(a+\tilde{z}-c)+\tilde{y}\,(c+\mathrm{i}\,w_j)]}.$$

Now, we notice that $\langle\boldsymbol{q}^*(s),\boldsymbol{q}(\theta)\rangle=1$, we can obtain

$$\begin{aligned}\langle\boldsymbol{q}^*(s),\boldsymbol{q}(\theta)\rangle&=\bar{D}\left(1,\bar{\alpha}^*,\bar{\beta}^*\right)(1,\alpha,\beta)^{\mathrm{T}}-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{D}\left(1,\bar{\alpha}^*,\bar{\beta}^*\right)\mathrm{e}^{-\mathrm{i}(\xi-\theta)w_j\tau_j}\mathrm{d}\boldsymbol{\eta}(\theta)\,(1,\alpha,\beta)^{\mathrm{T}}\,\mathrm{e}^{\mathrm{i}\,\xi w_j\tau_j}\mathrm{d}\xi\\&=\bar{D}\left\{1+\alpha\bar{\alpha}^*+\beta\bar{\beta}^*-k\tau_j\beta\bar{\beta}^*\mathrm{e}^{-\mathrm{i}\,w_j\tau_j}\right\}.\end{aligned}$$

Thus, we choose

$$D=\frac{1}{\left\{1+\bar{\alpha}\alpha^{*}+\bar{\beta}\beta^{*}-k\tau_{j}\bar{\beta}\beta^{*}\mathrm{e}^{\mathrm{i}\,w_{j}\tau_{j}}\right\}}.$$

In the following, we will compute the coordinates to describe the centre manifold $C_0$ at $\mu=0$. Using the same notations as those in Refs. [60] and [61], we suppose that $x_t$ is the solution of (A1) for $\mu=0$, and define

$$z\left(t\right)=\left\langle \boldsymbol{q}^{*},\boldsymbol{x}_{t}\right\rangle ,\quad \boldsymbol{W}\left(t,\theta\right)=\boldsymbol{x}_{t}\left(\theta\right)-z\left(t\right)\boldsymbol{q}\left(\theta\right)-\bar{z}\left(t\right)\bar{\boldsymbol{q}}\left(\theta\right), \tag{A8}$$

where $z$ and $\bar{z}$ are local coordinates for the centre manifold $\mathrm{C}_0$ in the directions of $\boldsymbol{q}^{*}$ and $\bar{\boldsymbol{q}}^{*}$. On the centre manifold $\boldsymbol{C}_0$, we have

$$\boldsymbol{W}\left(t,\theta\right)=\boldsymbol{W}\left(z\left(t\right),\bar{z}\left(t\right),\theta\right)=\boldsymbol{W}_{20}\left(\theta\right)\frac{z^{2}}{2}+\boldsymbol{W}_{11}\left(\theta\right)z\bar{z}+\boldsymbol{W}_{02}\left(\theta\right)\frac{\bar{z}^{2}}{2}+\boldsymbol{W}_{30}\left(\theta\right)\frac{z^{3}}{6}+\ldots$$

and

$$\begin{aligned}\dot{z}\left(t\right)&=\mathrm{i}\,w_{j}\tau_{j}z+\bar{\boldsymbol{q}}^{*}\left(\theta\right)\boldsymbol{f}\left(0,\boldsymbol{W}\left(z\left(t\right),\bar{z}\left(t\right),\theta\right)+2\mathrm{Re}\left\{z\left(t\right)\boldsymbol{q}\left(\theta\right)\right\}\right)\\&=\mathrm{i}\,w_{j}\tau_{j}z+\bar{\boldsymbol{q}}^{*}\left(0\right)\boldsymbol{f}\left(0,\boldsymbol{W}\left(z\left(t\right),\bar{z}\left(t\right),0\right)+2\mathrm{Re}\left\{z\left(t\right)\boldsymbol{q}\left(0\right)\right\}\right).\end{aligned}$$

Let $\boldsymbol{f}\left(\left(0,\boldsymbol{W}(z\left(t\right),\bar{z}\left(t\right),0)\right)+2\mathrm{Re}\left\{z\left(t\right)\boldsymbol{q}\left(0\right)\right\}=\boldsymbol{f}_{0}\left(z,\bar{z}\right)\right.$, then

$$\dot{z}=\mathrm{i}\,w_{j}\tau_{j}z+\bar{\boldsymbol{q}}^{*}\left(0\right)\boldsymbol{f}_{0}\left(z\left(t\right),\bar{z}\left(t\right)\right). \tag{A9}$$

Let $\bar{\boldsymbol{q}}^{*}\left(0\right)\boldsymbol{f}_{0}\left(z\left(t\right),\bar{z}\left(t\right)\right)=g\left(z,\bar{z}\right)$, then Eq. (A9) becomes

$$\dot{z}\left(t\right)=\mathrm{i}\,w_{j}\tau_{j}z+g\left(z,\bar{z}\right), \tag{A10}$$

where

$$g\left(z,\bar{z}\right)=g_{20}\frac{z^{2}}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^{2}}{2}+g_{21}(\theta)\frac{z^{2}\bar{z}}{2}+\ldots. \tag{A11}$$

Since $\boldsymbol{q}\left(\theta\right)=\left(1,\alpha,\beta\right)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\,\theta w_{j}\tau_{j}}$ and $\boldsymbol{x}_{t}\left(\theta\right)=\left(x_{1t}\left(\theta\right),y_{1t}\left(\theta\right),z_{1t}\left(\theta\right)\right)^{\mathrm{T}}=\boldsymbol{W}\left(t,\theta\right)+z\boldsymbol{q}\left(\theta\right)+\overline{z\boldsymbol{q}}\left(\theta\right)$, we have

$$\begin{aligned}x_{1t}\left(0\right)&=z+\bar{z}+W_{20}^{(1)}\left(0\right)\frac{z^{2}}{2}+W_{11}^{(1)}\left(0\right)z\bar{z}+W_{02}^{(1)}\left(0\right)\frac{\bar{z}^{2}}{2}+O\left(\left|(z,\bar{z})\right|^{3}\right),\\y_{1t}\left(0\right)&=\alpha z+\bar{\alpha}\bar{z}+W_{20}^{(2)}\left(0\right)\frac{z^{2}}{2}+W_{11}^{(2)}\left(0\right)z\bar{z}+W_{02}^{(2)}\left(0\right)\frac{\bar{z}^{2}}{2}+O\left(\left|(z,\bar{z})\right|^{3}\right),\\y_{1t}\left(0\right)&=\alpha z+\bar{\alpha}\bar{z}+W_{20}^{(2)}\left(0\right)\frac{z^{2}}{2}+W_{11}^{(2)}\left(0\right)z\bar{z}+W_{02}^{(2)}\left(0\right)\frac{\bar{z}^{2}}{2}+O\left(\left|(z,\bar{z})\right|^{3}\right).\end{aligned}$$

So, from Eq. (A11) it follows $g\left(z,\bar{z}\right)=\bar{\boldsymbol{q}}^{*}\left(0\right)\boldsymbol{f}_{0}\left(z,\bar{z}\right)=\bar{D}\tau_{j}\left(1,\bar{\alpha}^{*},\bar{\beta}^{*}\right)\begin{pmatrix}0\\-x_{1t}\left(0\right)x_{3t}\left(0\right)\\x_{1t}\left(0\right)x_{2t}\left(0\right)\end{pmatrix}$.

Let

$$\begin{aligned}\mathrm{tb1}&=z+\bar{z}+W_{20}^{(1)}\left(0\right)\frac{z^{2}}{2}+W_{11}^{(1)}\left(0\right)z\bar{z}+W_{02}^{(1)}\left(0\right)\frac{\bar{z}^{2}}{2}+O\left(\left|(z,\bar{z})\right|^{3}\right),\\\mathrm{tb2}&=\beta z+\overline{\beta}\,\bar{z}+W_{20}^{(3)}\left(0\right)\frac{z^{2}}{2}+W_{11}^{(3)}\left(0\right)z\bar{z}+W_{02}^{(3)}\left(0\right)\frac{\bar{z}^{2}}{2}+O\left(\left|(z,\bar{z})\right|^{3}\right),\\\mathrm{tb3}&=\alpha z+\bar{\alpha}\bar{z}+W_{20}^{(2)}(0)\frac{z^{2}}{2}+W_{11}^{(2)}(0)z\bar{z}+W_{02}^{(2)}(0)\frac{\bar{z}^{2}}{2}+O(|(z,\bar{z})|^{3}).\end{aligned}$$

Then $g\left(z,\bar{z}\right)==-\bar{D}\tau_{j}\bar{\alpha}^{*}\mathrm{tb1}\cdot\mathrm{tb2}++\bar{D}\tau_{j}\bar{\beta}^{*}\mathrm{tb1}\cdot\mathrm{tb3}$.

Comparing the coefficients with those of Eq. (A11), we have

$$g_{20}=2\bar{D}\tau_{j}\left(\bar{\beta}^{*}\alpha-\bar{\alpha}^{*}\beta\right), \tag{A12-1}$$

$$g_{11}=2\bar{D}\tau_{j}\left\{\bar{\beta}^{*}\mathrm{Re}\left\{\alpha\right\}-\bar{\alpha}^{*}\mathrm{Re}\left\{\beta\right\}\right\}, \tag{A12-2}$$

$$g_{02}=2\bar{D}\tau_{j}\left(\bar{\beta}^{*}\bar{\alpha}-\bar{\alpha}^{*}\bar{\beta}\right), \tag{A12-3}$$

$$g_{21}=-\bar{D}\tau_{j}\bar{\alpha}^{*}\mathrm{tc1}+\bar{D}\tau_{j}\bar{\beta}^{*}\mathrm{tc2}, \tag{A12-4}$$

where

$$\begin{aligned}\mathrm{tc1} &= 2W_{11}^{(3)}(0) + W_{20}^{(3)}(0) + 2\beta W_{11}^{(1)}(0) + \bar{\beta} W_{20}^{(1)}(0),\\ \mathrm{tc2} &= 2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + 2\alpha W_{11}^{(1)}(0) + \bar{\alpha} W_{20}^{(1)}(0).\end{aligned}$$

From Eqs. (A5) and (A8), we have

$$\dot{\boldsymbol{W}} = \dot{\boldsymbol{x}}_t - \dot{\boldsymbol{z}}\boldsymbol{q} - \dot{\bar{\boldsymbol{z}}}\bar{\boldsymbol{q}} = \begin{cases} \boldsymbol{A}\boldsymbol{W} - 2\mathrm{Re}\{\bar{\boldsymbol{q}}^*(\theta)\boldsymbol{f}_0(z,\bar{z})\boldsymbol{q}(\theta)\}, & \theta \in [-1,0);\\ \boldsymbol{A}\boldsymbol{W} - 2\mathrm{Re}\{\bar{\boldsymbol{q}}^*(0)\boldsymbol{f}_0(z,\bar{z})\boldsymbol{q}(0)\} + \boldsymbol{f}_0, & \theta = 0. \end{cases} \tag{A13}$$

Let

$$\boldsymbol{H}(z,\bar{z},\theta) = \begin{cases} 2\,\mathrm{Re}\{\bar{\boldsymbol{q}}^*(\theta)\boldsymbol{f}_0(z,\bar{z})\boldsymbol{q}(\theta)\}, & \theta \in [-1,0);\\ 2\,\mathrm{Re}\{\bar{\boldsymbol{q}}^*(0)\boldsymbol{f}_0(z,\bar{z})\boldsymbol{q}(0)\} + f_0, & \theta = 0. \end{cases}$$

we can rewrite Eq. (A13) as

$$\dot{\boldsymbol{W}} = \boldsymbol{A}\boldsymbol{W} + \boldsymbol{H}(z,\bar{z},\theta), \tag{A14}$$

where

$$\boldsymbol{H}(z,\bar{z},\theta) = \boldsymbol{H}_{20}\frac{z^2}{2} + \boldsymbol{H}_{11}z\bar{z} + \boldsymbol{H}_{02}\frac{\bar{z}^2}{2} + \ldots. \tag{A15}$$

From Eqs. (A13) and (A15) and the definition of $\boldsymbol{W}$, expanding the series and comparing the coefficients, we have

$$\begin{aligned}&(\boldsymbol{A} - 2\,\mathrm{i}\,w_j\tau_j)\boldsymbol{W}_{20}(\theta) = -\boldsymbol{H}_{20}(\theta),\\ &\boldsymbol{A}\boldsymbol{W}_{11}(\theta)) = -\boldsymbol{H}_{11}(\theta),\\ &\ldots.\end{aligned} \tag{A16}$$

From Eq. (A13), we have

$$\begin{aligned}\boldsymbol{H}(z,\bar{z},\theta) &= -\bar{\boldsymbol{q}}^*(0)\boldsymbol{f}_0\boldsymbol{q}(\theta) - \boldsymbol{q}^*(0)\bar{\boldsymbol{f}}_0\bar{\boldsymbol{q}}(\theta)\\ &= -g\boldsymbol{q}(\theta) - \bar{g}\bar{\boldsymbol{q}}(\theta).\end{aligned} \tag{A17}$$

So we obtain

$$\boldsymbol{H}_{20}(\theta) = -g_{20}\boldsymbol{q}(\theta) - \bar{g}_{02}\bar{\boldsymbol{q}}(\theta), \tag{A18}$$

$$\boldsymbol{H}_{11}(\theta) = -g_{11}\boldsymbol{q}(\theta) - \bar{g}_{11}\bar{\boldsymbol{q}}(\theta). \tag{A19}$$

From Eqs. (A16) and (A18), we have

$$\dot{\boldsymbol{W}}_{20}(\theta) = 2\,\mathrm{i}\,\tau_j w_j \boldsymbol{W}_{20}(\theta) + g_{20}\boldsymbol{q}(\theta) + \bar{g}_{02}\bar{\boldsymbol{q}}(\theta).$$

Substitute $\boldsymbol{q}(\theta) = (1,\alpha,\beta)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\,\theta w_j\tau_j}$ into the last equation, we can obtain the solution of it, which reads

$$\begin{aligned}\boldsymbol{W}_{20}(\theta) = {}& \frac{\mathrm{i}\,g_{20}}{w_j\tau_j}\boldsymbol{q}(0)\mathrm{e}^{\mathrm{i}\,\theta w_j\tau_j} + \frac{\mathrm{i}\,\bar{g}_{02}}{3w_j\tau_j}\bar{\boldsymbol{q}}(0)\mathrm{e}^{-\mathrm{i}\,\theta w_j\tau_j}\\ &+ \boldsymbol{E}_1\mathrm{e}^{2\,\mathrm{i}\,\theta w_j\tau_j};\end{aligned} \tag{A20}$$

and similarly

$$\begin{aligned}\boldsymbol{W}_{11}(\theta) = {}& -\frac{\mathrm{i}\,g_{11}}{w_j\tau_j}\boldsymbol{q}(0)\mathrm{e}^{\mathrm{i}\,\theta w_j\tau_j}\\ &+ \frac{\mathrm{i}\,\bar{g}_{11}}{w_j\tau_j}\bar{\boldsymbol{q}}(0)\mathrm{e}^{-\mathrm{i}\,\theta w_j\tau_j} + \boldsymbol{E}_2,\end{aligned} \tag{A21}$$

where $\boldsymbol{E}_1 = \left(E_1^{(1)}, E_1^{(2)}, E_1^{(3)}\right)^{\mathrm{T}} \in R^3$ and $\boldsymbol{E}_2 = \left(E_2^{(1)}, E_2^{(2)}, E_2^{(3)}\right)^{\mathrm{T}} \in R^3$ are constant vectors corresponding to the initial conditions of the differential equations respectively.

$$\begin{aligned}\dot{\boldsymbol{W}}_{20}(\theta) &= \int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\theta)\boldsymbol{W}_{20}(\theta)\\ &= 2\,\mathrm{i}\,\tau_j w_j \boldsymbol{W}_{20} - \boldsymbol{H}_{20}(0),\end{aligned} \tag{A22}$$

and

$$\dot{\boldsymbol{W}}_{11} = \int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\theta)\boldsymbol{W}_{11}(\theta) = -\boldsymbol{H}_{11}(0), \tag{A23}$$

where $\boldsymbol{\eta}(\theta) = \boldsymbol{\eta}(\theta,0)$. From Eq. (A13), we have

$$\begin{aligned}\boldsymbol{H}_{20}(0) = {}& -g_{20}\boldsymbol{q}(0) - \bar{g}_{02}\bar{\boldsymbol{q}}(0)\\ &+ 2\tau_j\begin{bmatrix}0 & -\beta & \alpha\end{bmatrix}^{\mathrm{T}},\end{aligned} \tag{A24}$$

$$\boldsymbol{H}_{11}(0) = -g_{11}\boldsymbol{q}(0) - \bar{g}_{11}\bar{\boldsymbol{q}}(0) + 2\tau_j\begin{pmatrix}0\\ -\mathrm{Re}\{\beta\}\\ \mathrm{Re}\{\alpha\}\end{pmatrix}. \tag{A25}$$

From the definition of the eigenvector, we know that

$$\begin{aligned}&\left(\mathrm{i}\,w_j\tau_j\boldsymbol{I} - \int_{-1}^{0}\mathrm{e}^{\mathrm{i}\,\theta w_j\tau_j}\mathrm{d}\boldsymbol{\eta}(\theta)\right)\boldsymbol{q}(0) = 0,\\ &\left(-\mathrm{i}\,w_j\tau_j\boldsymbol{I} - \int_{-1}^{0}\mathrm{e}^{-\mathrm{i}\,\theta w_j\tau_j}\mathrm{d}\boldsymbol{\eta}(\theta)\right)\bar{\boldsymbol{q}}(0) = 0.\end{aligned}$$

Keeping the above two equations in mind and substituting Eqs. (A20) and (A24) into Eq. (A22), we obtain

$$\begin{aligned}&\left(2\,\mathrm{i}\,w_j\tau_j\boldsymbol{I} - \int_{-1}^{0}\mathrm{e}^{2\,\mathrm{i}\,\theta w_j\tau_j}\mathrm{d}\boldsymbol{\eta}(\theta)\right)\boldsymbol{E}_1\\ &= 2\tau_j\begin{bmatrix}0 & -\beta & \alpha\end{bmatrix}^{\mathrm{T}}.\end{aligned}$$

Thereby

$$\begin{pmatrix} 2\,\mathrm{i}\,w_j + a & -a & 0 \\ a + \tilde{z} - c & 2\,\mathrm{i}\,w_j - c & \tilde{x} \\ -\tilde{y} & -\tilde{x} & 2\,\mathrm{i}\,w_i + b - K + K\mathrm{e}^{-2\,\mathrm{i}\,w_j\tau_j} \end{pmatrix} \boldsymbol{E}_1 = 2 \begin{pmatrix} 0 \\ -\beta \\ \alpha \end{pmatrix}.$$

So we have

$$E_1^{(1)} = \frac{-2a\left[2\beta\left(2\,\mathrm{i}\,w_j + b - K + K\mathrm{e}^{-2\,\mathrm{i}\,w_j\tau_j}\right) - 2\alpha\tilde{x}\right]}{A_a},$$

$$E_1^{(2)} = \frac{(2\,\mathrm{i}\,w_j + a)\left[-\beta\left(2\,\mathrm{i}\,w_j + b - K + K\mathrm{e}^{-2\,\mathrm{i}\,w_j\tau_j}\right) - \alpha\tilde{x}\right]}{A_a},$$

$$E_1^{(3)} = \frac{2}{A_a}\begin{vmatrix} 2\,\mathrm{i}\,w_j + a & -a & 0 \\ a + \tilde{z} - c & 2\,\mathrm{i}\,w_j - c & -\beta \\ -\tilde{y} & -\tilde{x} & \alpha \end{vmatrix},$$

where

$$A_a = \begin{vmatrix} 2\,\mathrm{i}\,w_j + a & -a & 0 \\ a + \tilde{z} - c & 2\,\mathrm{i}\,w_j - c & \tilde{x} \\ -\tilde{y} & -\tilde{x} & 2\,\mathrm{i}\,w_j + b - K + K\mathrm{e}^{-2\,\mathrm{i}\,w_j\tau_j} \end{vmatrix}.$$

Similarly, substituting Eqs. (A21) and (A25) into Eq. (A23), we have

$$\begin{pmatrix} a & -a & 0 \\ a + \tilde{z} - c & -c & \tilde{x} \\ -\tilde{y} & -\tilde{x} & b \end{pmatrix} \boldsymbol{E}_2 = 2 \begin{pmatrix} 0 \\ -\mathrm{Re}\,\{\beta\} \\ \mathrm{Re}\,\{\alpha\} \end{pmatrix}.$$

So we have

$$E_2^{(1)} = \frac{-2a\tilde{x}\mathrm{Re}\,\{\alpha\} - 2ab\mathrm{Re}\,\{\beta\}}{B},\ E_2^{(2)} = \frac{-2a\tilde{x}\mathrm{Re}\,\{\alpha\} - 2ab\mathrm{Re}\,\{\beta\}}{B},\ E_2^{(3)} = \frac{2}{B}\begin{vmatrix} a & -a & 0 \\ a + \tilde{z} - c & -c & -\mathrm{Re}\,\{\beta\} \\ -\tilde{y} & -\tilde{x} & \mathrm{Re}\,\{\alpha\} \end{vmatrix},$$

where

$$B = \begin{vmatrix} a & -a & 0 \\ a + \tilde{z} - c & -c & \tilde{x} \\ -\tilde{y} & -\tilde{x} & b \end{vmatrix}.$$

Consequently, we can determine $\boldsymbol{W}_{20}(0)$ and $\boldsymbol{W}_{11}(0)$, thus, all $g_{ij}$ can be determined. One can compute Eqs(13)–(16) in Section 3 and determine the property of the bifurcations.

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