Approximate solution for the Klein–Gordon–Schrödinger equation by the homotopy analysis method*

Wang Jia(王 佳)[a], Li Biao(李 彪)[a)b)†], and Ye Wang-Chuan(叶望川)[a)]

[a)] *Nonlinear Science Center and Department of Mathematics, Ningbo University, Ningbo* 315211, *China*
[b)] *MM Key Lab, Chinese Academy of Sciences, Beijing* 100080, *China*

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The Homotopy analysis method is applied to obtain the approximate solution of the Klein–Gordon–Schrödinger equation. The Homotopy analysis solutions of the Klein–Gordon–Schrödinger equation contain an auxiliary parameter which provides a convenient way to control the convergence region and rate of the series solutions. Through errors analysis and numerical simulation, we can see the approximate solution is very close to the exact solution.

Keywords: Klein–Gordon–Schrödinger equation, homotopy analysis method, approximate solution
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1. Introduction

The nonlinear Schrödinger equation coupled with a nonlinear Klein–Gordon equation in the (1+1)-dimensional case reads

$$\mathrm{i}\psi_t + \alpha\psi_{xx} + \rho\phi\psi = 0, \tag{1}$$

$$\phi_{tt} - c_0^2\phi_{xx} + \mu^2\phi + \gamma|\phi|^2\phi - \beta|\psi|^2 = 0. \tag{2}$$

The system of Eqs. (1) and (2), which is known as the Klein–Gordon–Schrödinger (KGS) equation model, is a classical example describing a system of conserved scalar nucleons interacting with neutral scalar mesons where the dynamics of these fields are coupled through the Yukawa interaction.[1] Both the nonlinear Schrödinger (NLS) and the KGS equations have been widely used to study the dynamics of small but finite amplitude nonlinearly interacting perturbations in many-body physics,[2] in nonlinear optics[3] and optical communications,[4] in nonlinear plasmas[5,6] and complex geophysical flows,[7] as well as in intense laser–plasma interactions and nonlinear quantum electrodynamics.[8] Furthermore, equations similar to (1) and (2) may describe the dynamics of coupled electrostatic upper-hybrid and ion-cyclotron waves in a uniform magnetoplasma.[9] It is noted that the solitary wave solution[10] for KGS equation while $\{\gamma = 0,\ \rho = c_0 = \beta = 1,\ \alpha = 1/2,\ \mu = M\}$ has been obtained. The existence and uniqueness of global solutions for rough data of the KGS system with quadratic coupling and cubic auto-interactions have been proved recently in Ref. [11].

The search for a new mathematical algorithm to discover the exact solutions or approximate solutions of nonlinear partial differential equations (PDEs) is an important and essential task in nonlinear science. Many methods have been developed by mathematicians and physicists to find special solutions of nonlinear PDEs, such as the inverse scattering method,[12] Hirota bilinear method,[13,14] Darboux transformation,[15] Bäcklund transformation,[16] Painleve expansion,[17] the tanh method,[18−20] various extended tanh methods,[21−25] variational iteration method,[26] homogenous balance method,[27] first-integral method,[28] Adomian's decomposition [29−31] and so on.

Currently, an analytical method for strongly nonlinear problems, namely the homotopy analysis method (HAM),[32,33] has been developed and successfully applied to many kinds of nonlinear problems in science and engineering. The homotopy analysis method (HAM) contains an auxiliary parameter h which provides us with a simple way to adjust the convergence region and rate of the series solution. Moreover, by means of the so-called h-curve, it is easy to find the valid region of h to gain a convergent se-

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†Corresponding author. E-mail: biaolee2000@yahoo.com.cn
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ries solution. Thus, through HAM, explicit analytic solutions of nonlinear problems are possible to obtain. Several investigators have already applied the HAM successfully to the solutions of various interesting problems.[34−36]

In this paper, the HAM is applied to solve KGS equations (1) and (2), here we take the arbitrary constants to be $\{\gamma = 0,\ \rho = c_0 = \beta = \mu = 1,\ \alpha = 1/2\}$ and using $\psi = u + \mathrm{i}v,\ \phi = w$, we can separate Eqs. (1) and (2) into real and imaginary parts. Therefore, one can obtain a (1+1)-dimensional tripled system in the following form

$$\begin{aligned} & w_{tt} - w_{xx} + w - u^2 - v^2 = 0, \\ & -v_t + \frac{1}{2}u_{xx} + wu = 0, \\ & u_t + \frac{1}{2}v_{xx} + wv = 0, \end{aligned} \tag{3}$$

whose exact solution[10] is

$$\begin{aligned} u &= \frac{3\sqrt{2}}{2}\mathrm{sech}\left(x - \frac{\sqrt{3}}{2}t\right)^2 \cos\left(\frac{\sqrt{3}}{2}x - \frac{13}{8}t\right), \\ v &= \frac{3\sqrt{2}}{2}\mathrm{sech}\left(x - \frac{\sqrt{3}}{2}t\right)^2 \sin\left(\frac{\sqrt{3}}{2}x + \frac{13}{8}t\right), \\ w &= 3\mathrm{sech}\left(x - \frac{\sqrt{3}}{2}t\right)^2. \end{aligned}$$

To examine the HAM for KGS equation, the above soliton solution will be used next.

The rest of this paper is organized as follows. In Section 2, the basic idea of the HAM is given briefly. In Section 3, the HAM is applied to the obtained approximate solution to Eq. (3). The last section is a short summary and conclusion.

2. Basic idea of HAM

In this article, we apply the HAM to the discussed problem. To show the basic idea, let us consider the following differential equation

$$\mathcal{N}[z(x,t)] = 0, \tag{4}$$

where $\mathcal{N}$ is a nonlinear differential operator, x and t denote independent variables and $z(x,t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM, one first constructs the zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(x,t;p) - z_0(x,t)] = ph\mathcal{N}[\phi(x,t;p)], \tag{5}$$

where $\mathcal{L}$ is an auxiliary linear operator, $\phi(x,t;p)$ is an unknown function, $z_0(x,t)$ is an initial guess of $z(x,t)$, $h \neq 0$ is an auxiliary parameter and $p \in [0,1]$ is the embedding parameter. Obviously, when $p = 0$ and $p = 1$, one has

$$\phi(x,t;0) = z_0(x,t), \quad \phi(x,t;1) = z(x,t), \tag{6}$$

respectively. Thus as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $z_0(x,t)$ to the solution $z(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to the embedding parameter p, one has

$$\phi(x,t;p) = z_0(x,t) + \sum_{m=1}^{+\infty} z_m(x,t)p^m, \tag{7}$$

where

$$z_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{8}$$

The convergence of the series (7) depends upon the auxiliary parameter h. If it is convergent at $p = 1$, one has

$$z(x,t) = z_0(x,t) + \sum_{m=1}^{+\infty} z_m(x,t), \tag{9}$$

which must be one of the solutions of the original nonlinear equation, as proved in Ref. [32].

Define the vectors

$$\boldsymbol{z}_n = \{z_0(x,t), z_1(x,t), \ldots, z_n(x,t)\}.$$

Differentiating the zeroth-order deformation Eq. (5) m-times with respect to p and then setting $p = 0$ and finally dividing them by $m!$, we obtain the following mth-order deformation equation

$$\mathcal{L}\left[z_m(x,t) - \chi_m z_{m-1}(x,t)\right] = h\mathcal{R}_m(\boldsymbol{z}_{m-1}), \tag{10}$$

where

$$\mathcal{R}_m(\boldsymbol{z}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0},$$

and

$$\chi_m = \begin{cases} 1, & m \geq 1, \\ 0, & m < 1. \end{cases} \tag{11}$$

It should be noted that $z_m(x,t)$ for $m \geq 1$ is governed by the linear equation (10) which can be solved by symbolic computation softwares, such as *Maple* and *Mathematica*.

3. Approximate solution to the KGS equation

To obtain the approximate solution of the KGS equation, we choose the linear operators

$$\mathcal{L}_1[\phi(x,t;p)] = \frac{\partial^2\phi(x,t;p)}{\partial t\partial t}, \quad \mathcal{L}_2[\phi(x,t;p)] = \frac{\partial\phi(x,t;p)}{\partial t},$$

with the property

$$\mathcal{L}_1[f(x)] = 0, \quad \mathcal{L}_2[c] = 0,$$

where f is an arbitrary function of x and c is a constant. From Eq. (3), we define a system of nonlinear operators as

$$\begin{aligned}
&\mathcal{N}_1[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)]\\
&= \frac{\partial^2\phi_3(x,t;p)}{\partial t\partial t} - \frac{\partial^2\phi_3(x,t;p)}{\partial x\partial x} + \phi_3(x,t;p) - \phi_1(x,t;p)^2 - \phi_2(x,t;p)^2,\\
&\mathcal{N}_2[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)]\\
&= -\frac{\partial\phi_2(x,t;p)}{\partial t} + \frac{1}{2}\frac{\partial^2\phi_1(x,t;p)}{\partial x\partial x} + \phi_3(x,t;p)\phi_1(x,t;p),\\
&\mathcal{N}_3[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)]\\
&= \frac{\partial\phi_1(x,t;p)}{\partial t} + \frac{1}{2}\frac{\partial^2\phi_2(x,t;p)}{\partial x\partial x} + \phi_3(x,t;p)\phi_2(x,t;p). \qquad (12)
\end{aligned}$$

Using the definition given in the above section, we construct the so-called zeroth-order deformation equations

$$\begin{aligned}
(1-p)\mathcal{L}_1[\phi_3(x,t;p) - z_{3,0}(x,t)] &= ph\mathcal{N}_1[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)],\\
(1-p)\mathcal{L}_2[\phi_2(x,t;p) - z_{2,0}(x,t)] &= ph\mathcal{N}_2[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)],\\
(1-p)\mathcal{L}_2[\phi_1(x,t;p) - z_{1,0}(x,t)] &= ph\mathcal{N}_3[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)]. \qquad (13)
\end{aligned}$$

Obviously, when $p = 0$ and $p = 1$,

$$\begin{aligned}
&\phi_1(x,t;0) = z_{1,0}(x,t) = u(x,0),\\
&\phi_1(x,t;1) = u(x,t),\\
&\phi_2(x,t;0) = z_{2,0}(x,t) = v(x,0),\\
&\phi_2(x,t;1) = v(x,t),\\
&\phi_3(x,t;0) = z_{3,0}(x,t) = w(x,0),\\
&\phi_3(x,t;1) = w(x,t).
\end{aligned}$$

Therefore, as the embedding parameter p increases from 0 to 1, $\phi_i(x,t;p)$ varies from the initial guess $z_i(x,t)$ to the exact solution $u(x,t)$, $v(x,t)$ and $w(x,t)$, for $i = 1,2,3$, respectively. Expanding ϕ_i in Taylor series with respect to p for $i = 1,2,3$, one has

$$\phi_i(x,t;p) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t)p^m, \qquad (14)$$

where

$$z_{i,m}(x,t) = \frac{1}{m!}\frac{\partial^m\phi_i(x,t;p)}{\partial p^m}\bigg|_{p=0}. \qquad (15)$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameters h are so properly chosen, the above series converge at $p = 1$, and one has

$$\begin{aligned}
u(x,t) &= z_{1,0}(x,t) + \sum_{m=1}^{+\infty} z_{1,m}(x,t),\\
v(x,t) &= z_{2,0}(x,t) + \sum_{m=1}^{+\infty} z_{2,m}(x,t),\\
w(x,t) &= z_{3,0}(x,t) + \sum_{m=1}^{+\infty} z_{3,m}(x,t), \qquad (16)
\end{aligned}$$

which must be one of approximate solution of the original nonlinear equation. Define the vectors

$$\boldsymbol{z}_{i,n} = \{z_{i,0}(x,t), z_{i,1}(x,t), \ldots, z_{i,n}(x,t)\}, \quad i = 1,2,3. \qquad (17)$$

We obtain the mth-order deformation equations

$$\begin{aligned}
&\mathcal{L}_1[z_{3,m}(x,t)-\chi_m z_{3,m-1}(x,t)]=h\mathcal{R}_{3,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1}),\\
&\mathcal{L}_2[z_{2,m}(x,t)-\chi_m z_{2,m-1}(x,t)]=h\mathcal{R}_{2,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1}),\\
&\mathcal{L}_2[z_{1,m}(x,t)-\chi_m z_{1,m-1}(x,t)]=h\mathcal{R}_{1,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1}),
\end{aligned}\tag{18}$$

where

$$\begin{aligned}
\mathcal{R}_{1,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1})&=\frac{\partial^2 z_{3,m-1}(x,t)}{\partial t\partial t}-\frac{\partial^2 z_{3,m-1}(x,t)}{\partial x\partial x}\\
&\quad+\sum_{n=0}^{m-1}z_{3,n}(x,t)\Big(\sum_{n=0}^{m-1}z_{1,n}(x,t)\Big)^2-\Big(\sum_{n=0}^{m-1}z_{2,n}(x,t)\Big)^2,\\
\mathcal{R}_{2,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1})&=-\frac{\partial z_{2,m-1}(x,t)}{\partial t}\\
&\quad+\frac{1}{2}\frac{\partial^2 z_{1,m-1}(x,t)}{\partial x\partial x}+\sum_{n=0}^{m-1}z_{3,n}(x,t)\cdot\sum_{n=0}^{m-1}z_{1,n}(x,t),\\
\mathcal{R}_{3,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1})&=\frac{\partial z_{1,m-1}(x,t)}{\partial t}+\frac{1}{2}\frac{\partial^2 z_{2,m-1}(x,t)}{\partial x\partial x}\\
&\quad+\sum_{n=0}^{m-1}z_{3,n}(x,t)\cdot\sum_{n=0}^{m-1}z_{2,n}(x,t).
\end{aligned}\tag{19}$$

Now, the solution of the mth-order deformation equations (18) for $m\geq 1$ becomes

$$\begin{aligned}
&z_{1,m}(x,t)=\chi_m z_{1,m-1}(x,t)+h\mathcal{L}_2^{-1}[\mathcal{R}_{3,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1})],\\
&z_{2,m}(x,t)=\chi_m z_{2,m-1}(x,t)+h\mathcal{L}_2^{-1}[\mathcal{R}_{2,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1})],\\
&z_{3,m}(x,t)=\chi_m z_{3,m-1}(x,t)+h\mathcal{L}_1^{-1}[\mathcal{R}_{1,m}(\boldsymbol{z}_{1,m-1},\boldsymbol{z}_{2,m-1},\boldsymbol{z}_{3,m-1})].
\end{aligned}\tag{20}$$

First, we consider the solution of Eq. (3) under the initial conditions

$$\begin{aligned}
u(x,0)&=\frac{3\sqrt{2}}{2}\mathrm{sech}(x)^2\cos\left(\frac{\sqrt{3}}{2}x\right),\\
v(x,0)&=\frac{3\sqrt{2}}{2}\mathrm{sech}(x)^2\sin\left(\frac{\sqrt{3}}{2}x\right),\\
w(x,0)&=3\mathrm{sech}(x)^2.
\end{aligned}\tag{21}$$

According to Eqs. (20) and (21), we now successively obtain

$$\begin{aligned}
z_{1,1}(x,t)&=\frac{3ht[13\sqrt{2}\sin\left(\frac{\sqrt{3}}{2}x\right)\cosh(x)-8\sqrt{6}\cos\left(\frac{\sqrt{3}}{2}x\right)\sinh(x)]}{4[\cosh(3x)+3\cosh(x)]},\\
z_{1,2}(x,t)&=\frac{3ht[13\sqrt{2}\cos\left(\frac{\sqrt{3}}{2}x\right)\cosh(x)+8\sqrt{6}\sin\left(\frac{\sqrt{3}}{2}x\right)\sinh(x)]}{4[\cosh(3x)+3\cosh(x)]},\\
z_{1,3}(x,t)&=-9t^2h[\frac{1}{4}\mathrm{sech}(x)^4-\frac{1}{2}\mathrm{sech}(x)^2+\mathrm{sech}(x)^2\tanh(x)^2],\\
z_{2,1}(x,t)&=\frac{9h^2t^3\sqrt{2}\sin\left(\frac{\sqrt{3}}{2}x\right)}{4096\cosh(x)^{15}}[51\cosh(5x)-\cosh(7x)+174\cosh(3x)\\
&\quad-5\cosh(9x)-\cosh(11x)+294\cosh(x)]+\ldots,\\
z_{2,2}(x,t)&=\frac{9h^2t^3\sqrt{2}\cos\left(\frac{\sqrt{3}}{2}x\right)}{4096\cosh(x)^{15}}[51\cosh(5x)-\cosh(7x)+174\cosh(3x)\\
&\quad-5\cosh(9x)-\cosh(11x)+294\cosh(x)]+\ldots,\\
z_{2,3}(x,t)&=-\frac{9h^2t^4}{512\cosh(x)^9}[-\cosh(7x)+31\cosh(5x)+9\cosh(3x)-135\cosh(x)]+\ldots,
\end{aligned}$$

$$
\begin{aligned}
z_{3,1}(x,t) = {} & \frac{27h^3t^5}{5}\sqrt{2}\sin\left(\frac{\sqrt{3}}{2}x\right)[70968\cosh(6x) + 198198 + 4376\cosh(8x) \\
& - 6963\cosh(12x) + 335556\cosh(2x) - 334\cosh(16x) + 197938\cosh(4x) \\
& - 11176\cosh(10x) - 18\cosh(18x) - 2114\cosh(14x) + \cosh(20x)]/[\cosh(24x) \\
& + 24\cosh(22x) + 276\cosh(20x) + 2024\cosh(18x) + 10626\cosh(16x) \\
& + 42504\cosh(14x) + 134596\cosh(12x) + 346104\cosh(10x) + 735471\cosh(8x) \\
& + 1307504\cosh(6x) + 1961256\cosh(4x) + 2496144\cosh(2x) + 1352078] + \ldots, \\
z_{3,2}(x,t) = {} & \frac{27h^3t^5}{5}\sqrt{2}\cos\left(\frac{\sqrt{3}}{2}x\right)[70968\cosh(6x) + 198198 + 4376\cosh(8x) \\
& - 6963\cosh(12x) - 334\cosh(16x) + 335556\cosh(2x) + 197938\cosh(4x) \\
& - 11176\cosh(10x) - 18\cosh(18x) - 2114\cosh(14x) + \cosh(20x)]/[\cosh(24x) \\
& + 24\cosh(22x) + 276\cosh(20x) + 2024\cosh(18x) + 10626\cosh(16x) \\
& + 42504\cosh(14x) + 134596\cosh(12x) + 346104\cosh(10x) + 735471\cosh(8x) \\
& + 1307504\cosh(6x) + 1961256\cosh(4x) + 2496144\cosh(2x) + 1352078] + \ldots, \\
z_{3,3}(x,t) = {} & \frac{9h^3t^6}{83886080\cosh(x)^{23}}[5582928\cosh(x) + 1037334\cosh(5x) \\
& - 71403\cosh(13x) - 335184\cosh(7x) + 177\cosh(19x) - 7472\cosh(15x) \\
& - \cosh(21x) - 270501\cosh(11x) + 3505866\cosh(3x) + 672\cosh(17x) \\
& - 529520\cosh(9x)] + \ldots, \\
& \vdots \quad \vdots \quad \vdots
\end{aligned}
$$

Finally, for the limitation of computer we obtain 4th-order approximate solution

$$
\begin{aligned}
u(x,t) &\cong z_{1,0}(x,t) + z_{1,1}(x,t) + z_{1,2}(x,t) + z_{1,3}(x,t), \\
v(x,t) &\cong z_{2,0}(x,t) + z_{2,1}(x,t) + z_{2,2}(x,t) + z_{2,3}(x,t), \\
w(x,t) &\cong z_{3,0}(x,t) + z_{3,1}(x,t) + z_{3,2}(x,t) + z_{3,3}(x,t).
\end{aligned}
$$

As pointed out by Liao,[32] in general, by means of the so-called h-curve, it is straightforward to choose a proper value of h which ensures that the solution series is convergent. In this way, it is found that our series converge when $h = -0.95$.

To examine the accuracy and reliablility of the HAM for the KGS equation, we compare the 4th-order approximate solution with an exact solution in Section 1. The following figures can help us to express the comparison between the 4th-order approximate solution and an exact solution in Section 1.

In Fig. 1, the absolute error for the KGS equation by the 4th-order HAM approximation of $|u_{\rm ham} - u_{\rm exact}|$, $|v_{\rm ham} - v_{\rm exact}|$ and $|w_{\rm ham} - w_{\rm exact}|$ are given when $h = -0.95$, $t = 0.2$ and the interval $8 \leq x \leq 13$. In Fig. 2, the exact distribution of $u(x,t)$ and the approximated distribution of $u(x,t)$ are plotted for the intervals $-4 \leq x \leq 8$ and $0 \leq t \leq 0.03$. In Fig. 3, the exact distribution of $v(x,t)$ and the approximated distribution of $v(x,t)$ are plotted for the same intervals. In Fig. 4, the exact distribution of $w(x,t)$ and the approximated distribution of $w(x,t)$ are plotted for the same intervals. Through these figures, we can see the approximate solution is very close to the exact solution.

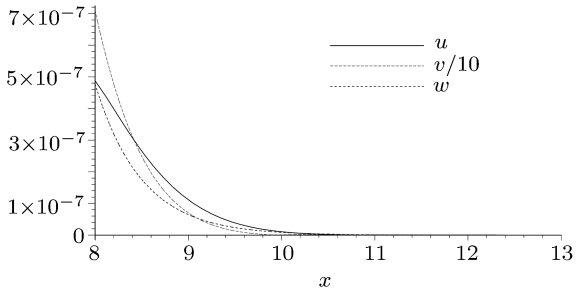


Fig. 1. The absolute error for the KGS equation by the 4th-order HAM approximation of $u(x,t)$, $v(x,t)$ and $w(x,t)$.

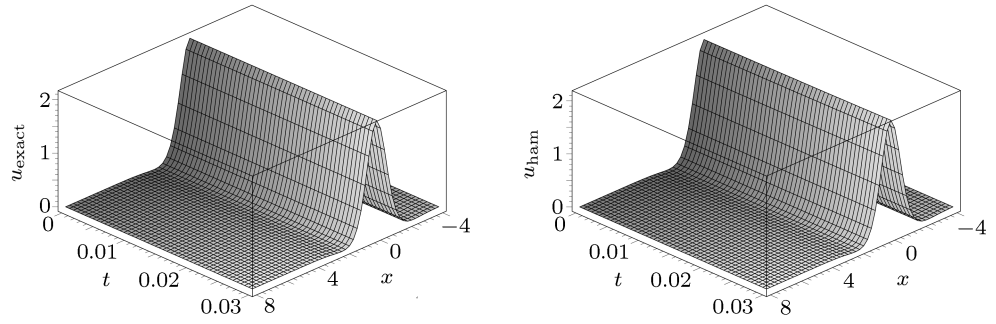


Fig. 2. Exact solution and 4th-order HAM approximation solution of $u(x,t)$.

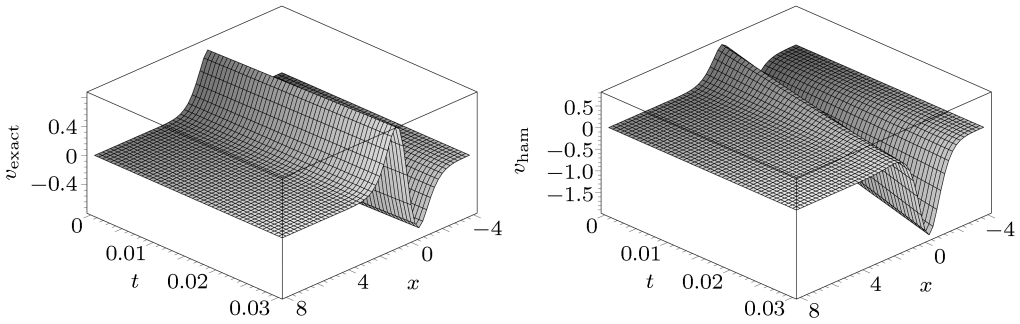


Fig. 3. Exact solution and 4th-order HAM approximation solution of $v(x,t)$.

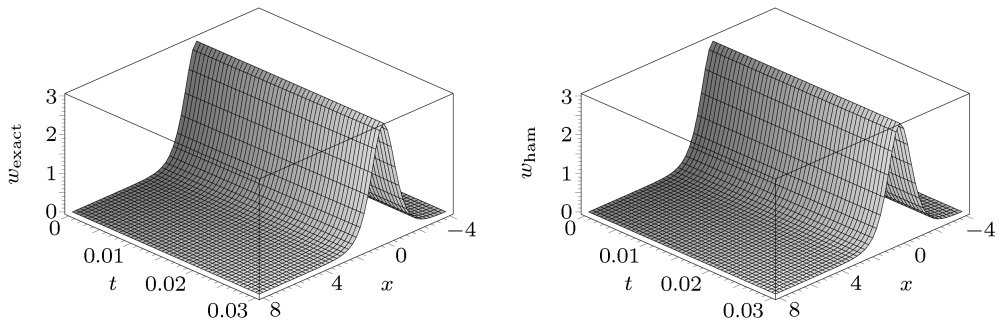


Fig. 4. Exact solution and 4th-order HAM approximation solution of $w(x,t)$.

4. Summary and discussion

In this paper, the HAM is applied to obtain the approximate solution of the KGS equation. HAM provides us with a convenient way to control the convergence of approximate series solution, which is a fundamental qualitative difference in analysis between HAM and other methods. To control the convergence of the solution, we can choose the proper values of h, in this paper we choose $h = -0.95$. From the absolute error between the approximate series solution and the exact solution of KGS equation and the 3D plots of the approximate series solution and the exact solution, we can see that our analysis solution agrees well with the exact solution.

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