Nonlinear feedback control of a novel hyperchaotic system and its circuit implementation*

Wang Hao-Xiang(汪浩祥), Cai Guo-Liang(蔡国梁)†, Miao Sheng(缪 盛), and Tian Li-Xin(田立新)

Nonlinear Scientific Research Center, Jiangsu University, Zhenjiang 212013, *China*

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This paper reports a new hyperchaotic system by adding an additional state variable into a three-dimensional chaotic dynamical system. Some of its basic dynamical properties, such as the hyperchaotic attractor, Lyapunov exponents, bifurcation diagram and the hyperchaotic attractor evolving into periodic, quasi-periodic dynamical behaviours by varying parameter k are studied. An effective nonlinear feedback control method is used to suppress hyperchaos to unstable equilibrium. Furthermore, a circuit is designed to realize this new hyperchaotic system by electronic workbench (EWB). Numerical simulations are presented to show these results.

Keywords: hyperchaos, nonlinear feedback control, Lyapunov exponents, chaos circuit

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1. Introduction

Recently, hyperchaotic dynamical systems have begun to attract considerable attention from many physical scientists. Many systems display hyperchaotic dynamics, such as viscoelastic systems, electromagnetic waves, and atmospheric dynamical systems. Hyperchaos is characterized as a chaotic system with more than one positive exponent, implying that its dynamics are expended in several different directions simultaneously. The hyperchaotic system has more complex dynamical behaviours than the chaotic system and has been studied with increasing interest due to its theoretical and practical applications in technological fields, such as secure communications, lasers, nonlinear circuits, neural networks. Thus, generation, control, synchronization and application of hyperchaos have recently become a central topic in research.[1−9] In recent years, there are various hyperchaotic systems discovered in high-dimensional systems. However, the two most well known hyperchaotic systems are the four-dimensional hyperchaotic Rössler system[10] and the four-dimensional hyperchaotic Chua's circuit.[11] Very recently, hyperchaos was found numerically and experimentally by adding a simple state feedback controller.[12−17]

This paper reports a new hyperchaotic system which is constructed based on a three-dimensional system by introducing a state feedback controller, and stabilization of the hyperchaotic system. The properties and dynamics of the hyperchaotic system are investigated numerically via bifurcation diagram, Lyapunov exponents. A circuit is designed to realize the new hyperchaotic system. By using nonlinear feedback methods, effective controllers are designed for stabilizing the hyperchaotic system to unstable equilibrium. Conclusions are given in the final section.

2. Construction of the new hyperchaotic system

Recently, Cai *et al.* constructed the third-dimensional autonomous chaotic system.[18] The system is described by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = bx + cy - xz, \\ \dot{z} = y^2 - hz, \end{cases} \tag{1}$$

where a, b, c, h are constants and x, y, z are variables. When $a = 27.5$, $b = 3$, $c = 19.3$, $h = 2.9$, the Lyapunov exponents of this chaotic system are 1.9506, 0, −13.0503, and the Lyapunov dimension of the attractor is 2.1495. Apparently, system (1) is a three-dimensional autonomous chaotic system.

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†Corresponding author. E-mail: glcai@ujs.edu.cn

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Based on the system (1), a new hyperchaotic system which is judged by two positive Lyapunov exponents can be generated by adding an additional state variable u. The four-dimensional autonomous differential equations have the form as follows:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = bx + cy - xz + u, \\ \dot{z} = y^2 - hz, \\ \dot{u} = -kx, \end{cases} \tag{2}$$

where k is a parameter to be determined, its value can be varied within a certain range.

We know that the new system (2) is four-dimensional quadratic autonomous system. To create hyperchaos from system (2) should have two positive Lyapunov exponents along with one zero and one negative Lyapunov exponent.

Let $a = 27.5$, $b = 3$, $c = 19.3$, $h = 2.9$ and $k = 3.3$, the new hyperchaotic system can be obtained from the system (2). Its Lyapunov exponents can be obtained

$$\mathrm{LE}_1 = 1.6170, \quad \mathrm{LE}_2 = 0.1123,$$
$$\mathrm{LE}_3 = 0, \quad \mathrm{LE}_4 = -12.8245.$$

The Lyapunov dimension of this system (2) is

$$D_{\mathrm{L}} = j + \frac{1}{|\mathrm{LE}_{j+1}|}\sum_{i=1}^{j}\mathrm{LE}_i = 3 + \frac{1.6170 + 0.1123}{12.8245} = 3.1348.$$

Obviously, the Lyapunov dimension of this new hyperchaotic system is also a fractional dimension. The new system (2) shows hyperchaotic behaviour, as depicted in Fig. 1(a)–1(c).

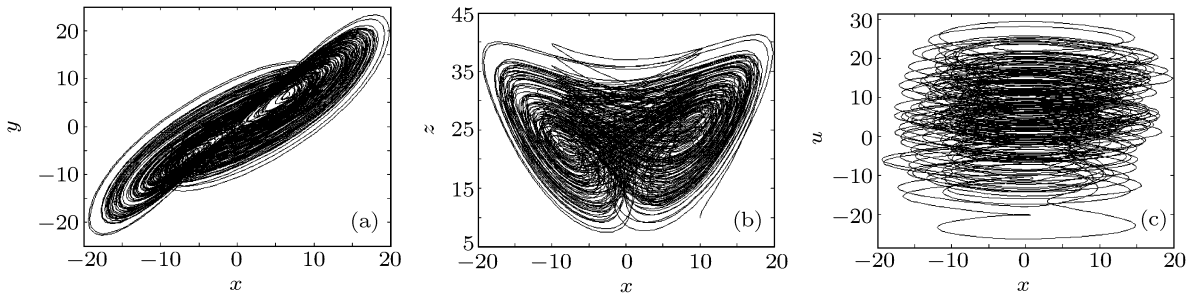


Fig. 1. Some phase portraits of hyperchaotic system (2) with $a = 27.5$, $b = 3$, $c = 19.3$, $h = 2.9$ and $k = 3.3$: (a) $x - y$ plan; (b) $x - z$ plan; (c) $x - u$ plan.

3. Basic dynamical behaviours and circuit implementation of the new hyperchaotic system (2)

In this section, basic properties and complex dynamics of the new system (2) are investigated. Lyapunov exponent, bifurcation diagram, phase portraits and circuit implementation are employed to illustrate how the dynamics of the system (2) alters with the increasing value of k. The new system (2) has the following basic properties.

3.1. Symmetry

The new system (2) is symmetric with respect to the z-axis, which is invariant under the following coordinate transformations $(x, y, z, u) \to (-x, -y, z, -u)$.

3.2. Dissipation

The divergence of the new four-dimensional system (2) is

$$\nabla \dot{V} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{u}}{\partial u} = -(a - c + h).$$

Thus, when $a - c + h > 0$, the new system (2) is dissipative. That is to say, a volume element v_0 is apparently contracted by flowing into a volume element $v_0 e^{-(a-c+h)t}$ in time t. It means that each volume containing the trajectory of this dynamical system (2) shrinks to zero as $t \to \infty$ at an exponential rate of $(a - c + h)$. Consequently, all the trajectories of the new system ultimately arrive at an attractor.

3.3. Equilibrium

Obviously, the new system (2) has a unique equilibrium E_0 (0, 0, 0, 0) for any parameters. Linearizing the new system (2) at the equilibrium E_0 yields the Jacobian matrix:

$$\boldsymbol{J} = \begin{pmatrix} -a & a & 0 & 0 \\ b & c & 0 & 1 \\ 0 & 0 & -h & 0 \\ -k & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

and its characteristic equation:

$$(\lambda + h)(\lambda^3 + (a - c)\lambda^2 - a(b + c)\lambda + ak) = 0. \tag{4}$$

From Eq. (4), we could obtain $\lambda_1 = -h$ and the following equation:

$$f(\lambda) = \lambda^3 + (a - c)\lambda^2 - a(b + c)\lambda + ak = 0. \tag{5}$$

Let $c_1 = a - c$, $c_2 = -a(b + c)$, $c_3 = ak$. According to the Routh–Hurwitz criterion, the real numbers of the roots λ of Eq. (5) are negative if and only if $a - c > 0$, $ak > 0$, $-a(a - c)(b + c) - ak > 0$.

However, it is impossible under the conditions of a, b, c, $k > 0$, $c_1 > 0$. It means that the equilibrium E_0 is unstable and thus it is possible to generate chaos or hyperchaos in the new system (2).

3.4. Bifurcation analysis

The new dynamical system (2) can be characterized with its Lyapunov exponents computed by Wolf method.[19] For equilibrium points: $\mathrm{LE}_4 < \mathrm{LE}_3 < \mathrm{LE}_2 < \mathrm{LE}_1 < 0$; for periodic orbits: $\mathrm{LE}_1 = 0, \mathrm{LE}_4 < \mathrm{LE}_3 < \mathrm{LE}_2 < 0$; for quasi-periodic orbits:$\mathrm{LE}_1 = \mathrm{LE}_2 = 0, \mathrm{LE}_4 < \mathrm{LE}_3 < 0$; while for chaotic obits: $\mathrm{LE}_1 > 0, \mathrm{LE}_2 = 0, \mathrm{LE}_4 < \mathrm{LE}_3 < 0, \mathrm{LE}_1 + \mathrm{LE}_3 + \mathrm{LE}_4 < 0$; and for hyperchaotic obits: $\mathrm{LE}_1 > \mathrm{LE}_2 > 0$, $\mathrm{LE}_3 = 0$, $\mathrm{LE}_4 < 0, \mathrm{LE}_1 + \mathrm{LE}_2 + \mathrm{LE}_4 < 0$

In order to discover the effect of the parameters on the dynamics of the new four-dimensional dynamical system (2), we fix the parameters $a = 27.5$, $b = 3$, $c = 19.3$, $h = 2.9$, and let the parameter k vary in the interval [0, 50].

The bifurcation diagram and the three Lyapunov exponents are shown in Figs. 2 and 3.

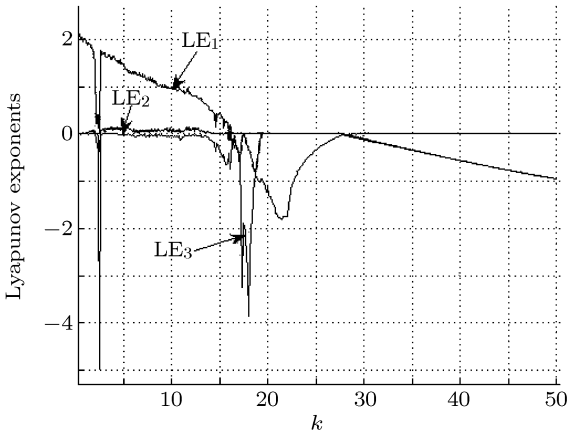


Fig. 2. Lyapunov exponents spectrum of the new system (2) versus parameter k.

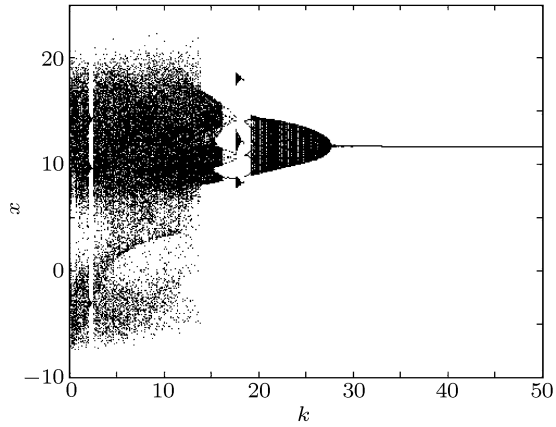


Fig. 3. Bifurcation diagram of the new system (2) versus parameter k.

From Figs. 2 and 3, we can observe that the Lyapunov exponents spectrum gives results completely consistent with the bifurcation diagram. Figure 2 shows that the system (2) is hyperchaotic for a range of k, and the system can also evolve into chaotic orbits, periodic orbits, and quasi-periodic orbits. With the variation of the parameter k, the dynamical behaviours of the four-dimensional system (2) are as follows:

When $0 < k \leq 2.20$ and $2.60 \leq k \leq 14.10$, the system (2) has two positive Lyapunov exponents, representing that the hyperchaos occurs. When $k = 3.3$, the corresponding phase portraits are shown in Fig. 1(a)–1(c).

When $14.20 \leq k \leq 16.10$, the largest Lyapunov exponent is positive, implying that the system shows chaotic behaviour. Figure 4 displays chaotic attractors for $k = 14.8$.

When $19.20 \leq k \leq 27.70$, the first and the second largest Lyapunov exponents are almost zero, which means that the system (2) is quasi-periodic. The phase portraits of special k are shown in Fig. 5.

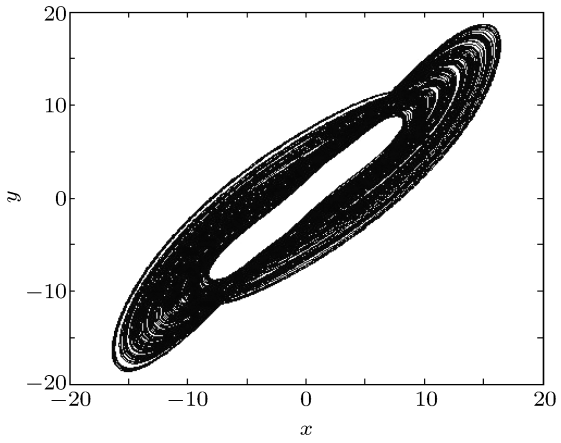


Fig. 4. Chaos phase portraits of the system (2) in the $x - y$ plane for $k = 14.8$.

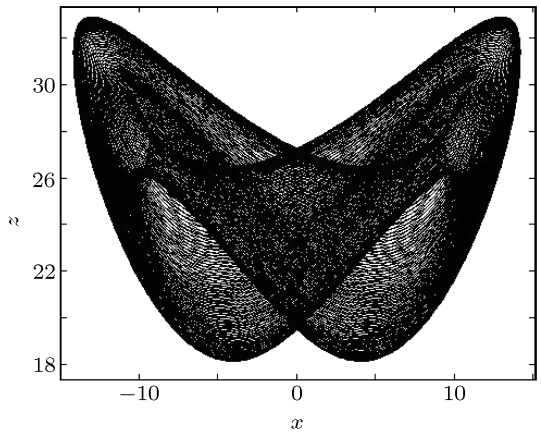


Fig. 5. Quasi-periodic phase portraits of the system (2) in in the $x - z$ plane for $k = 21.3$.

When $2.30 \leq k \leq 2.50, 16.20 \leq k \leq 19.10$ and $27.70 \leq k \leq 50.00$, the largest Lyapunov exponent almost equals zero, which means that system (2) is periodic. Especially, there are some chaotic obits in the periodic region. Some periodic phase portraits for special values of k are depicted in Figs.6(a) and 6(b).

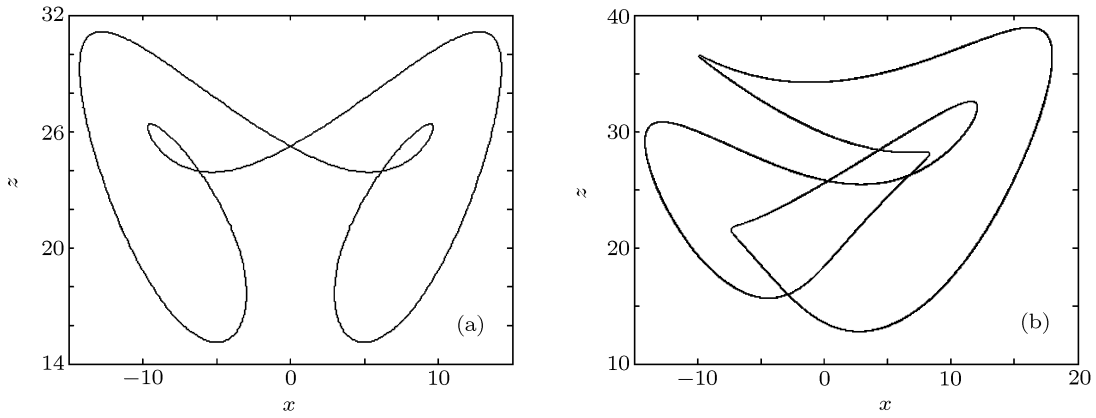


Fig. 6. Periodic phase portraits of the system (2) in the $x - z$ plane for different k: (a) $k = 2.4$; (b) $k = 18.4$.

3.5. Circuit implementation

By EWB, a fourth-order electronic circuit is designed to realize the previous hyperchaotic system (2). Figure 7 shows the diagram of the circuit which comprises linear resistors and linear capacitors. We choose operational ampliers LF347 chip and analogue multipliers AD633 chip. Considering the error of the circuit experiment, we select the circuit parameter as follows: resistance: $R_3 = R_4 = R_5 = R_6 = R_{10} = R_{11} = R_{15} = R_{17} = R_{20} = R_{21} = 10$ kΩ; $R_1 = R_2 = 3.6$ kΩ; $R_7 = 33.3$ kΩ; $R_8 = 5.2$ kΩ; $R_9 = 100$ kΩ; $R_{13} = 34.5$ kΩ; $R_{19} = 30.3$ kΩ; $R_{12} = R_{18} = 1$ kΩ; Capacitance: $C_1 = C_2 = C_3 = C_4 = 10$ μF. Utilizing EWB to simulate the 4D hyperchaotic system (2), we obtain some phase portraits as shown in Fig. 8.

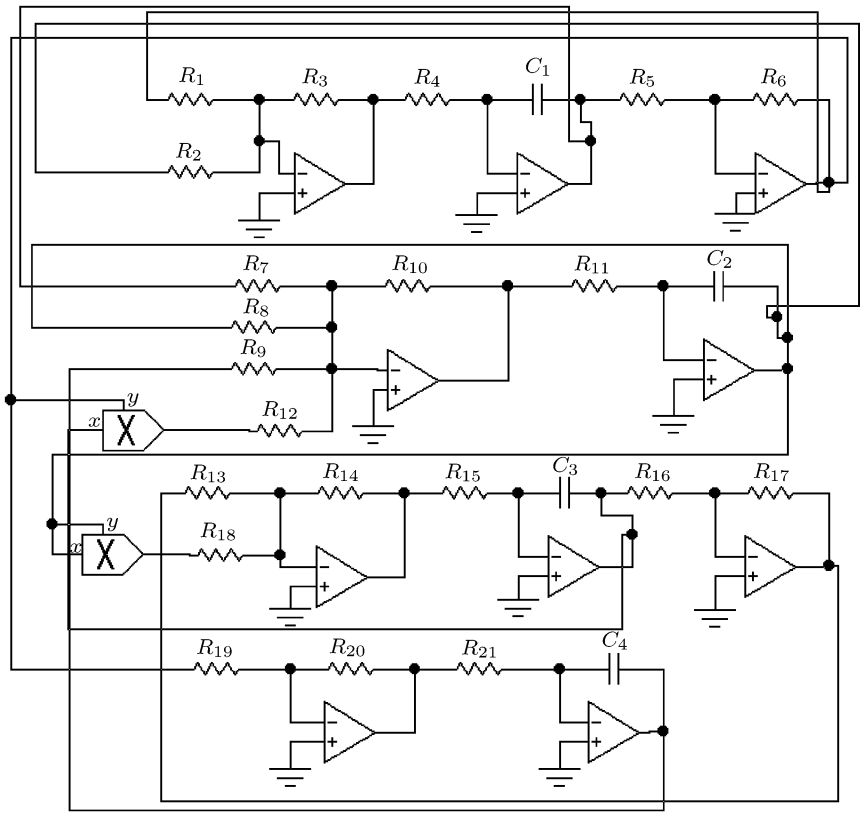


Fig. 7. Circuit diagram for implementing the 4D hyperchaotic system (2).

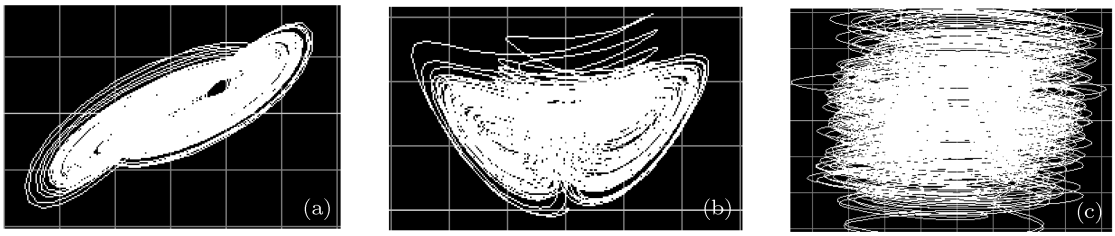


Fig. 8. The hyperchaotic phase portraits of system (2) simulated by EWB. (a) $x-y$ plan; (b) $x-z$ plan; (c) $x-u$ plan.

4. Controlling hyperchaos in the new hyperchaotic system

In this section, we will control the hyperchaotic system to equilibrium $E(0,0,0,0)$ periodic and quasi-periodic orbits by using feedback control methods. Suppose that the control system has the following form:

$$\begin{cases} \dot{x} = a(y-x) + u_1, \\ \dot{y} = bx + cy - xz + u + u_2, \\ \dot{z} = y^2 - hz + u_3, \\ \dot{u} = -kx + u_4. \end{cases} \tag{6}$$

In which u_i $(i = 1,2,3,4)$ are external control inputs. By choosing the suitable input functions u_i $(i = 1,2,3,4)$, the hyperchaotic trajectory (x, y, z, u) of system (2) will be dragged to $E(0,0,0,0)$ periodic and quasi-periodic orbits.

4.1. Controlling hyperchaotic attractor to equilibrium $E(0,0,0,0)$

In this section, we will control the hyperchaotic system to equilibrium $E(0,0,0,0)$ by using nonlinear feedback methods. We assume that $u_2 = -\hat{k}_1 y, u_4 = -\hat{k}_2 u$ and $u_1 = u_3 = 0$, where $\dot{\hat{k}}_1 = \beta_1 y^2$, $\dot{\hat{k}}_2 = \beta_2 u^2$, ($\beta_1$, β_2 are positive feedback gains), and the speed of hyperchaotic system controlled to the equilibrium E

will varies when values of $\beta_1,\ \beta_2$ are altered.

The controlled hyperchaotic system (6) is rewritten as follows:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx + cy - xz + u - \hat{k}_1 y, \\ \dot{z} = y^2 - hz, \\ \dot{u} = -kx - \hat{k}_2 u. \end{cases} \tag{7}$$

Lyapunov function is constructed as follows:

$$V = \frac{1}{2}(x^2+y^2+z^2+u^2) + \frac{1}{2\beta_1}(\hat{k}_1 - k^*)^2 + \frac{1}{2\beta_2}(\hat{k}_2 - k^*)^2,$$

where k^* is a parameter to be determined.

The derivation of V is:

$$\begin{aligned} \dot{V} &= x\dot{x} + y\dot{y} + z\dot{z} + u\dot{u} + (\hat{k}_1 - k^*)y^2 + (\hat{k}_2 - k^*)u^2 \\ &= -ax^2 - (k^* - c)y^2 - hz^2 - k^*u^2 + (a+b)xy - kxu + uy \\ &= -(x,y,z,u)\begin{pmatrix} a & -\frac{a+b}{2} & 0 & \frac{k}{2} \\ -\frac{a+b}{2} & k^* - c & 0 & -\frac{1}{2} \\ 0 & 0 & h & 0 \\ \frac{k}{2} & -\frac{1}{2} & 0 & k^* \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = -\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}. \end{aligned}$$

To make the system (6) asymptotically stable, the matrix $\boldsymbol{P}$ should be positive definite, based matrix theory, and when $\boldsymbol{P}$ satisfy

$$\begin{cases} a > 0, \\ a(k^* - c) - \left(\frac{a+b}{2}\right)^2 > 0, \\ h > 0, \\ ak^* - \left(ac + \frac{k^2}{4} + \frac{(a+b)^2}{4}\right)k^* + \frac{k^2}{4}c - \frac{1}{4}a > 0. \end{cases}$$

Even if

$$k^* > \max\left\{\frac{(a+b)^2 + 4ac}{4}, \frac{4ac + k^2 + (a+b)^2 + \sqrt{(4ac + k^2 + (a+b)^2)^2 - 16(ack^2 - a^2)}}{8a}\right\}$$

$\boldsymbol{P}$ is a positive definite matrix, thus $\dot{V}$ is negative semi-definite. However, we cannot obtain immediately the controlled system (7) being asymptotically stable at equilibrium $E(0,0,0,0)$.

In fact,

$$\dot{V} \le 0, \quad x,y,z,u \in L_\infty, \quad \hat{k}_1 - k^*, \quad \hat{k}_2 - k^* \in L_\infty,$$

for this reason , $\dot{x},\dot{y},\dot{z},\dot{u} \in L_\infty$ as a result of $\dot{V} = -\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, we arrive at

$$\int_0^t \lambda_{\min}(\boldsymbol{P})\left\|\boldsymbol{x}\right\|^2 d_t \le \int_0^t \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}\mathrm{d}_t \le \int_0^t -\dot{V}d_t = V(0) - V(t) \le V(0),$$

where $\lambda_{\min}(\boldsymbol{P})$ is the minimum eigenvalue of definite matrix $\boldsymbol{P}$, and so $x,y,z,u \in L_2$, according to the Barbalat Lemma, $\lim\limits_{x\to\infty}\|x(t)\| = 0$. Even x, y, z, u will be asymptotically converged to zero, therefore the controlled hyperchaotic system (7) is asymptotically converged to equilibrium $E(0,0,0,0)$.

To verify the effectiveness and feasibility of the control approach, by using the Matlab program, the numerical simulations have been completed. In the simulations, we choose the parameters as follows. $a = 27.5$, $b = 3$, $c = 19.3$, $h = 2.9$ and $k = 3.3$. The initial states of the controlled system (6) are selected as (5, 4, 3, 1) and the

corresponding feedback gain coefficients are given by $\beta_1 = 1$, $\beta_2 = 4$ and $\beta_1 = 5$, $\beta_2 = 8$, The behaviours of the state (x, y, z, u) of the controlled hyperchaotic system (7) with time are displayed in Fig. 9.

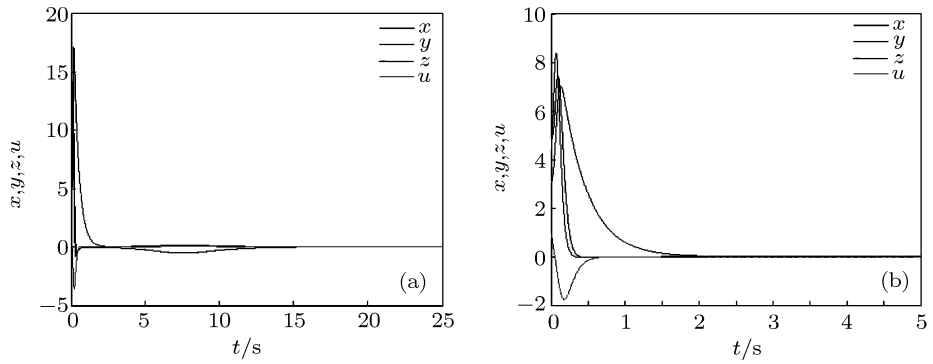


Fig. 9. The evolution of the controlled system (7) for the feedback control gain β_1, β_2. (a) $\beta_1 = 1$, $\beta_2 = 4$; (b) $\beta_1 = 5$, $\beta_2 = 8$.

4.2. Suppress hyperchaos to periodic and quasi-periodic orbits

In this subsection, the hyperchaotic system (2) will be controlled to a periodic orbit and quasi-periodic orbit with simple linear feedback.

Considering $u_2 = l(x + y)$ and $u_1 = u_3 = u_4 = 0$, the controlled hyperchaotic system (6) is rewritten as follows:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = bx + cy - xz + u + l(x + y), \\ \dot{z} = y^2 - hz, \\ \dot{u} = -kx, \end{cases} \tag{8}$$

where $l(x + y)$ is the linear feedback, the control gain l will be determined to suppress hyperchaos.

Choosing the control gain $l = 4.10$, Lyapunov exponents are obtained by using the Wolf method,[19] the four Lyapunov exponents are 0, -0.7347, -0.7350 and -5.5309. From the Lyapunov exponents above, one can judge that the controller $4.10(x + y)$ can stabilize the hyperchaotic system (2) to have a periodic orbit as shown in Fig. 10(a).

Choosing the control gain $l = 3.48$, the four Lyapunov exponents of the controlled system (8) are 0, 0, -1.0517 and -6.5512. Thus, we obtain that the hyperchaotic system (2) can be controlled to a quasi-periodic orbit with the gain $l = 3.48$ as shown in Fig. 10(b).

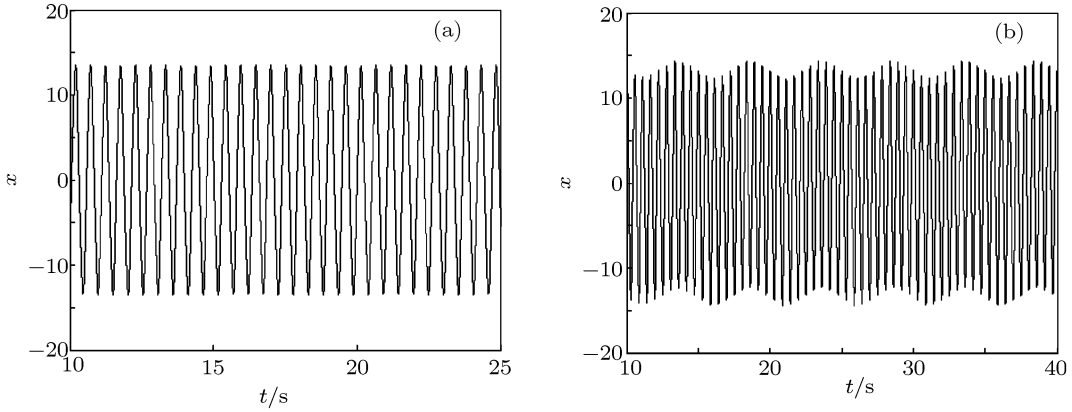


Fig. 10. (a) Control hyperchaos to a periodic orbit with the control gain $l = 4.10$. (b) Quasi-periodic orbit of the controlled system (8) with the control gain $l = 3.48$.

5. Conclusion

In this paper, a new hyperchaotic system is introduced. Some basic dynamic behaviours are further studied by calculating its Lyapunov exponent spectrum and bifurcation diagrams. Furthermore, based on nonlinear feedback control, the new hyperchaotic system is stabilized to unstable equilibrium, periodic and quasi-periodic orbits. Numerical simulations are shown to verify and illustrate the effectiveness of these controllers. A circuit is designed to realize the new hyperchaotic system by EWB. The EWB results show agreement with numerical simulations. More detailed and thorough analysis will be provided in the near future.

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