Statistic properties of Fermi gas in a strong magnetic field

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Based on the thermodynamic potential function of Fermi gas in a strong magnetic field, using the thermodynamics method, the integrated analytical expressions of thermodynamic quantities of the system at low temperatures are derived, and the effects of the magnetic field on the statistic properties of the system are analysed. It is shown that, as long as the temperature is not zero, the effects of the magnetic field on the thermodynamic quantities of the system contain both oscillatory and non-oscillatory parts. For the non-oscillatory part, compared with the situation of Fermi gas in a weak magnetic field, the influence of the magnetic field on the thermodynamic quantities is not exactly the same. For the oscillatory part, the period and amplitude of the oscillation are all related to the magnetic field. Due to the oscillation, the chemical potential may be greater than Ferim energy of the system, but the oscillation does not affect the thermodynamic stability of the system.

Keywords: strong magnetic field, Fermi gas, statistic property

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1. Introduction

Over the last few years, the experimental and theoretical studies of ultracold Fermi gases have been a very active field, and have achieved exciting advances.[1−6] For instance, Regal *et al.*[1] investigated the creation of ultracold molecules from a Fermi gas of atoms, and indicated the prospect of coupling fermionic atoms to bosonic molecules, thus altering the quantum statistics of the system. The investigation greatly excitated the enthusiasm to study Fermi gas. Xiong *et al.*[2] studied theoretically the ultracold two-component fermionic gases when a gradient magnetic field was used to tune the scattering length between atoms. It was shown that for ^{6}Li at the narrow resonance $B_0 = 543.25 \times 10^{-4}$ T, the gases would be in a coexistence of the regimes of Bardeen–Cooper–Schrieffer (BCS), Bose–Einstein–Condensate (BEC), and unitarity limit. Dong and Ma[3] using quantum hydrodynamic approaches, studied the quantum pressure correction to the collective excitation spectrum of the interacting trapped superfluid Fermi gases in the BEC–BCS crossover, etc.

In research on ultracold Fermi gases, regulating the magnetic field is an important experimental measure, therefore to investigate the effects of magnetic field on Fermi gases is significant. Recently, researches on the Fermi gas trapped in an external magnetic field have increased gradually.[7−10] For example, the oscillatory part of the thermodynamic potential function of Fermi gas in a strong magnetic field was given in Ref. [7], and the oscillatory period and amplitude of the magnetic moment were analysed. In Ref. [8], the orbital motion and the spin motion of fermions were considered simultaneously, and the thermodynamic properties of a weakly interacting Fermi gas in a weak magnetic field were also studied. Xiong *et al.*[9] had investigated Pauli's paramagnetism of an ideal Fermi gas with finite number of particles trapped in a weakly magnetic field.

In the present paper, based on the thermodynamic potential function of Fermi gas trapped in a strong magnetic field, by using the thermodynamic theory and method, we investigate the statistic properties of the system at low temperatures, and give the integrated analytical expressions of the thermodynamic quantities. We also analyse the effects of magnetic field on the statistic properties.

2. Analytical expression of thermodynamic potential function

The condition of strong magnetic field is

$$T \leq \sigma B \ll \mu, \tag{1}$$

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where σ is the Bohr magneton, B is the magnetic induction strength, T is temperature of the system, μ is the chemical potential of the free system, and the Boltzmann constant is set to $k = 1$. The free system is the one without external field.

Considering the orbital motion and the spin motion of particles simultaneously, according to Ref. [7], we have for the thermodynamic potential function of the system

$$\Omega = \Omega_0(\mu) + \frac{mTV}{\pi^2\hbar^3}\mathrm{Re}\sum_{k=1}^{\infty} I_k, \tag{2}$$

where $\Omega_0(\mu)$ is the thermodynamic potential function of the free system and

$$I_k = -2\sigma B\int_{-\infty}^{\infty}\int_{0}^{\infty}\log\left[1+\mathrm{e}^{\left(\mu/T-p_z^2/2mT-2x\sigma B/T\right)}\right]\mathrm{e}^{\mathrm{i}2\pi kx}\,\mathrm{d}x\,\mathrm{d}p_z. \tag{3}$$

Using the method described in Ref. [7], we can obtain the following by calculation

$$\begin{aligned}\Omega \;=\; & \Omega_0(\mu) - \frac{(m\sigma B)^{3/2}TV}{\pi^3\hbar^3}\log\left[1+\mathrm{e}^{\mu/T}\right]\sum_{k=1}^{\infty}\frac{1}{k^{3/2}} \\ & -\frac{m^{3/2}(\sigma B)^{5/2}V}{\pi^4\hbar^3}\frac{\mathrm{e}^{\mu/T}}{1+\mathrm{e}^{\mu/T}}\sum_{k=1}^{\infty}\frac{1}{k^{5/2}} + \frac{\sqrt{2}(m\sigma B)^{3/2}TV}{\pi^2\hbar^3}\sum_{k=1}^{\infty}\frac{\cos\alpha}{k^{3/2}\sinh\beta},\end{aligned} \tag{4}$$

where $\alpha = \pi\mu k/\sigma B - \pi/4$, $\beta = \pi^2 kT/\sigma B$, the oscillatory part $\Omega_3 = [\sqrt{2}(m\sigma B)^{3/2}TV/\pi^2\hbar^3]\sum_{k=1}^{\infty}\cos\alpha/k^{3/2}\sinh\beta$ is just the result given in Ref. [7].

From Eq. (4), by using thermodynamic relations, we can obtain the total energy, heat capacity, chemical potential and stability condition of the system, which can be expressed respectively as

$$\begin{aligned}& U = -T^2\frac{\partial}{\partial T}\left[\frac{\Omega}{T}\right]_{V,N},\quad C = \left[\frac{\partial U}{\partial T}\right]_{V,N}, \\ & u = \left[\frac{\partial\Omega}{\partial N}\right]_{V,T},\quad \left[\frac{\partial P}{\partial V}\right]_{T,N} = -\left[\frac{\partial^2\Omega}{\partial V^2}\right] < 0.\end{aligned} \tag{5}$$

3. The statistic properties of the system at low temperatures

When $T \ll T_{\mathrm{F}}$, $\mu = \varepsilon_{\mathrm{F}}\left[1-(\pi^2/12)\,(T/T_{\mathrm{F}})^2\right]$, where $\varepsilon_{\mathrm{F}} = (h^2/2m)\,(3n/8\pi)^{2/3}$ is the Fermi energy of the free system, n is the density of particle number, and

$$\frac{\partial\mu}{\partial T} = -\frac{\pi^2 T}{6T_{\mathrm{F}}},\quad \frac{\partial^2\mu}{\partial T^2} = -\frac{\pi^2}{6T_{\mathrm{F}}},$$

$$\begin{aligned}\frac{\partial\mu}{\partial V} &= -\frac{2\varepsilon_{\mathrm{F}}}{3V}\left[1-\left(\frac{T}{T_{\mathrm{F}}}\right)^2\right], \\ \frac{\partial^2\mu}{\partial V^2} &= \frac{10\varepsilon_{\mathrm{F}}}{9V^2}\left[1-\frac{1}{5}\left(\frac{T}{T_{\mathrm{F}}}\right)^2\right].\end{aligned} \tag{6}$$

For $\mu \gg T$, $\log[1+\mathrm{e}^{\mu/T}] \approx \log\mathrm{e}^{\mu/T} = \mu/T$, $\mathrm{e}^{\mu/T}/(1+\mathrm{e}^{\mu/T}) \approx 1$. Thus, the corresponding thermodynamic potential of the system can be expressed as

$$\begin{aligned}\Omega_1 \;&=\; -\frac{(m\sigma B)^{3/2}TV}{\pi^3\hbar^3}\log\left[1+\mathrm{e}^{\mu/T}\right]\sum_{k=1}^{\infty}\frac{1}{k^{3/2}} \\ &=\; -\frac{(m\sigma B)^{3/2}\mu V}{\pi^3\hbar^3}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}},\end{aligned} \tag{7}$$

$$\begin{aligned}\Omega_2 \;&=\; -\frac{m^{3/2}(\sigma B)^{5/2}V}{\pi^4\hbar^3}\frac{\mathrm{e}^{\mu/T}}{1+\mathrm{e}^{\mu/T}}\sum_{k=1}^{\infty}\frac{1}{k^{5/2}} \\ &=\; -\frac{m^{3/2}(\sigma B)^{5/2}V}{\pi^4\hbar^3}\sum_{k=1}^{\infty}\frac{1}{k^{5/2}}.\end{aligned} \tag{8}$$

The total energy $U = U_0 + U_1 + U_2 + U_3$, where

$$U_1 = -T^2\frac{\partial}{\partial T}\left[\frac{\Omega_1}{T}\right]_{V,N} = -\frac{5}{2\sqrt{2}\pi}\left(\frac{\sigma B}{\varepsilon_{\mathrm{F}}}\right)^{3/2}E_0\left[1+\frac{\pi^2}{12}\left(\frac{T}{T_{\mathrm{F}}}\right)^2\right]\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}, \tag{9}$$

$$U_2 = -T^2\frac{\partial}{\partial T}\left[\frac{\Omega_2}{T}\right]_{V,N} = -\frac{5}{2\sqrt{2}\pi^2}\left(\frac{\sigma B}{\varepsilon_{\mathrm{F}}}\right)^{5/2}E_0\sum_{k=1}^{\infty}\frac{1}{k^{5/2}}, \tag{10}$$

$$U_3 = -T^2\frac{\partial}{\partial T}\left[\frac{\Omega_3}{T}\right]_{V,N} = E_0\frac{5\pi}{2}\left(\frac{\sigma B}{\varepsilon_{\rm F}}\right)^{1/2}\left(\frac{T}{T_{\rm F}}\right)^2\left[-\frac{\pi}{6}\frac{T}{T_{\rm F}}\sum_{k=1}^{\infty}\frac{\sin\alpha}{k^{1/2}\sinh\beta}+\sum_{k=1}^{\infty}\frac{\cos\alpha\cosh\beta}{k^{1/2}\sinh^2\beta}\right]. \quad (11)$$

Here $E_0 = 3N\varepsilon_{\rm F}/5$ is the ground state energy of the free system.

The heat capacity $C = C_0 + C_1 + C_2 + C_3$, where

$$C_1 = \frac{\partial U_1}{\partial T} = -\frac{C_0}{2\sqrt{2}\pi}\left(\frac{\sigma B}{\varepsilon_{\rm F}}\right)^{3/2}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}, \quad (12)$$

$$C_2 = \frac{\partial U_2}{\partial T} = 0, \quad (13)$$

$$\begin{aligned} C_3 &= \frac{\partial U_3}{\partial T} \\ &= 3\pi^2\left(\frac{\varepsilon_{\rm F}}{\sigma B}\right)^{1/2}C_0\frac{T}{T_{\rm F}}\left\{\left[\frac{\pi^2}{36}\left(\frac{T}{T_{\rm F}}\right)^2-1\right]\sum_{k=1}^{\infty}k^{1/2}\frac{\cos\alpha}{\sinh\beta}+\frac{\pi T}{3T_{\rm F}}\sum_{k=1}^{\infty}k^{1/2}\frac{\sin\alpha\cosh\beta}{\sinh^2\beta}-2\sum_{k=1}^{\infty}k^{1/2}\frac{\cos\alpha}{\sinh^3\beta}\right\} \\ &\quad +6\left(\frac{\sigma B}{\varepsilon_{\rm F}}\right)^{1/2}C_0\left[\sum_{k=1}^{\infty}\frac{\cos\alpha\cosh\beta}{k^{1/2}\sinh^2\beta}-\frac{\pi}{4}\frac{T}{T_{\rm F}}\sum_{k=1}^{\infty}\frac{\sin\alpha}{k^{1/2}\sinh\beta}\right]. \end{aligned} \quad (14)$$

Here $C_0 = \pi^2 NT/2T_{\rm F}$ is the heat capacity of the free system at low temperatures.

The chemical potential is

$$u = \left[\frac{\partial\Omega}{\partial N}\right]_{T,V} \approx \mu\left[1-\frac{1}{\sqrt{2}\pi}\left(\frac{\sigma B}{\varepsilon_{\rm F}}\right)^{3/2}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}-\frac{\pi(\sigma B)^{1/2}T}{\varepsilon_{\rm F}^{3/2}}\sum_{k=1}^{\infty}\frac{\sin\alpha}{k^{1/2}\sinh\beta}\right]. \quad (15)$$

The pressure of the system,

$$\begin{aligned} P &= -\left[\frac{\partial\Omega_0}{\partial V}\right]_{T,V}-\left[\frac{\partial\Omega_1}{\partial V}\right]_{T,V} \\ &\quad -\left[\frac{\partial\Omega_2}{\partial V}\right]_{T,V}-\left[\frac{\partial\Omega_3}{\partial V}\right]_{T,V} \\ &= P_0+P_1+P_2+P_3, \end{aligned}$$

where

$$P_1 \approx \frac{1}{3}\frac{(m\sigma B)^{3/2}}{\pi^3\hbar^3}\mu\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}, \quad (16)$$

$$P_2 = \frac{m^{3/2}(\sigma B)^{5/2}}{\pi^4\hbar^3}\sum_{k=1}^{\infty}\frac{1}{k^{5/2}}, \quad (17)$$

$$\begin{aligned} P_3 \approx &-\frac{\sqrt{2}(m\sigma B)^{3/2}T}{\pi^2\hbar^3}\sum_{k=1}^{\infty}\frac{\cos\alpha}{k^{3/2}\sinh\beta} \\ &-\frac{2\sqrt{2}m^{3/2}(\sigma B)^{1/2}T}{3\pi\hbar^3}\mu\sum_{k=1}^{\infty}\frac{\sin\alpha}{k^{1/2}\sinh\beta}. \end{aligned} \quad (18)$$

The stability condition of the system can be expressed as

$$\begin{aligned} \frac{\partial P}{\partial V} = \frac{\partial P_0}{\partial V} &-\frac{2(m\sigma B)^{3/2}}{9\pi^3\hbar^3}\frac{\mu}{V}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}} \\ &-\frac{2\sqrt{2}m^{3/2}(\sigma B)^{1/2}T}{9\pi\hbar^3}\frac{\mu}{V}\sum_{k=1}^{\infty}\frac{\sin\alpha}{k^{1/2}\sinh\beta} \\ &+\frac{4\sqrt{2}m^{3/2}T}{9\hbar^3(\sigma B)^{1/2}}\frac{\mu^2}{V}\sum_{k=1}^{\infty}k^{1/2}\frac{\cos\alpha}{\sinh\beta} < 0, \end{aligned} \quad (19)$$

with P_0 being the pressure of the free system.

When $T \to 0$, from Eqs. (9)–(15), we can obtain the ground state energy, Fermi energy and heat capacity of Fermi gas in a strong magnetic field. They can be respectively written as

$$\begin{aligned} E = E_0\Bigg[1&-\frac{5}{2^{3/2}\pi}\left(\frac{\sigma B}{\varepsilon_{\rm F}}\right)^{3/2}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}} \\ &-\frac{5}{2^{3/2}\pi^2}\left(\frac{\sigma B}{\varepsilon_{\rm F}}\right)^{5/2}\sum_{k=1}^{\infty}\frac{1}{k^{5/2}}\Bigg], \end{aligned} \quad (20)$$

$$E_{\rm F} \approx \varepsilon_{\rm F}\left[1-\frac{1}{\sqrt{2}\pi}\left(\frac{\sigma B}{\varepsilon_{\rm F}}\right)^{3/2}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}\right], \quad (21)$$

$$C = C_0 = 0. \quad (22)$$

4. Analysis and discussion

(i) As long as $T \neq 0$, the effects of the magnetic field on the statistic properties of the system contain

both oscillatory and non-oscillatory parts. For the non-oscillatory part, compared with the situation of Fermi gas in a weak magnetic field, the influences of the magnetic field on the chemical potential, Fermi energy and stability condition are changed. For the oscillatory part, the magnetic field affects the period and amplitude of the oscillation. The oscillatory period is small and proportional to $\sigma B/\mu$. Under the condition of $T \sim \sigma B$, the oscillatory amplitude is proportional to the factor $\sigma B/\varepsilon_{\rm F}$.

(ii) The ground state energy and Fermi energy of the system do not oscillate. Compared with the situation of free Fermi gas, the magnetic field always decreases the ground state energy and the Fermi energy of the system. When $T \neq 0$, because the chemical potential has oscillation, under specifical conditions of magnetic field and temperature, the chemical potential of the system may be greater than the Fermi energy of the system, this characteristic can be seen easily from Eq. (15).

(iii) From the analytical expressions of thermodynamic quantities, it is obvious that, compared with the respective non-oscillatory parts, the oscillatory amplitudes of the total energy, heat capacity and chemical potential are smaller, bigger and the same in their magnitude, respectively. Under the condition of $T \sim \sigma B$, due to $\sigma B/\varepsilon_{\rm F} \ll 1$, the system can be kept in thermodynamic stability even though the oscillation exists.

5. Conclusions

In this paper, using thermodynamic methods, we have studied the statistic properties of Fermi gas in a strong magnetic field at low temperatures, and given the integrated analytical expressions of thermodynamic quantities. It is shown that the thermodynamic quantities of Fermi gas in a strong magnetic field, such as the total energy, heat capacity, chemical potential, pressure, and stability condition of the system will oscillate with the magnetic field varying under the condition of $T \neq 0$. The oscillatory period is small, and proportional to $\sigma B/\mu$. Under the condition of $T \sim \sigma B$, the oscillatory amplitude is proportional to the factor $\sigma B/\varepsilon_{\rm F}$. The oscillation could make the chemical potential greater than Ferim energy, but does not affect the thermodynamic stability of the system.

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