A new approach of solving Green's function for wave propagation in an inhomogeneous absorbing medium*

Li Wei(李 维), Liu Shi-Bing(刘世炳)†, and Yang Wei(杨 巍)

Institute of Laser Engineering, Beijng University of Technology, Beijing 100124, *China*

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A new approach is developed to solve the Green's function that satisfies the Hehmholtz equation with complex refractive index. Especially, the Green's function for the Helmholtz equation can be expressed in terms of a one-dimensional integral, which can convert a Helmholtz equation into a Schrödinger equation with complex potential. And the Schrödinger equation can be solved by Feynman path integral. The result is in excellent agreement with the previous work.

Keywords: Green's function, wave propagation, Helmholtz equation, inhomogeneous absorbing medium

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1. Introduction

Wave propagation in an inhomogeneous medium is an important subject and has many applications,[1] such as wave propagation in the atmosphere and underwater. It is well-known that the Helmholtz equation often arises in the study of wave propagation problems. Until now, only a few analytic solutions of the Helmholtz equation are obtained for wave propagation in inhomogeneou medium. For example, Perkeris[2] found an analytic solution for a point source in a layered medium, where the refractive index is given by $n = z^{-1}$. Kormilitsin[3] and Holford[4] obtained solutions in media specified by $n = x^{1/2}$ and $n = \left(A + Bx + Cy + Dx^2 + Exy + Fy^2\right)^{1/2}$ respectively, in which the solutions were expressed as a onefold integral. Using the path integral[5,6] and other methods, Li *et al.*[7−9] found some solutions of the Helmholtz equation. All the approaches, however, are not applicable to study the wave propagation in an inhomogeneous absorbing medium, for the wave propagation in the media with real refractive index other than complex refractive index. Nevertheless, the wave propagation in the media with complex refractive index has many special properties, which has attracted extensive attention.[10−14]

Therefore, in this paper, a new approach is developed to solve the Green's function which satisfies the Hehmholtz equation with a complex refractive index. In particular, the Green's function for the Helmholtz equation can be expressed in terms of a onefold integral,[6,9] which can convert the Helmholtz equation into a Schrödinger equation with complex potential. Then Feyman path integral[15−19] is employed to solve the Schrödinger equation. Finally, we study a Green's function in an inhomogeneous absorbing medium, in which complex refractive index has a simple form. The result is compared to that in an inhomogeneous non-absorbing medium.[9]

2. Helmholtz equation of complex refractive index

In this paper, we only consider the one-dimensional inhomogeneous absorbing medium with a complex refractive index profile $n^2(x) = 1 + V(x)$, where $V(x)$ is a complex function in arbitrary form. The one-dimensional Green's function satisfies the Helmholtz equation

$$\frac{\partial^2}{\partial x^2} G(x, x') + k_0^2 n^2(x) G(x, x') = -\delta(x - x') \quad (1)$$

with $k_0 = \omega/c_0$, ω is the angular frequency of the incident light and c_0 is the velocity of the light in vacuum.

According to Refs. [6] and [9], the Green's function in Eq. (1) can be expressed as

$$G(x, x') = \frac{\mathrm{i}}{2k_0} \int_0^\infty G(x, x', t) \exp\left(\frac{\mathrm{i}\, k_0 t}{2}\right) \mathrm{d}t, \quad (2)$$

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†Corresponding author. E-mail: sbliu@bjut.edu.cn

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where t is only a variable of integration which does not have any physical meaning. Considering the initial condition and the radiation conditions

$$G(x,x',0)=\delta(x-x'), \tag{3}$$

$$G(\pm\infty,x',t)=0, \tag{4}$$

and substituting Eq. (2) into Eq. (1), a time-dependent Schrödinger-type equation is obtained as below[9]

$$\frac{\partial^2}{\partial x^2}G(x,x',t)+k_0^2V(x)G(x,x',t)+2\,\mathrm{i}\,k_0\frac{\partial G(x,x',t)}{\partial t}=0. \tag{5}$$

Multiplying this equation by $1/2k_0^2$, we find that

$$\frac{1}{2k_0^2}\frac{\partial^2}{\partial x^2}G(x,x',t)+\frac{V(x)}{2}G(x,x',t)+\mathrm{i}\,\frac{1}{k_0}\frac{\partial G(x,x',t)}{\partial t}=0. \tag{6}$$

Our research concentrates on analytic solution of the Helmholtz equation for light propagation in an inhomogeneous absorbing medium, in which the reciprocal of k_0 is much less than $1(1/k_0\ll 1)$. Then we let $\eta=1/k_0$, the Eq. (6) can be changed into a complete time-dependent Schrödinger equation

$$\frac{\eta^2}{2}\frac{\partial^2}{\partial x^2}G(x,x',t)+\frac{V(x)}{2}G(x,x',t)+\mathrm{i}\,\eta\frac{\partial G(x,x',t)}{\partial t}=0, \tag{7}$$

where η corresponds to constant $\hbar$ and the mass m is reduced into 1. It is noted that function $V(x)$ in Eq. (7) is complex, therefore Eq. (7) can be regarded as a Schrödinger equation with complex potential.

3. Schrödinger equation with complex potential

The solution of time-dependent Schrödinger equation with complex potential can be expressed in terms of a path integral. According to Refs. [18] and [19], the expression of Green's function which satisfies Eq. (7) is

$$G(x,x',t)=A_NF_N\exp\left[\frac{\mathrm{i}}{\eta}S^{(N)}(z)\right], \tag{8}$$

$$F_N=\int_{\Gamma_1}\mathrm{d}z_1\cdots\int_{\Gamma_{N-1}}\mathrm{d}z_{N-1}\exp\left(\frac{\mathrm{i}}{\eta}\Delta t\xi^T\sigma\xi\right), \tag{9}$$

where all the definitions of the variables can be found in Refs. [18] and [19]. Here we must pay special attention to the matrix $\boldsymbol{\sigma}$, which is given by[13,14]

$$\boldsymbol{\sigma}=\left(\frac{1}{2\Delta t^2}\right)J-\frac{1}{2}\boldsymbol{W}, \tag{10}$$

where $\boldsymbol{W}$ is

$$W_{ij}=\delta_{ij}\left(\partial^2V/\partial z^2\right)\big|_{z=Z_j} \tag{11}$$

and $J_{ii}=2, i=1,2,\ldots,N-1$; $J_{i,i+1}=-1, i=1,2,\ldots,N-2$; $J_{i-1,i}=-1, i=2,3,\ldots,N-1$; all other $J_{ij}=0$. There will be a problem that the matrix $\boldsymbol{\sigma}$ does not guarantee a complete set of eigenvectors for complex potentials and paths, thus Eq. (9) evaluated by the standard technique will not work. In order to avoid this issue, we choose complex function $V(x)$ which has the following form

$$V(x)=V_{\mathrm{R}}(x)-\mathrm{i}\,gV_{\mathrm{I}}(x), \tag{12}$$

where $V_{\mathrm{R}}(x)$ and $V_{\mathrm{I}}(x)$ are real functions, g is a non-negative real whose order of magnitude is close to η. Under this assumption, the Green's function in Eq. (7) can be determined completely by following equations[18,19]

$$G(x,x',t)=\left[\frac{1}{2\pi\,\mathrm{i}\,\eta f(t)}\right]^{1/2}\exp\left[\frac{\mathrm{i}}{\eta}S(Z(\cdot))\right], \tag{13}$$

$$\frac{\mathrm{d}^2f(\tau)}{\mathrm{d}\tau^2}+W(\tau)f(\tau)=0, \tag{14}$$

$$f(0)=0, \tag{15}$$

$$\left.\frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}\right|_{\tau=0}=0. \tag{16}$$

This approach can be applied to the light propagation in a quadratic continuous absorbing medium. In order to compare with a previous work,[9] complex function $V(x)$ is given by a quadratic form

$$V(x)=\Omega^2x^2-\mathrm{i}\,gx^2, \tag{17}$$

inserting Eq. (17) into Eq. (11) and combining Eqs. (14)–(16), the normalizing factor $f(t)$ in Eq. (13) can be obtained

$$f(t)=\frac{\sin(\sqrt{\Omega^2-\mathrm{i}\,g})}{\sqrt{\Omega^2-\mathrm{i}\,g}}, \tag{18}$$

and the action $S(Z(\cdot))$ is given by

$$S\left(Z\left(\cdot\right)\right)=\frac{\sqrt{\Omega^2-\mathrm{i}\,g}}{2\sin\left(t\sqrt{\Omega^2-\mathrm{i}\,g}\right)}\left[\left(x^2+x'^2\right)\cos\left(t\sqrt{\Omega^2-\mathrm{i}\,g}\right)-2xx'\right]. \tag{19}$$

In this case, the solution of the time-dependent Schrödinger equation with complex potential is

$$G\left(x,x',t\right)=\left[\frac{k_0\sqrt{\Omega^2-\mathrm{i}\,g}}{2\pi\,\mathrm{i}\sin(\sqrt{\Omega^2-\mathrm{i}\,g})}\right]^{1/2}\exp\left[\frac{\mathrm{i}\,k_0\sqrt{\Omega^2-\mathrm{i}\,g}}{2\sin\left(t\sqrt{\Omega^2-\mathrm{i}\,g}\right)}\left[\left(x^2+x'^2\right)\cos\left(t\sqrt{\Omega^2-\mathrm{i}\,g}\right)-2xx'\right]\right]. \tag{20}$$

Finally, taking Eq. (20) back to Eq. (2), we easily obtain the Green's function which satisfies the Helmholtz equation

$$G\left(x,x'\right)=\frac{\mathrm{i}}{2k_0}\int_0^\infty\left[\frac{k_0\sqrt{\Omega^2-\mathrm{i}\,g}}{2\pi\,\mathrm{i}\sin(\sqrt{\Omega^2-\mathrm{i}\,g})}\right]^{1/2}\times\exp\frac{\mathrm{i}\,k_0}{2}\left\{t+\frac{\sqrt{\Omega^2-\mathrm{i}\,g}}{\sin\left(t\sqrt{\Omega^2-\mathrm{i}\,g}\right)}\left[\left(x^2+x'^2\right)\cos\left(t\sqrt{\Omega^2-\mathrm{i}\,g}\right)-2xx'\right]\right\}\mathrm{d}t. \tag{21}$$

Therefore, the form of the light field at an arbitrary observation point x can be determined by the Eq. (21), which describes the propagation behaviour of light field in the quadratic continuous absorbing medium. A numerical solution of the Eq. (21) is planned for the future. In addition, it is also interesting to note that by setting $g=0$, the Green's function which satisfies Helmholtz equation of complex refractive index can be rewritten as

$$G\left(x,x'\right)=\frac{\mathrm{i}}{2k_0}\int_0^\infty\left[\frac{k_0\Omega}{2\pi\,\mathrm{i}\sin(\Omega t)}\right]^{1/2}\exp\frac{\mathrm{i}\,k_0}{2}\left\{t+\frac{\Omega}{\sin\left(\Omega t\right)}\left[\left(x^2+x'^2\right)\cos\left(\Omega t\right)-2xx'\right]\right\}\mathrm{d}t. \tag{22}$$

Thus the Green's function in Eq. (21) can be easily reduce to the Green's function for the inhomogeneous non-absorbing medium, which is consistent with Eq. (9) in Ref. [9].

4. Summary and conclusions

In this paper, a new approach is developed to solve the Green's function that satisfies the Helmholtz equation with a complex refractive index. To be specific, we first show that the Green's function for the Helmholtz equation in a general inhomogeneous absorbing medium can be expressed in terms of a onefold integral. Then the Green's function which satisfies the Helmholtz equation with a complex refractive index can be converted into a Schrödinger equation with complex potential. Subsequently, path integral approach is employed to solve the Schrödinger equation with complex potential. Finally we find that Green's function which satisfies the Helmholtz equation with complex refractive index. The result is in agreement with the previous work. It should be noted that our work is based upon $1/k_0 \ll 1$, which means that our approach is suitable only for solving Helmholtz equation of high frequency vibration wave, such as infrared light, x-ray, etc. Besides, we restrict our attention to purely absorptive potentials, thus $V_{\mathrm{I}}\left(x\right)$ in Eq. (12) must greater than or equal to 0. The applications along with generalizations of the present work to three-dimensions will be presented in the future.

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