# The series solution to the metric of stationary vacuum with axisymmetry

Gu Ying-Qiu(辜英求)†

*School of Mathematical Science, Fudan University, Shanghai* 200433*, China*

(Received 22 March 2009; revised manuscript received 1 August 2009)

The multipole moment method not only conduces to the understanding of the deformation of the space–time, but also serves as an effective tool to approximately solve the Einstein field equation with. However, the usual multipole moments are recursively determined by a sequence of symmetric and trace-free tensors, which is inconvenient for practical resolution. In this paper, we develop a simplified procedure to generate the series solutions to the metric of the stationary vacuum with axisymmetry, and show its validity. In order to understand the free parameters in the solution, we propose to take the Schwarzschild metric as a standard ruler, and some well- known examples are analysed and compared with the series solutions in detail.

**Keywords:** stationary metric, multipole moments, asymptotically flat, series solution
**PACC:** 0420J, 0420, 0490, 9710K

## 1. Introduction

The asymptotically flat vacuum solutions with axisymmetry play an important role in general relativity. To understand the global structure of the metric caused by stars, we have the exact solution families of Schwarzschild, Curzon and Kerr metrics and some of their extensions to the electro-vacuum solutions such as Reissner–Nordström and Kerr–Newman metrics.[1−4] Manko *et al.* obtained an axisymmetric solution with five free parameters.[5] This solution is suitable for the study on a rapidly rotating neutron star, in which the strong magnetic field and the oblate shape should be taken into account.[6]

Owing to the fact that it is more and more difficult to obtain a realistic exact solution, the effective approximate method becomes a good alternative. The multipole moment expansion was introduced to interpret the physical meaning of the solutions to the Einstein equation. Geroch determined a recursive relation of multipole moments for the asymptotically flat, static vacuum with axisymmetry by means of the time-like Killing vector.[7,8] The norm of the Killing vector satisfies the Laplace equation in the three-dimensional hypersurface orthogonal to the Killing vector, which is derived from the Einstein equation. For the asymptotically flat space–time, the norm of the Killing vector and the metric are analytic near the spatial infinity, so the solution can be expanded in series of $r^{-1}$ with multipole moments. The multipole moments can be recursively determined by a sequence of symmetric and trace-free tensors. This idea was generalized to the stationary case by Hansen in Ref. [9], where two different sets of multipole moments were employed. One is the mass moments and the other is the angular momentum moments. The two sets of moments can be generated from two potential functions determined by the norm and the twist of the time-like Killing vector. The formalism proposed in Refs. [8] and [9] is in the covariant form. Some theorems are proved based on this formalism. In Ref. [10], Xanthopoulos proved that a stationary space–time was static if and only if all of its current moments vanished, and static space–time was flat if and only if all of its mass moments vanished. Beig–Simon[11] and Kundu[12] proved that, two spacetimes with the same multipole moments have the same space–time geometry at large radii, where the multipole expansion of the metric converges. This formalism was realized in the Weyl–Lewis–Papapetrou canonical metric.[13−15] In Ref. [16], the extended hyperbolic complex function method was used to study the stationary axisymmetric Einstein–Maxwell theory with $p$ Abelian gauge field, and two structural Riemann–Hilbert transformations were constructed and then shown to give an infinite-dimensional symmetry group, which was verified to have the structure of the semidirect product of Kac–Moody group and Virasoro group.

Another formalism of multipole moments for realistic calculation of slowly changing systems or ex-

†Corresponding author. E-mail: yqgu@fudan.edu.cn
© 2010 Chinese Physical Society and IOP Publishing Ltd

actly stationary ones was reviewed and developed by Thorne in Ref. [17]. This formalism is more manifest for a practical solution, which is associated with a special kind of coordinate system called the 'Asymptotical Cartesian and Mass Centered'(ACMC) coordinate system. In such an ACMC-$N$ coordinate system, the metric can be expressed by multipole moment expansion, and the coefficients of $r^{-(l+1)}(l \leq N)$ terms are linear combinations of the spherical harmonics of orders $l, l-1, \ldots, 0$. The de Donder coordinate system belongs to ACMC-$\infty$. The comparison and the relation between the two formalisms were discussed in Ref. [18]. This formalism the in de Donder coordinate system and quotient harmonic one was developed into an effective tool for solving the approximate global solutions to the Einstein equation.[19−21] The junction condition and equation in the special cases of Robertson–Walker and the Schwarzschild–de Sitter metrics were discussed in Ref. [22]. However, most of the previous studies are concerned mainly with the theoretical aspect of the multipole moments, which are very inconvenient for practically solving the Einstein's field equation. Besides, the interpretation of the physical meanings of the parameters is still a general problem in general relativity,[23,24] although the primary motivation of Geroch to introduce the multipole moments was to clarify this problem.

In the present paper, we consider how to effectively solve the series solutions to the stationary, asymptotically flat vacuum with axisymmetry. This case is of physical significance. First, we show that the canonical form of the Weyl–Lewis–Papapetrou coordinate system is a system of ACMC-$\infty$. Then we solve the series solution straightforwardly in this coordinate system. To understand the meanings of the free parameters in the solution, we set the Schwarzschild solution as a standard ruler. By comparison with this solution, we find that the bigger the absolute value of the dimensionless free coefficient is, the larger the deformation and the convection of the gravitating source will be. Noticing its generality, the series solution is an effective alternative to the exact one, and it is helpful to understand the abundant structure of the space–time.

## 2. Series solution in Weyl–Lewis–Papapetrou metric

To describe the axisymmetric space–time, the canonical form of the Weyl–Lewis–Papapetrou metric is simplest.[1] The line element is equivalent to

$$\mathrm{d}s^2 = U(\mathrm{d}t + W\mathrm{d}\varphi)^{\,2} - V(\mathrm{d}\rho^2 + \mathrm{d}z^2) - U^{-1}\rho^2\mathrm{d}\varphi^2, \quad (1)$$

where $(t, \rho, z, \varphi)$ are the coordinates of the geometrical meaning near the cylindrical coordinates in Minkowski space. $(U, V, W)$ depend on $(\rho, z)$ only. In this paper we take $G$ and $c$ to be unity, i.e. $G = c = 1$. Since the following calculations take the dimensionless form, the units will not be confused. The canonical form (1) has an important property, that is,

**Lemma 1** Except for the translation $t = \widetilde{t} + t_0, \varphi = \widetilde{\varphi} + \varphi_0$ related to two Killing vectors $(\partial_t, \partial_\varphi)$ and $z = \pm(\widetilde{z} + z_0)$, the form of metric (1) removes other uncertainty caused by coordinate condition.

This property can be checked as follows. The transformation $(\rho, z) \to (\widetilde{\rho}, \widetilde{z})$ with keeping $V(\mathrm{d}\rho^2 + \mathrm{d}z^2) = \widetilde{V}(\mathrm{d}\widetilde{\rho}^2 + \mathrm{d}\widetilde{z}^2)$ is a conformal transformation

$$\rho = \Re[f(\widetilde{\rho} \pm \widetilde{z}\,\mathrm{i})], \quad z = \Im[f(\widetilde{\rho} \pm \widetilde{z}\,\mathrm{i})], \quad (2)$$

where $f$ is any given analytic function. But the restriction $\rho^2\mathrm{d}\varphi^2 = \widetilde{\rho}^2\mathrm{d}\varphi^2$ and $\rho \geq 0$ give $f = \widetilde{\rho} \pm (\widetilde{z} + z_0)\,\mathrm{i}$, then we have $z = \pm(\widetilde{z} + z_0)$.

Except for the axial symmetry, the celestial body in equilibrium and the relevant stationary metric still have top–bottom reflection symmetry, which means that the metric will be the even functions of $z$ by setting the origin $(\rho = 0, z = 0)$ at the mass centre. In what follows we discuss only this case, that is, $(U, V, W)$ are even functions of $z$, although the discussion is also valid in the case including odd terms of $z$. So if we set the direction of $z$, the relation between coordinate and metric is completely fixed.

The coordinates $(\rho, z)$ in expression (1) is not convenient for the following calculation, so we make a polar coordinate transformation

$$\rho = r\sin\theta, \qquad z = r\cos\theta, \quad (3)$$

then the line element (1) becomes

$$\mathrm{d}s^2 = U(\mathrm{d}t + W\mathrm{d}\varphi)^{\,2} - V(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) - U^{-1}r^2\sin^2\theta\mathrm{d}\varphi^2. \quad (4)$$

Calculating the Ricci tensor $\boldsymbol{R}_{\mu\nu}$, we obtain the independent components in the Einstein equations for the axisymmetric vacuum[1]

$$\partial_r^2 U + \frac{2}{r}\partial_r U + \frac{\cot\theta}{r^2}\partial_\theta U + \frac{1}{r^2}\partial_\theta^2 U - \frac{|\nabla U|^2}{U} + \frac{U^3|\nabla W|^2}{r^2\sin^2\theta} \equiv -2V\boldsymbol{R}_{tt} = 0, \tag{5}$$

$$\partial_r^2 W - \frac{\cot\theta}{r^2}\partial_\theta W + \frac{1}{r^2}\partial_\theta^2 W + \frac{2}{U}(\nabla U\cdot\nabla W) \equiv \frac{2V}{U}(W\boldsymbol{R}_{tt} + \boldsymbol{R}_{t\varphi}) = 0, \tag{6}$$

where $\nabla = \left(\partial_r, r^{-1}\partial_\theta\right)$. And then $V$ can be integrated from

$$\begin{aligned}
&\frac{\partial_r V}{V} + \frac{\partial_r U}{U} - \frac{r\sin^2\theta}{2U^2}\left((\partial_r U)^2 + \frac{2\cot\theta}{r}\partial_r U\partial_\theta U - \frac{1}{r^2}(\partial_\theta U)^2\right)\\
&+ \frac{U^2}{2r}\left((\partial_r W)^2 + \frac{\cot\theta}{r}\partial_r W\partial_\theta W - \frac{1}{r^2}(\partial_\theta W)^2\right)\\
&\equiv -\frac{1}{r}\sin^2\theta(r^2\boldsymbol{R}_{rr} - \boldsymbol{R}_{\theta\theta}) - 2\cos\theta\sin\theta\boldsymbol{R}_{r\theta} = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\frac{\partial_\theta V}{V} + \frac{\partial_\theta U}{U} + \frac{r^2\sin^2\theta}{2U^2}\left(\cot\theta\left[(\partial_r U)^2 - \frac{1}{r^2}(\partial_\theta U)^2\right] - \frac{2}{r}\partial_r U\partial_\theta U\right)\\
&- \frac{U^2}{2}\left(\cot\theta\left[(\partial_r W)^2 - \frac{1}{r^2}(\partial_\theta W)^2\right] - \frac{1}{r}\partial_r W\partial_\theta W\right)\\
&\equiv \cos\theta\sin\theta(r^2\boldsymbol{R}_{rr} - \boldsymbol{R}_{\theta\theta}) - 2r\sin^2\theta\boldsymbol{R}_{r\theta} = 0.
\end{aligned} \tag{8}$$

Substituting $\boldsymbol{R}_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha)$ into the above equations, we obtain the dynamical equations for interior metric, which can also be solved by series.

The solutions to Eqs. (5)–(8) of most physical interest are obtained in the asymptotically flat cases, that is, the solutions satisfy the boundary conditions as $r\to\infty$

$$\begin{aligned}
&U \to 1 - \frac{2m}{r} + O(r^{-2}),\\
&W \to \frac{4J}{r}\sin^2\theta + O(r^{-2}),\\
&V \to 1 + \frac{2m}{r} + O(r^{-2}),
\end{aligned} \tag{9}$$

in which $m$ is the total gravitational mass, and $J$ the angular momentum of the source. Expression (9) means that $r=\infty$ is the regular point of the solution. On the other hand, the main parts of derivatives in Eqs. (5) and (6) are the linear elliptic operators. So the vacuum solutions $(U, W)$ are analytic functions in the neighbourhood of $r=\infty$,[11,12] and can be expressed as Taylor series of $r^{-1}$, then we generally have

$$U = 1 - \frac{R_{\rm S}}{r} + \frac{1}{r^2}\sum_{n=0}^{\infty}\frac{\widetilde{A}_n}{r^n}, \quad W = \sum_{n=1}^{\infty}\frac{\widetilde{B}_n}{r^n}, \tag{10}$$

where $(R_{\rm S}, \widetilde{A}_n, \widetilde{B}_n)$ are to-be-determined functions of $\theta$. For the regular metric, $(U, V)$ are bounded even functions of $z$, or equivalently, $(R_{\rm S}, \widetilde{A}_n, \widetilde{B}_n)$ are bounded functions of $\cos^2\theta$.

Substituting Eq. (10) into Eq. (5), and then expanding it into Taylor series of $r^{-1}$, we can obtain the equations for the coefficients, i.e.

$$R_{\rm S}'' + \cot\theta R_{\rm S}' = 0, \tag{11}$$

$$\widetilde{A}_0'' + \cot\theta\widetilde{A}_0' + 2\widetilde{A}_0 = (R_{\rm S}')^2 + R_{\rm S}^2. \tag{12}$$

The bounded solution of reflecting symmetry with respect to $z=0$ or $\theta=\pi/2$, which we call regular solution in the following, is given by

$$R_{\rm S} = 2m, \quad \widetilde{A}_0 = \frac{1}{2}R_{\rm S}^2, \tag{13}$$

where $m$ is a constant, which refers to the total mass of the source. $m=0$ corresponds to the Papapetrou class of solutions or flat space–time. The case of $m<0$ is nonphysical. So only the case with positive mass $m>0$ is of the most physical significance, and we discuss only this case later. Obviously, $R_{\rm S}$ is the corresponding Schwarzschild radius in length dimension.

By virtue of the excellent structure of Eqs. (5) and (6), we have the following results.

**Theorem 2** For the regular solution (10) with starting value (13), it takes the following form

$$\begin{aligned}
U = 1 &- \frac{R_{\rm S}}{r} + \frac{1}{2}\left(\frac{R_{\rm S}}{r}\right)^2\\
&+ \left(\frac{R_{\rm S}}{r}\right)^2\sum_{n=1}^{\infty}\widetilde{A}_n\left(\frac{R_{\rm S}}{r}\right)^n,
\end{aligned} \tag{14}$$

$$W = R_{\rm S}\sum_{n=1}^{\infty}\widetilde{B}_n\left(\frac{R_{\rm S}}{r}\right)^n, \tag{15}$$

in which

$$\widetilde{A}_n = \sum_{k=0}^{N_n} \widetilde{a}_{nk} \cos(2k\theta), \quad \widetilde{B}_n = \sum_{k=1}^{N_n} \widetilde{b}_{nk} [\cos(2k\theta) - 1], \tag{16}$$

where $(\widetilde{a}_{nk}, \widetilde{b}_{nk})$ are to-be-determined dimensionless constants, and $N_n = [(n+1)/2]$ represents the integer part of $(n+1)/2$, namely, $N_{2k-1} = N_{2k} = k$.

**Proof** Substituting Eqs.(14) and (15) into Eqs.(5) and (6) and expanding them into Taylor series of $r^{-1}$, we obtain the following equations

$$\widetilde{A}_1'' + \cot\theta \widetilde{A}_1' + 2 \cdot 3 \widetilde{A}_1 = -1, \tag{17}$$

$$\widetilde{A}_2'' + \cot\theta \widetilde{A}_2' + 3 \cdot 4 \widetilde{A}_2 = -\frac{1}{2} - 6\widetilde{A}_1 - \frac{1}{\sin^2\theta} \left( (\widetilde{B}_1')^2 + \widetilde{B}_1^2 \right), \tag{18}$$

$$\widetilde{B}_1'' - \cot\theta \widetilde{B}_1' + 1 \cdot 2 \widetilde{B}_1 = 0, \tag{19}$$

$$\widetilde{B}_2'' - \cot\theta \widetilde{B}_2' + 2 \cdot 3 \widetilde{B}_2 = 2\widetilde{B}_1. \tag{20}$$

Solve the equations, then we will find that the regular solutions take the forms (16). We check the succeeding equations by mathematical induction.

Assume that the forms (16) hold for $n = 2N-1$ and $n = 2N$. Since Eqs. (5) and (6) are homogeneous with respect to $r$, and the highest derivatives with respect to $r$ is linear. Calculation shows that these equations for $n = 2N+1$ and $n = 2N+2$ take the following forms:

$$\widetilde{A}_{2N+1}'' + \cot\theta \widetilde{A}_{2N+1}' + (2N+2)(2N+3)\widetilde{A}_{2N+1} = P_1 - 4(N+1)\widetilde{A}_{2N}, \tag{21}$$

$$\begin{aligned} \widetilde{A}_{2N+2}'' + \cot\theta \widetilde{A}_{2N+2}' + (2N+3)(2N+4)\widetilde{A}_{2N+2} - 4\widetilde{b}_{11}[2\cot\theta \widetilde{B}_{2N+1}' + (2N+1)\widetilde{B}_{2N+1}] \\ = P_2 - 2(2N+3)\widetilde{A}_{2N+1}, \end{aligned} \tag{22}$$

$$\widetilde{B}_{2N+1}'' - \cot\theta \widetilde{B}_{2N+1}' + (2N+1)(2N+2)\widetilde{B}_{2N+1} = Q_1 + 4N\widetilde{B}_{2N}, \tag{23}$$

$$\widetilde{B}_{2N+2}'' - \cot\theta \widetilde{B}_{2N+2}' + (2N+2)(2N+3)\widetilde{B}_{2N+2} = Q_2 + 2(2N+1)\widetilde{B}_{2N+1}, \tag{24}$$

where $(P_k, Q_k)$ are polynomials determined by functions $(\widetilde{A}_{2j-1}, \widetilde{A}_{2j}, \widetilde{B}_{2j-1}, \widetilde{B}_{2j}), (j \leq N)$ and their derivatives, which all take the form (16). Since $\cos(2n\theta) = \frac{1}{2}(\mathrm{e}^{2n\theta\,\mathrm{i}} + \mathrm{e}^{-2n\theta\,\mathrm{i}})$, under the constraint of the index of $(R_\mathrm{S}/r)^n$, we find

$$\begin{aligned} P_k &= \sum_{n=0}^{N+1} p_{kn} \cos(2n\theta), \\ Q_k &= \sum_{n=0}^{N+1} q_{kn} \cos(2n\theta), \end{aligned} \tag{25}$$

with constants $(p_{kn}, q_{kn})$. Substituting Eq. (25) into Eqs. (21)–(24) and solving the solutions with finite and symmetrical conditions, the solutions $(\widetilde{A}_{2N+2}, \widetilde{B}_{2N+1})$ have the form (16), but $(\widetilde{A}_{2N+1}, \widetilde{B}_{2N+2})$ will introduce extra terms similar to the second kind Legendre functions, which include the following factor

$$F(\theta) = \ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right) \sin\theta\cos\theta. \tag{26}$$

$F(\theta)$ also satisfies the boundary condition and symmetry. Nevertheless, the terms including this factor will soon be cancelled at the next step of resolution. Then we prove that the regular solutions always take the form (16).

Equation (16) means that the Weyl–Lewis–Papapetrou coordinate system (1) is a system of ACMC-$\infty$. Equation (16) can be transformed into the Legendre polynomials, but the Legendre polynomials are less convenient for calculating the nonlinear terms than the trigonometric functions. If the solutions of $(U, W)$ are determined, $V$ can be solved only from Eq. (7) with boundary condition $V(\infty, \theta) = 1$. Then the stationary axisymmetrical metric of vacuum is determined in principle. We determine the coefficients of the series in the next section.

## 3. Coefficients of the series and the examples

In the way similar to determining the coefficients of the special functions by recursion, the theorem 2 implies that, the resolution of Eqs. (5) and (6) can

be converted into a problem of solving coefficients by recursive relation. For convenience of calculation, we make transformation $U = \exp(u/2)$, then Eqs. (5) and (6) become

$$r^2\partial_r^2 u + 2r\partial_r u + \cot\theta\partial_\theta u + \partial_\theta^2 u + \frac{2\mathrm{e}^u}{r^2\sin^2\theta}|r^2(\partial_r W)^2 + (\partial_\theta W)^2| = 0, \tag{27}$$

$$r^2\partial_r^2 W - \cot\theta\partial_\theta W + \partial_\theta^2 W + (r^2\partial_r u\partial_r W + \partial_\theta u\partial_\theta W) = 0. \tag{28}$$

And Eqs. (14)–(16) become

$$u = \sum_{n=1}^{\infty} A_n\left(\frac{R_\mathrm{S}}{r}\right)^n,$$

$$W = R_\mathrm{S}\sum_{n=1}^{\infty} B_n\left(\frac{R_\mathrm{S}}{r}\right)^n, \tag{29}$$

$$A_n = \sum_{k=0}^{N_n-1} a_{nk}\cos(2k\theta),$$

$$B_n = \sum_{k=1}^{N_n} b_{nk}[\cos(2k\theta) - 1], \tag{30}$$

where $N_n = (n+1)/2$.

Substituting Eqs. (29) and (30) into the original equation Eqs. (27) and (28), and expanding them into series, we obtain the relations among the parameters $(a_{kn}, b_{kn})$ (see appendix). In each term, the parameters $(a_{kn}, b_{kn})$ with the larger indexes always take linear form with bigger coefficients, which correspond to the linear Laplace-like operator in Eqs. (27) and (28). So it is easy to solve the coefficients. However, the equations are underdetermined for the parameters $(a_{kn}, b_{kn})$, and we have the following two sequences of free parameters. Let

$$a_{10} = -2, \quad a_{31} = m_1, \quad a_{52} = m_2, \quad a_{2k+1,k} = m_k, \tag{31}$$

$$b_{11} = -\frac{1}{2}w_1, \quad b_{32} = -\frac{1}{4}w_2, \quad b_{2k-1,k} = -\frac{1}{2k}w_k, \quad (k = 1, 2, 3, \ldots), \tag{32}$$

or in general forms

$$a_{2k+1,k} = m_k, \quad (k = 1, 2, 3, \ldots), \tag{33}$$

$$b_{2k-1,k} = -\frac{1}{2k}w_k, \quad (k = 1, 2, 3, \ldots), \tag{34}$$

where $\{m_k, w_k\}$ are dimensionless numbers determined by the multipole moments of the energy–momentum tensor of the source, $a_{10} = -2$ is determined from expression (9), and the factor $1/(2k)$ is introduced to scale the coefficients in the recursive relation. The free parameters (31) and (32) correspond to 'null data' in Refs. [25] and [26], but the definitions are different. The coefficients can be recursively expressed as the polynomials of the free parameters(see appendix). Solving the coefficients, we obtain the series solutions of the metric

$$u = -2\frac{R_\mathrm{S}}{r} + \frac{1}{3}m_1(1 + 3\cos(2\theta))\left(\frac{R_\mathrm{S}}{r}\right)^3 - \frac{1}{2}w_1^2(1 + \cos(2\theta))\left(\frac{R_\mathrm{S}}{r}\right)^4 + \left[m_2\cos(4\theta) + \left(\frac{2}{7}{w_1}^2 + \frac{4}{7}m_2\right)\cos(2\theta) + \frac{8}{35}{w_1}^2 + \frac{9}{35}m_2\right]\left(\frac{R_\mathrm{S}}{r}\right)^5 + O(r^{-6}), \tag{35}$$

$$U = 1 - \frac{R_\mathrm{S}}{r} + \frac{1}{2}\left(\frac{R_\mathrm{S}}{r}\right)^2 + \left(\frac{1}{2}m_1\cos(2\theta) + \frac{1}{6}m_1 - \frac{1}{6}\right)\left(\frac{R_\mathrm{S}}{r}\right)^3 + \left[\left(-\frac{1}{4}{w_1}^2 - \frac{1}{2}m_1\right)\cos(2\theta) + \frac{1}{24} - \frac{1}{4}{w_1}^2 - \frac{1}{6}m_1\right]\left(\frac{R_\mathrm{S}}{r}\right)^4 + O(r^{-5}), \tag{36}$$

$$W = R_\mathrm{S}\sin^2\theta\left\{w_1\frac{R_\mathrm{S}}{r} + \frac{w_1}{2}\left(\frac{R_\mathrm{S}}{r}\right)^2 + (w_2\cos(2\theta) + \frac{3}{5}w_2 + \frac{1}{5}w_1)\left(\frac{R_\mathrm{S}}{r}\right)^3 + \left[\left(\frac{3}{4}w_2 + \frac{1}{8}m_1w_1\right)\cos(2\theta) + \frac{1}{15}w_1 + \frac{5}{24}m_1w_1 + \frac{9}{20}w_2\right]\left(\frac{R_\mathrm{S}}{r}\right)^4 + O(r^{-5})\right\}, \tag{37}$$

$$V = 1 + \frac{R_\mathrm{S}}{r} + \left(\frac{1}{8}\cos(2\theta) + \frac{3}{8}\right)\left(\frac{R_\mathrm{S}}{r}\right)^2 + \left[\left(-\frac{1}{2}m_1 + \frac{1}{8}\right)\cos(2\theta) - \frac{1}{6}m_1 + \frac{1}{24}\right]\left(\frac{R_\mathrm{S}}{r}\right)^3 + \left[\left(\frac{9}{64}{w_1}^2 - \frac{5}{32}m_1 + \frac{1}{256}\right)\cos(4\theta) + (-\frac{3}{8}m_1 + \frac{3}{16}{w_1}^2 + \frac{3}{64})\cos(2\theta) + \frac{11}{64}{w_1}^2 - \frac{7}{768} - \frac{13}{96}m_1\right]\left(\frac{R_\mathrm{S}}{r}\right)^4 + O(r^{-5}). \tag{38}$$

Comparing Eqs. (36) and (37) with expression (9), we have the physical meaning of $R_{\rm S}$ and $w_1$,

$$R_{\rm S} = 2m, \quad J = \frac{1}{4} w_1 R_{\rm S}^2 = w_1 m^2. \tag{39}$$

That is, $R_{\rm S}$ corresponds to the Schwarzschild radius, and $w_1$ corresponds to the angular momentum.

Now we compare the series solution with some exact metrics, i.e. the Curzon, Schwarzschild and Kerr solutions. The Curzon metric is diagonal,

$$U = \exp\left(-\frac{R_{\rm S}}{r}\right), \quad V = \exp\left(\frac{R_{\rm S}}{r} - \frac{1}{4}\left(\frac{R_{\rm S}}{r}\right)^2 \sin^2\theta\right), \quad W = 0. \tag{40}$$

Expanding Eqs. (40) into Taylor series and then comparing them with Eqs. (36) and (37), we find that all free parameters vanish

$$m_k = w_k \equiv 0, \qquad (\forall k). \tag{41}$$

Evidently, such a result is caused by the coordinate system.

For the Kerr and Schwarzschild solutions, under some parameter transformation, in the canonical coordinate system (4) they become

$$U = 1 - \frac{2(R_+ + R_- + R_{\rm S}\cosh\omega) R_{\rm S}\cosh\omega}{[(R_+ - R_-)^2 + {R_{\rm S}}^2]\cosh^2\omega + 2(R_+ + R_-)R_{\rm S}\cosh\omega + 4R_+R_-}, \tag{42}$$

$$V = 1 + \frac{[(R_+ - R_-)^2 + R_{\rm S}^2]\cosh^2\omega + 2(R_+ + R_-)R_{\rm S}\cosh\omega}{4R_+R_-}, \tag{43}$$

$$W = \frac{[(R_+ - R_-)^2 - R_{\rm S}^2](R_+ + R_- + R_{\rm S}\cosh\omega)\sinh\omega}{[(R_+ - R_-)^2 - {R_{\rm S}}^2]\cosh^2\omega + 4R_+R_-}, \tag{44}$$

in which

$$R_\pm = \sqrt{r^2\cosh^2\omega \pm 2rm\cos\theta\cosh\omega + m^2}, \quad R_{\rm S} = 2m, \quad \tanh\omega = \frac{2J}{m^2}. \tag{45}$$

$\omega = 0$ corresponds to the Schwarzschild solution in the Weyl–Lewis–Papapetrou coordinate system. The above formalism is introduced for the convenience of the following comparison. In contrast to the Taylor series of the above functions with (36) and (37), we obtain the following sequence of the free parameters,

$$\begin{aligned}
m_1 &= -\frac{1}{8} + \frac{3}{8}\alpha^2,\\
m_2 &= -\frac{7}{2^9} + \frac{17}{2^8}\alpha^2 - \frac{5\cdot 7}{2^9}\alpha^4,\\
m_3 &= -\frac{3\cdot 11}{2^{14}} + \frac{11\cdot 19}{2^{14}}\alpha^2 - \frac{17\cdot 23}{2^{14}}\alpha^4 + \frac{3\cdot 7\cdot 11}{2^{14}}\alpha^6,\\
m_4 &= -\frac{5\cdot 11\cdot 13}{2^{21}} + \frac{3\cdot 5\cdot 7\cdot 13}{2^{19}}\alpha^2 - \frac{3\cdot 5^3\cdot 19}{2^{20}}\alpha^4\\
&\quad + \frac{59\cdot 67}{2^{19}}\alpha^6 - \frac{3^2\cdot 5\cdot 11\cdot 13}{2^{21}}\alpha^8,\\
w_1 &= \frac{1}{2}\alpha, \quad w_2 = \frac{3}{32}\alpha - \frac{5}{32}\alpha^3,\\
w_3 &= \frac{3^4}{2^{12}}\alpha - \frac{3\cdot 41}{2^{11}}\alpha^3 + \frac{3^3\cdot 7}{2^{12}}\alpha^5,\\
w_4 &= \frac{11\cdot 13}{2^{15}}\alpha - \frac{13\cdot 47}{2^{15}}\alpha^3\\
&\quad + \frac{881}{2^{15}}\alpha^5 - \frac{3\cdot 11\cdot 13}{2^{15}}\alpha^7, \quad \ldots
\end{aligned} \tag{46}$$

where $\alpha = \tanh\omega < 1$. The sequences vanish quite fast.

The parameter $\alpha = 0$ corresponds to the Schwarzschild solution. In this case, the 'mass moments' $m_k \neq 0$, which is caused by the coordinates. So to understand these data we need a unified standard.

Evidently, the above calculation shows that any stationary metric with axisymmetry is identical with two sequences $\{m_n, w_n\}$, in which $\{m_n\}$ is mainly related to the multipole moments of mass density, and $\{w_n\}$ is related to the current distribution. On the contrary, for any given sequences $\{m_n, w_n\}$ with suitable upper bounds, the series is a solution to the Einstein equation of vacuum in the region of its convergence. So the relation between the radius of convergence of the series and the values of free parameters $\{m_n, w_n\}$ is important. The convergences of the multipole moment series were discussed in different context. The conditions of convergence for the static solutions were established in Refs. [14], [25] and [27], and the conditions for the stationary ones were given

in Refs. [26] and [28]. Although the results ensure the convergence of the above series, how to derive the concrete constraint for free parameters is quite complicated. In the static case, the radius of convergence can be directly derived from the exact solution. We perform some analyses, because the result is enlightening. Since $W = 0$, Eq. (27) becomes Laplace equation, and its general asymptotically flat solution is of the Weyl class(see Ch.18 in Ref. [2]), given as

$$u = \sum_{n=0}^{N} m_n P_n(\cos\theta) \left(\frac{R_{\rm S}}{r}\right)^{n+1}, \quad (m_0 = -2,\ N \leq \infty), \tag{47}$$

where $P_n$ are the Legendre polynomials of $n$ degree. For any value of $N$, the function $V$ can also be exactly solved from Eq. (7).[2] The Curzon solution is the simplest case with $N = 0$. In the case of $N = \infty$, we have

**Theorem 3** For the series solution (47) with $N = \infty$, if the free parameter $m_n$ satisfies the following condition

$$|m_n| \leq C(n+1)^K \lambda^n, \tag{48}$$

where $C > 0$ and $\lambda > 0, K$ are the given numbers independent of $n$, then the series solution (47) and all its derivatives converge in the region

$$r > r_0 \equiv \lambda R_{\rm S}. \tag{49}$$

**Proof** According to the property of the Legendre polynomials, for all values of $n$ we have

$$|P_n| \leq 1, \tag{50}$$

$$\frac{\rm d}{{\rm d}\theta} P_n(\cos\theta) = -(2n-1)\sin\theta P_{n-1}(\cos\theta) + \frac{\rm d}{{\rm d}\theta} P_{n-2}(\cos\theta). \tag{51}$$

From Eq.(51) we obtain

$$\begin{aligned}\left|\frac{\rm d}{{\rm d}\theta} P_n(\cos\theta)\right| &\leq (2n-1) + \left|\frac{\rm d}{{\rm d}\theta} P_{n-2}(\cos\theta)\right| \\ &\leq (2n-1) + (2n-5) + (2n-9) + \ldots \\ &= \frac{1}{2} n(n+1).\end{aligned} \tag{52}$$

Substituting Eqs. (48), (50) and (52) into Eq. (47), we find that the series and its derivatives are controlled by

$$|u| < C \sum_{n=1}^{\infty} (n+1)^K \left(\frac{\lambda R_{\rm S}}{r}\right)^{n+1}, \tag{53}$$

$$|\partial_r u| < \frac{C}{r} \sum_{n=1}^{\infty} (n+1)^{K+1} \left(\frac{\lambda R_{\rm S}}{r}\right)^{n+1}, \tag{54}$$

$$|\partial_\theta u| < \frac{C}{2} \sum_{n=1}^{\infty} (n+1)^{K+2} \left(\frac{\lambda R_{\rm S}}{r}\right)^{n+1}. \tag{55}$$

The series on the right hand side all converge in the region $r > r_0$, so $(u, \partial_r u, \partial_\theta u)$ are absolutely convergent in the region $r > r_0$, and uniformly absolutely convergent in the region $r \geq r_1$ for any given $r_1 > r_0$. Similarly we can check the results for higher order derivatives. Integrating Eq. (7), we find that the radius of convergence of $V$ is also $r_0$. The proof is finished.

In the stationary case with $W \neq 0$, the situation becomes more complicated due to the nonlinear terms, because $(a_{nk}, b_{nk})$ are polynomials of $(m_j, w_j)$ of $j \leq [n/2]$. However, from the recursive relations, we find that the following inequalities seem to be valid:

$$\begin{aligned}&|a_{nk}| < CM_n(n+1)^K, \\ &|b_{nk}| < CM_n(n+1)^K, \quad \forall (n,k),\end{aligned} \tag{56}$$

where $K > 0$ and $C > 0$, which are constants independent of $n$, and

$$M_n = \max_{j \leq [n/2]} \left\{|m_j|, |w_j|, |m_j|^{n/2}, |w_j|^{n/2}\right\}. \tag{57}$$

In principle, the relation (56) can be derived from the results in Refs. [26] and [28]. If inequalities in expression (56) hold, the series solutions (14)–(16) will be controlled by

$$\begin{aligned}|U| &< C_0 \sum_{n=0}^{\infty} (n+1)^{K+1} \left(\frac{\lambda R_{\rm S}}{r}\right)^{n}, \\ |W| &< C_1 \sum_{n=0}^{\infty} (n+1)^{K+1} \left(\frac{\lambda R_{\rm S}}{r}\right)^{n+1},\end{aligned} \tag{58}$$

where $C_0$ and $C_1$ are constants independent of $n$, and

$$\lambda = \lim_{n\to\infty} \sqrt[n]{M_n}. \tag{59}$$

Then the inequalities in expression (58) also imply that the series and their derivatives converge in the region $r > \lambda R_{\rm S}$.

# 4. Interpretation of the multipole moments

From Eqs. (41) and (46) we find that in the canonical form of the Weyl–Lewis–Papapetrou coordinate

system (4), the Curzon metric has no 'multipole moment' (where it means the free parameters), but the Schwarzschild metric has infinite ones. These results are somewhat unnatural and puzzling. Evidently, such results are caused by the coordinate system, although the canonical form (4) is most convenient for solving the metric. So how to extract the understandable information from the solutions is also an important problem.

Similar to the cases of point charge and dipole in the electromagnetism, an ideal explanation for the multipole moments should be expressed in the forms of some conserved spatial integrals of the source.[23] However, this idea is associated with how to define the covariant generalized functions for nonlinear differential equations, which may notm have a general solution for the higher order moments. A realistic explanation for these free parameters is to solve them by associating the exterior solution with the interior one, the results will endow the parameters with concrete values.[21] However, the influence of the coordinate system still exists. In Refs. [24], the authors suggested two ways of comparing the approximate solution with the exact solution: one is to calculate the multipole structure of the Ernst complex potentials for the solutions, and the second is to generate approximate solutions from exact ones by expanding the latter into Taylor series with respect to a small parameter.

To interpret the physical meanings of these free parameters, introducing a standard ruler may be a convenient option. As an approach of the first step, we find that the Schwarzschild metric is a good ruler, because its properties are simplest and have been well understood. By comparing the other solutions with this ruler, we can have some definite and understandable meanings of these free parameters.

The standard exterior Schwarzschild space–time is described by

$$\mathrm{d}s^2 = \left(1-\frac{R_\mathrm{S}}{R}\right)\mathrm{d}t^2 - \left(1-\frac{R_\mathrm{S}}{R}\right)^{-1}\mathrm{d}R^2 - R^2\mathrm{d}\Theta^2 - R^2\sin^2\Theta\mathrm{d}\varphi^2. \tag{60}$$

The transformation between (60) and (61) reads[1]

$$r = \sqrt{R^2-2mR+m^2\cos^2\Theta}, \quad \cos\theta = \frac{(R-m)\cos\Theta}{\sqrt{R^2-2mR+m^2\cos^2\Theta}}, \tag{61}$$

which is valid in the region $R > \frac{1}{2}R_\mathrm{S}(1+\sin\Theta)$. In the coordinate system $(t, R, \Theta, \varphi)$, the line element (4) becomes

$$\mathrm{d}s^2 = U(\mathrm{d}t+W\mathrm{d}\varphi)^2 - \widetilde{V}[\mathrm{d}R^2+R(R-R_\mathrm{S})\mathrm{d}\Theta^2] - U^{-1}R(R-R_\mathrm{S})\sin^2\Theta\mathrm{d}\varphi^2, \tag{62}$$

$$\widetilde{V} = \left(1+\frac{R_\mathrm{S}^2\sin^2\Theta}{4R(R-R_\mathrm{S})}\right)V. \tag{63}$$

$(t, R, \Theta, \varphi)$ also form an ACMC-$\infty$ coordinate system.

For the Curzon solution (40), $(U\to 0, V\to\infty)$ corresponds to the surface

$$R = \frac{1}{2}R_\mathrm{S}(1+\sin\Theta), \quad \frac{1}{2}R_\mathrm{S} \le R \le R_\mathrm{S}, \tag{64}$$

which is the surface of an oblate spheroid. This implies that the solution is an analytic extension of the vacuum produced by an ellipsoid. However, by $\widetilde{V}$ in Eq. (63), we find that the solutions are valid only in the region $R > R_\mathrm{S}$. Expanding $(U, \widetilde{V})$ into the Taylor series, we obtain the free parameters for the Curzon solution in the form of Legendre polynomials

$$U = 1-\frac{R_\mathrm{S}}{R}+\frac{1}{12}P_2\left(\frac{R_\mathrm{S}}{R}\right)^3+\frac{1}{24}P_2\left(\frac{R_\mathrm{S}}{R}\right)^4+\left(\frac{1}{56}P_2-\frac{3}{560}P_4\right)\left(\frac{R_\mathrm{S}}{R}\right)^5 + \left(\frac{1}{1440}P_0+\frac{1}{144}P_2-\frac{1}{160}P_4\right)\left(\frac{R_\mathrm{S}}{R}\right)^6+O(r^{-7}), \tag{65}$$

$$\widetilde{V} = 1+\frac{R_\mathrm{S}}{R}+\left(\frac{R_\mathrm{S}}{R}\right)^2+\left(P_0-\frac{1}{12}P_2\right)\left(\frac{R_\mathrm{S}}{R}\right)^3+\left(P_0-\frac{29}{168}P_2-\frac{1}{28}P_4\right)\left(\frac{R_\mathrm{S}}{R}\right)^4 + \left(P_0-\frac{41}{168}P_2-\frac{57}{560}P_4\right)\left(\frac{R_\mathrm{S}}{R}\right)^5+O(r^{-6}). \tag{66}$$

For the Kerr solution (42) and (44), the multipole moments in the coordinate system $(t, R, \Theta, \varphi)$ are expressed as follows

$$U = 1 - \frac{R_{\rm S}}{R} + \frac{1}{4}\alpha^2 P_2\left(\frac{R_{\rm S}}{R}\right)^3 + \alpha^2\left(-\frac{1}{24}P_0 + \frac{1}{24}P_2\right)\left(\frac{R_{\rm S}}{R}\right)^4$$
$$+ \alpha^2\left[\left(-\frac{1}{16}\alpha^2 + \frac{1}{140}\right)P_4 - \frac{1}{40}P_0 + \frac{1}{56}P_2\right]\left(\frac{R_{\rm S}}{R}\right)^5 + O(r^{-7})$$
$$+ \alpha^2\left[\left(\frac{3}{280} - \frac{47}{1120}\alpha^2\right)P_4 + (\frac{1}{28}\alpha^2 + \frac{1}{168})P_2 + (-\frac{1}{60} + \frac{1}{160}\alpha^2)P_0\right]\left(\frac{R_{\rm S}}{R}\right)^6, \tag{67}$$

$$W = \alpha R_{\rm S}\sin^2\Theta\left\{\frac{1}{2}\frac{R_{\rm S}}{R} + \frac{1}{2}\left(\frac{R_{\rm S}}{R}\right)^2 + \left[-\frac{5}{24}\alpha^2 P_2 + \left(\frac{1}{2} - \frac{1}{24}\alpha^2\right)P_0\right]\left(\frac{R_{\rm S}}{R}\right)^3\right.$$
$$+ \left[-\frac{7}{16}\alpha^2 P_2 + \left(\frac{1}{2} - \frac{1}{16}\alpha^2\right)P_0\right]\left(\frac{R_{\rm S}}{R}\right)^4 + O(r^{-7})$$
$$+ \left[\left(\frac{9}{160}\alpha^4 - \frac{3}{280}\alpha^2\right)P_4 + \left(\frac{1}{32}\alpha^4 - \frac{113}{168}\alpha^2\right)P_2 + \frac{1}{2} + \frac{1}{160}\alpha^4 - \frac{1}{15}\alpha^2\right]\left(\frac{R_{\rm S}}{R}\right)^5$$
$$\left. + \left[\left(\frac{79}{448}\alpha^4 - \frac{19}{560}\alpha^2\right)P_4 + \left(\frac{113}{1344}\alpha^4 - \frac{305}{336}\alpha^2\right)P_2 + \frac{1}{2} + \frac{1}{48}\alpha^4 - \frac{7}{120}\alpha^2\right]\left(\frac{R_{\rm S}}{R}\right)^6\right\}. \tag{68}$$

Comparing the dimensionless coefficients of the term $P_2(\cos\Theta)\,(R_{\rm S}/R)^3$ in (65) and (67), namely, $1/12$ and $\alpha^2/4$, we obtain a definite view about the relative deformations of each space–time. The bigger the absolute values of the free coefficients are, the larger the deformation and the convection of the gravitating source will be. In the viewpoint of series, the Kerr solution has not any speciality. However, besides ($R_{\rm S} = 2m, w_1 = \alpha/2 = J/m^2$), it is unclear whether the other coefficients have some relations with the general physical concepts.

The solutions in the metric (62) can be directly obtained from the following equations

$$R(R - R_{\rm S})\partial_R^2 u + (2R - R_{\rm S})\partial_R u + \cot\Theta\partial_\Theta u + \partial_\Theta^2 u + \frac{2e^u}{\sin^2\Theta}\left[(\partial_R W)^2 + \frac{(\partial_\Theta W)^2}{R(R - R_{\rm S})}\right] = 0, \tag{69}$$
$$R(R - R_{\rm S})\partial_R^2 W - \cot\Theta\partial_\Theta W + \partial_\Theta^2 W + [R(R - R_{\rm S})\partial_R u\partial_R W + \partial_\Theta u\partial_\Theta W] = 0. \tag{70}$$

The solutions are equivalent to those in expressions (35)–(38).

## 5. Discussion and conclusion

The above procedure provides a simple but effective method to solve the series solution of stationary and asymptotically flat metric with axisymmetry to any precision required. The solution is identical with two sequences of free dimensionless parameters, which correspond to the usual multipole moments. The free parameters are determined by the energy-momentum distribution of gravitating source. For a wide class of given parameters with suitable upper bound, the series solution converges in the region $r > R_{\rm S}$. For a normal star, we always have $r \gg R_{\rm S}$, so the series provides highly accurate solutions to the stationary metric with axisymmetry.

To interpret the meanings of the free parameters, using dimensionless form and setting up a standard ruler are meaningful. Only comparing our solutions with a simple ruler, we can find the difference between the solutions, and obtain a clear view about the free parameters. For a given star, from the free parameters obtained by associating the exterior metric with the interior solution, the results will have concrete physical meanings. However, the matching conditions on the surface of the star should be carefully discussed. In Ref. [29], we find $U \in C^1$ but $V \in C^0$.

## Acknowledgment

The author is grateful to his supervisors Prof. Ta-Tsien Li and Prof. Han-Ji Shang for their encouragement.

# Appendix

Substituting Eqs. (29) and (30) into the original equations (27) and (28), and expanding them into series, we have the relations among the parameters $(a_{kn}, b_{kn})$ as follows:

$$\begin{aligned}0 = {} & 22a_{20}\left(\frac{R_{\rm S}}{r}\right)^2 + (-2a_{3,1} + 6a_{30})\left(\frac{R_{\rm S}}{r}\right)^3 \\ & + [(6a_{41} + 12b_{11}^2)\cos(2\theta) + (-2a_{41} + 12a_{40}) + 20b_{11}^2]\left(\frac{R_{\rm S}}{r}\right)^4 \\ & + [((-8a_{5,2} + 14a_{5,1}) + 4b_{1,1}(3a_{1,0}b_{1,1} + 4b_{2,1}))\cos(2\theta) \\ & + (-4a_{5,2} - 2a_{5,1} + 20a_{5,0}) + 4b_{1,1}(5a_{1,0}b_{1,1} + 12b_{2,1})]\left(\frac{R_{\rm S}}{r}\right)^5 + O(r^{-6}), \end{aligned} \tag{A1}$$

$$\begin{aligned}0 = \Bigg\{ & (8b_{21} + 2a_{10}b_{11})\left(\frac{R_{\rm S}}{r}\right)^2 + [(16b_{32} + 20b_{31}) + 4a_{20}b_{11} + 4a_{10}b_{21}]\left(\frac{R_{\rm S}}{r}\right)^3 \\ & + [(32b_{4,2} - 2a_{3,1}b_{1,1} + 12a_{1,0}b_{3,2})\cos(2\theta) + (48b_{4,2} + 36b_{4,1}) \\ & + 6b_{1,1}a_{3,0} - 8a_{3,1}b_{1,1} + 12a_{1,0}b_{3,2} + 8a_{2,0}b_{2,1} + 6a_{1,0}b_{3,1}]\left(\frac{R_{\rm S}}{r}\right)^4 \\ & + [((48b_{5,3} + 72b_{5,2}) + 16a_{1,0}b_{4,2} + 4a_{3,1}b_{2,1} + 24a_{2,0}b_{3,2})\cos(2\theta) \\ & + (72b_{5,3} + 88b_{5,2} + 56b_{5,1}) + 8b_{1,1}a_{4,0} + 16a_{1,0}b_{4,2} - 8a_{3,1}b_{2,1} + 24a_{2,0}b_{3,2} \\ & + 12a_{2,0}b_{3,1} + 12b_{2,1}a_{3,0} - 8a_{4,1}b_{1,1} + 8a_{1,0}b_{4,1}]\left(\frac{R_{\rm S}}{r}\right)^5 + O(r^{-6})\Bigg\} R_{\rm S}\sin^2\theta. \end{aligned} \tag{A2}$$

The values of the parameters are given by the following recursive relations:

$$\begin{aligned} & a_{10} = -2, \quad a_{20} = 0, \quad a_{31} = m_1, \quad a_{30} = \frac{1}{3}m_1, \quad a_{41} = -\frac{1}{2}w_1^2, \quad a_{40} = -\frac{1}{2}w_1^2, \\ & a_{52} = m_2, \quad a_{51} = \frac{2}{7}w_1^2 + \frac{4}{7}m_2, \quad a_{50} = \frac{8}{35}w_1^2 + \frac{9}{35}m_2, \\ & a_{62} = -\frac{1}{2}w_1w_2, \quad a_{61} = -\frac{1}{10}{w_1}^2 - \frac{4}{5}w_1w_2, \quad a_{60} = -\frac{1}{10}{w_1}^2 - \frac{3}{10}w_1w_2, \ \ldots \\ & b_{11} = -\frac{1}{2}w_1, \quad b_{21} = -\frac{1}{4}w_1, \quad b_{32} = -\frac{1}{4}w_2, \quad b_{31} = \frac{1}{5}w_2 - \frac{1}{10}w_1, \\ & b_{42} = -\frac{1}{32}m_1w_1 - \frac{3}{16}w_2, \quad b_{41} = -\frac{1}{30}w_1 + \frac{3}{20}w_2 - \frac{1}{24}m_1w_1, \\ & b_{53} = -\frac{1}{6}w_3, \quad b_{52} = -\frac{1}{12}w_2 + \frac{1}{9}w_3, \quad b_{51} = -\frac{1}{21}m_1w_1 - \frac{1}{105}w_1 + \frac{1}{15}w_2 + \frac{5}{126}w_3, \\ & b_{63} = -\frac{5}{36}w_3 - \frac{1}{16}w_1m_2 + \frac{1}{96}m_1w_2, \\ & b_{62} = -\frac{1}{168}{w_1}^3 - \frac{3}{56}w_1m_2 + \frac{1}{240}m_1w_2 + \frac{1}{120}m_1w_1 - \frac{1}{36}w_2 + \frac{5}{54}w_3, \\ & b_{61} = \frac{11}{480}m_1w_2 - \frac{1}{420}{w_1}^3 + \frac{51}{560}w_1m_2 - \frac{1}{35}m_1w_1 + \frac{1}{45}w_2 + \frac{25}{756}w_3 - \frac{1}{420}w_1 \ldots . \end{aligned} \tag{A3}$$

The solutions to Eqs. (69) and (70) in metric (62) are given by

$$\begin{aligned} U = {} & 1 - \frac{R_{\rm S}}{R} + \left(\frac{1}{2}M_1\cos(2\Theta) + \frac{1}{6}M_1\right)\left(\frac{R_{\rm S}}{R}\right)^3 \\ & + \left[\left(\frac{1}{4}M_1 - \frac{1}{4}w_1^2\right)\cos(2\Theta) + \frac{1}{12}M_1 - \frac{1}{4}w_1^2\right]\left(\frac{R_{\rm S}}{R}\right)^4 + O(r^{-5}), \end{aligned} \tag{A4}$$

$$\begin{aligned} \widetilde{V} = {} & 1 + \frac{R_{\rm S}}{R} + \left(\frac{R_{\rm S}}{R}\right)^2 + (1 - \frac{1}{2}M_1\cos(2\Theta) - \frac{1}{6}M_1)\left(\frac{R_{\rm S}}{R}\right)^3 + O(r^{-5}) + \left(1 - \frac{37}{96}M_1 + \frac{11}{64}{w_1}^2\right. \\ & \left. + \left(-\frac{5}{32}M_1 + \frac{9}{64}{w_1}^2\right)\cos(4\Theta) + \left(-\frac{9}{8}M_1 + \frac{3}{16}{w_1}^2\right)\cos(2\Theta)\right)\left(\frac{R_{\rm S}}{R}\right)^4, \end{aligned} \tag{A5}$$

$$W = R_{\rm S}\sin^2\Theta\left\{w_1\frac{R_{\rm S}}{R} + w_1\left(\frac{R_{\rm S}}{R}\right)^2 + O(r^{-5})\right.$$
$$+\left[\left(w_2 - \frac{3}{16}w_1\right)\cos(2\Theta) + \frac{3}{5}w_2 + \frac{71}{80}w_1\right]\left(\frac{R_{\rm S}}{R}\right)^3$$
$$\left.+\left[\left(-\frac{27}{64}w_1 + \frac{1}{8}M_1w_1 + \frac{9}{4}w_2\right)\cos(2\Theta) + \frac{239}{320}w_1 + \frac{5}{24}M_1w_1 + \frac{27}{20}w_2\right]\left(\frac{R_{\rm S}}{R}\right)^4\right\}, \tag{A6}$$

where

$$M_1 = m_1 + \frac{1}{8}, \quad M_2 = m_2 + \frac{7}{2^9}, \quad M_3 = m_3 + \frac{3\cdot 11}{2^{14}}, \ \ldots \tag{A7}$$

are somewhat 'pure mass' multipole moments induced by the influence of the coordinates (see Eq. (46)). In the case of Schwarzschild metric, we have ($M_k = w_k = 0, \forall k$).

# References


[1] Islam J N 1985 *Rotating Fields in General Relativity* (Cambridge: Cambridge University Press)
[2] Kramer D, Stephani H, Herlt E, MacCallum M and Schmutzer E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)
[3] Bicak J 2006 *Einstein Equations: Exact Solutions, in Encyclopedia of Mathematical Physics* Vol. 2 165 (Oxford: Elsevier)
[4] Hawking S and Ellis G 1999 *The Large Scale Structure of the Space–time* (Cambridge: Cambridge University Press)
[5] Manko V S, Sanabria-Gómez J D and Manko O V 2000 *Phys. Rev.* D **62** 044048
[6] Stute M and Camenzind M 2002 *Mon. Not. Roy. Astron. Soc.* **336** 831
[7] Geroch R 1970 *J. Math. Phys.* **11** 1955
[8] Geroch R 1970 *J. Math. Phys.* **11** 2580
[9] Hansen R O 1974 *J. Math. Phys.* **15** 46
[10] Xauthopoulos B C 1979 *J. Phys.* A **12** 1025
[11] Beig R and Simon W 1980 *Commun. Math. Phys.* **78** 75
[12] Kundu P 1981 *J. Math. Phys.* **22** 1236
[13] Fodor G, Hoenselaers C and Perjés Z 1989 *J. Math Phys.* **30** 2252
[14] Bäckdahl T and Herberthson M 2005 *Class. Quant. Grav.* **22** 1607
[15] Bäckdahl T and Herberthson M 2005 *Class. Quant. Grav.* **22** 3585
[16] Gao Y J 2006 *Chin. Phys.* **15** 66
[17] Thorne K S 1980 *Rev. Mod. Phys.* **52** 299
[18] Gürsel Y 1983 *Gen. Rel. Grav.* **15** 8
[19] Martín J and Ruiz E 1985 *Phys. Rev.* D **32** 2550
[20] Aguirregabiria J M, Bel Ll, Martín J, Molina A and Ruiz E 2001 *Gen. Rel. Grav.* **33** 1809
[21] Cabezas J A, Martín J, Molina A and Ruiz E 2006 arXiv:gr-qc/0611013
[22] Gong T X and Wang Y J 2008 *Chin. Phys.* B **17** 2356
[23] Kozameh C, Newman E T and Silva-Ortigoza G 2008 *Gen. Rel. Grav.* **40** 2043
[24] Manko V S and Ruiz E 2004 *Class. Quant. Grav.* **21** 5849
[25] Friedrich H 2007 *Ann. Henri Poincare* **8** 817
[26] Aceña A E 2008 arXiv:0810.4256
[27] Bäckdahl T and Herberthson M 2006 *Class. Quant. Grav.* **23** 5997
[28] Bäckdahl T 2007 *Class. Quant. Grav.* **24** 2205
[29] Gu Y Q 2007 arXiv:0712.0219