A connection between the (G'/G)-expansion method and the truncated Painlevé expansion method and its application to the mKdV equation*

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Recently the (G'/G)-expansion method was proposed to find the traveling wave solutions of nonlinear evolution equations. This paper shows that the (G'/G)-expansion method is a special form of the truncated Painlevé expansion method by introducing an intermediate expansion method. Then the generalized (G'/G)–(G/G') expansion method is naturally derived from the standpoint of the nonstandard truncated Painlevé expansion. The application of the generalized method to the mKdV equation shows that it extends the range of exact solutions obtained by using the (G'/G)-expansion method.

Keywords: (G'/G)-expansion method, truncated Painlevé expansion method, mKdV equation, traveling wave solutions

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1. Introduction

During the past four decades or so, many powerful methods to construct exact solutions of nonlinear evolution equations have been established and developed.[1−3] Among them, the function expansion methods such as the truncated Painlevé expansion,[4−6] the tanh-function expansion and its various extension,[7,8] the Jacobi elliptic function expansion,[9] the F-expansion,[10−12] and the recently proposed (G'/G)-expansion method[13] are distinguished by directness, simplicity and effectivity of computation.

A natural question to ask is: does there exist a unified point of view of these various function expansion methods or what is the relationship among them? We make a small step in this direction. In this paper, we construct a connection between the (G'/G)-expansion method and the truncated Painlevé expansion method, which says that the (G'/G)-expansion method is a special form of the truncated Painlevé expansion method. With this in mind, the (G'/G)–(G/G') expansion method is naturally derived from the standpoint of the nonstandard truncated Painlevé expansion (we declare that the (G'/G)–(G/G') expansion method has been suggested by Wang from experiential knowledge).

The outline of this paper is as follows: in Section 2, we describe the (G'/G)-expansion method and the truncated Painlevé expansion method; in Section 3, we show that the (G'/G)-expansion method is a special form of the truncated Painlevé expansion method in two steps; as an application, in Section 4 we present the (G'/G)–(G/G') expansion method from the standpoint of the nonstandard truncated Painlevé expansion and apply it to the mKdV equation; in Section 5 some conclusions are given.

2. The methods

Suppose a nonlinear partial differential equation (PDE), say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, ...) = 0, \tag{1}$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In what follows we highlight the main features of the two methods which we focus on in this paper. More details can be found in Refs. [4–6] and [13].

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2.1. The (G'/G)-expansion method

Recently the pioneer work of Wang *et al.*[13] proposed the (G'/G)-expansion method which is proved to be powerful in finding traveling wave solutions of nonlinear evolution equations.[14−16] The main idea of the method is as follows. First, convert Eq. (1) into an ordinary differential equation (ODE) for $u = u(\xi)$

$$P(u, -\upsilon u', u', \upsilon^2 u'', -\upsilon u'', u'', \ldots) = 0, \tag{2}$$

by using a wave variable

$$\xi = x - \upsilon t, \tag{3}$$

where υ is the wave speed. Then suppose the solution of ODE (2) can be expressed by a polynomial in (G'/G)

$$u(\xi) = \sum_{k=0}^{m} \alpha_k \left(\frac{G'}{G}\right)^k, \quad \alpha_m \neq 0 \tag{4}$$

where $'$ denotes the derivative with respect to ξ (the same below), $G = G(\xi)$ satisfies a second order linear ODE (LODE) in the form

$$G'' + \lambda G' + \mu G = 0. \tag{5}$$

The degree m of Eq. (4) is determined by considering the homogeneous balance[17,18] between the highest order derivatives and nonlinear terms appearing in Eq. (2). Substituting Eq. (4) into Eq. (2) and using Eq. (5), and collecting all terms with the same order of (G'/G) together, we can convert the left-hand side of Eq. (2) into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero yields a set of algebraic equations determining the constants $\alpha_m, \ldots, \alpha_0, \upsilon, \lambda$ and μ. The traveling wave solutions of Eq. (1) can be obtained by substituting $\alpha_m, \ldots, \alpha_0, \upsilon, \lambda, \mu$ and the well known general solutions of Eq. (5) into Eq. (4).

2.2. The truncated Painlevé expansion method

The truncated Painlevé expansion method proposed by Conte[4] and developed by Pickering[5] and Lou[6] can be used to construct exact solutions for both integrable and non-integrable differential equations. For a given PDE, say Eq. (1), suppose all the solutions of Eq. (1) have an expansion of the form

$$u = \sum_{j=0}^{\infty} u_j \chi^{j-p}, \quad u_j \neq 0, \tag{6}$$

where $p \in Z^+$ is the leading order, $u_j = u_j(x,t)$ is the expansion coefficient, $\chi = \chi(x,t)$ is the expansion variable (different from the arbitrary singular manifold ϕ, see Refs. [4] and [19]). The gradient of χ is given by

$$\chi_x = \sum_{j=0}^{N} S_j \chi^j, \quad \chi_t = \sum_{j=0}^{N} T_j \chi^j, \tag{7}$$

where $S_j = S_j(x,t)$ and $T_j = T_j(x,t)$ are functions to be determined later. The main idea of the truncated Painlevé expansion method is to seek a finite expansion of u, more precisely, truncate Eq. (6) at the constant level (called standard truncated Painlevé expansion)

$$u = \sum_{j=0}^{p} u_j \chi^{j-p}, \tag{8}$$

or at a higher level (called nonstandard truncated Painlevé expansion)

$$u = \sum_{j=0}^{p+\overline{p}} u_j \chi^{j-p}, \tag{9}$$

where p is determined from leading order analysis[20] of Eq. (1), $\overline{p} \in Z^+$ is determined by Eqs. (1) and (7) (see Refs. [5] and [6]). Substituting Eq. (8) (or Eq. (9)) into Eq. (1) and using Eq. (7), collecting all terms with the same order of χ together, we can convert the left-hand side of Eq. (1) into another plolynomial in χ. Equating each coefficient of this polynomial to zero, we can got a set of PDEs (called Painlevé–Darboux equations[19]) determining the expansion coefficients u_j, S_j and T_j. The exact solutions of Eq. (1) can be obtained by substituting u_j and the solutions of Eq. (7) into Eq. (8) (or Eq. (9)).

3. The connection between the methods

In order to show that the (G'/G)-expansion method is a special form of the truncated Painlevé expansion method, we first construct an intermediate expansion method which is almost the same as the (G'/G)-expansion method except that the ansatz (4) is changed into

$$u(\xi) = \sum_{k=0}^{m} \alpha_k Z^k, \quad \alpha_m \neq 0, \tag{10}$$

where $Z = Z(\xi)$ satisfies a Riccati equation

$$Z' + Z^2 + \lambda Z + \mu = 0, \tag{11}$$

instead of LODE (5). Of course the remaining parts concerning (G'/G) must change into Z accordingly. Let us call it the Riccati expansion method.

Proposition 1 The (G'/G)-expansion method is equivalent to the above Riccati expansion method.

Proof Note that the transformation $Z = G'/G$ linearizes Eq. (11) into Eq. (5), meanwhile Eq. (11) can be recovered from Eq. (5) under the transformation $G = \mathrm{e}^{\int^{\xi} Z(s)\,\mathrm{d}s}$. So if a solution of Eq. (2) has a (G'/G)-expansion form Eq. (4), then it correspondingly has a Riccati expansion form Eq. (10) and vise versa. In addition, from the construction of the Riccati expansion method, the coefficients α_k and m of the two expansions shall be determined by the same set of algebraic equations. Hence, the sets of traveling wave solutions obtained respectively by the (G'/G)-expansion method and the Riccati expansion method are identically the same. This completes the proof.

Now we only need to show that the Riccati expansion method is a special form of the truncated Painlevé expansion method. In fact, we have the following result:

Proposition 2 The Riccati expansion method is equivalent to the standard truncated Painlevé expansion method when

$$\begin{aligned} &N = 2, S_0 = 1, S_1 = \lambda, S_2 = \mu, \\ &T_i = -\upsilon S_i, \quad i = 0, 1, 2, \end{aligned} \tag{12}$$

in Eq. (7), and u_j, $j = 0, \ldots, p$, are constants in Eq. (8).

Proof Under the condition Eq. (12), Eq. (7) becomes

$$\chi_x = 1 + \lambda\chi + \mu\chi^2, \quad \chi_t = -\upsilon\chi_x. \tag{13}$$

By using the wave variable $\xi = x - \upsilon t$, we can incorporate Eq. (13) into

$$\chi' = 1 + \lambda\chi + \mu\chi^2, \tag{14}$$

which is equivalent to Eq. (11) via the transformation $\chi = Z^{-1}$. In addition, though p in Eq. (8) is determined by the leading order analysis while m in Eq. (10) by the homogeneous balance, actually p is equal to m (see Ref. [21]). Note also that when S_i, T_i and u_j are constants the Painlevé–Darboux equations for S_i, T_i and u_j become algebraic equations which are just the determining equations for λ, μ and α_k in the (G'/G)-expansion method. At this point, Eq. (8) is precisely Eq. (10). That is, the exact solutions obtained by using the Riccati expansion method can also be obtained by using the standard truncated Painlevé expansion method in the above conditions and vise versa. The proof is complete.

From propositions 1 and 2, we draw the expected conclusion.

Proposition 3 The (G'/G)-expansion method is a special form of the truncated Painlevé expansion method.

4. The (G'/G)–(G/G') expansion method and its application

Since the (G'/G)-expansion method is a special form of the truncated Painlevé expansion method, the existence of the nonstandard truncated Painlevé expansion[5,6] for some solutions of the nonlinear differential equations suggests the possibility of the (G'/G)–(G/G') expansion method, which will probably extend the range of exact solutions we are able to obtain by using the the (G'/G)-expansion method.

In fact, when χ satisfies Eq. (13), $\overline{p}$ must equal p (see Ref. [5]) in the nonstandard truncated Painlevé expansion Eq. (9), that is, u has the following finite expansion

$$u = \sum_{j=0}^{2p} u_j \chi^{j-p}. \tag{15}$$

From the course of proof of propositions 1 and 2 we can seek the solutions of ODE (2) in a polynomial of (G'/G)

$$\begin{aligned} u(\xi) &= \sum_{j=0}^{2p} u_j \left(\frac{G'}{G}\right)^{j-p}, \quad \text{or equivalently} \\ u(\xi) &= \sum_{k=0}^{m} \alpha_k \left(\frac{G'}{G}\right)^{k} + \sum_{k=1}^{m} \beta_k \left(\frac{G}{G'}\right)^{k}, \ \alpha_m \neq 0, \end{aligned} \tag{16}$$

where G satisfies the same LODE (5), α_k and β_k are expansion coefficients to be determined similarly as that in the (G'/G)-expansion method. Now let us take an example.

Consider the mKdV equation[13,22]

$$u_t - u^2 u_x + \delta u_{xxx} = 0, \quad \delta > 0. \tag{17}$$

Using the wave variable Eq. (3) and integrating once, we can convert Eq. (17) into an ODE for $u = u(\xi)$

$$C - \upsilon u - \frac{1}{3}u^3 + \delta u'' = 0, \tag{18}$$

where C is an integration constant to be determined later. Considering the homogeneous balance between u'' and u^3 in Eq. (18) ($m+2 = 3m \Rightarrow m = 1$), we can suppose that the solution of Eq. (18) is of the form

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0 + \beta_1 \left(\frac{G}{G'}\right), \quad \alpha_1 \neq 0, \tag{19}$$

where $G = G(\xi)$ satisfies the second order LODE (5), $\alpha_1, \alpha_0, \beta_1$, λ and μ are to be determined later.

Using Eqs. (5) and (19) we get

$$\begin{aligned} u'' =\ & 2\alpha_1 \left(\frac{G'}{G}\right)^3 + 3\alpha_1\lambda \left(\frac{G'}{G}\right)^2 + (\alpha_1\lambda^2 + 2\alpha_1\mu)\left(\frac{G'}{G}\right) + \alpha_1\lambda\mu + \beta_1\lambda \\ & + (\beta_1\lambda^2 + 2\beta_1\mu)\left(\frac{G}{G'}\right) + 3\beta_1\lambda\mu \left(\frac{G}{G'}\right)^2 + 2\beta_1\mu^2 \left(\frac{G}{G'}\right)^3, \end{aligned} \tag{20}$$

$$\begin{aligned} u^3 =\ & \alpha_1^3 \left(\frac{G'}{G}\right)^3 + 3\alpha_1^2\alpha_0 \left(\frac{G'}{G}\right)^2 + (3\alpha_1\alpha_0^2 + 3\alpha_1^2\beta_1)\left(\frac{G'}{G}\right) + \alpha_0^3 + 6\alpha_1\alpha_0\beta_1 \\ & + (3\alpha_0^2\beta_1 + 3\alpha_1\beta_1^2)\left(\frac{G}{G'}\right) + 3\alpha_0\beta_1^2 \left(\frac{G}{G'}\right)^2 + \beta_1^3 \left(\frac{G}{G'}\right)^3. \end{aligned} \tag{21}$$

Substituting Eqs. (19–21) into Eq. (18) and collecting all terms with the same power of (G'/G) together, we can convert the left-hand side of Eq. (18) into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_1, \alpha_0, \beta_1, \upsilon, \lambda, \mu$ and C as follows:

$$\begin{aligned} & 2\delta\beta_1\mu^2 - \frac{1}{3}\beta_1^3 = 0, \quad -\alpha_0\beta_1^2 + 3\delta\beta_1\lambda\mu = 0, \\ & -\upsilon\beta_1 - \alpha_0^2\beta_1 - \alpha_1\beta_1^2 + \delta(2\beta_1\mu + \beta_1\lambda^2) = 0, \\ & -\frac{1}{3}\alpha_0^2 - 2\alpha_1\alpha_0\beta_1 + C - \upsilon\alpha_0 + \delta(\beta_1\lambda + \alpha_1\lambda\mu) = 0, \\ & \delta(\alpha_1\lambda^2 + 2\alpha_1\mu) - \alpha_1^2\beta_1 - \alpha_1\alpha_0^2 - \upsilon\alpha_1 = 0, \\ & -\alpha_1^2\alpha_0 + 3\delta\alpha_1\lambda = 0, \quad -\frac{1}{3}\alpha_1^3 + 2\delta\alpha_1 = 0. \end{aligned}$$

Solving them by Maple we have the following sets of solutions

$$\begin{aligned} & \upsilon = 2\delta\mu - \frac{1}{2}\delta\lambda^2, \quad C = 0, \\ & \alpha_0 = \pm\frac{1}{2}\lambda\sqrt{6\delta}, \quad \alpha_1 = \pm\sqrt{6\delta}, \quad \beta_1 = 0, \end{aligned} \tag{22}$$

and

$$\begin{aligned} & \upsilon = -4\delta\mu - \frac{1}{2}\delta\lambda^2, \quad C = \pm 2\delta\sqrt{6\delta}\lambda\mu, \\ & \alpha_0 = \pm\frac{1}{2}\lambda\sqrt{6\delta}, \quad \alpha_1 = \pm\sqrt{6\delta}, \\ & \beta_1 = \pm\mu\sqrt{6\delta}, \end{aligned} \tag{23}$$

where λ and μ are free.

Note that when $\beta_1 = 0$, Eq. (19) degenerates into

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \tag{24}$$

which is the (G'/G)-expansion of $u(\xi)$. We do not list the exact solutions of the mKdV equation (17) resulting from Eq. (22) here, since they are the same as that obtained by using the (G'/G)-expansion method.

Substituting Eq. (23) into Eq. (19) yields

$$\begin{aligned} & u(\xi) = \pm\sqrt{6\delta}\left(\frac{G'}{G}\right) \pm \frac{1}{2}\lambda\sqrt{6\delta} \pm \mu\sqrt{6\delta}\left(\frac{G}{G'}\right), \\ & \xi = x + \left(\frac{1}{2}\delta\lambda^2 + 4\delta\mu\right)t, \end{aligned} \tag{25}$$

which is the formula of a solution of Eq. (18), provided that the integration constant C in Eq. (18) is taken as that in Eq. (23).

Substituting the general solutions of Eq. (5) into Eq. (25), we have three types of traveling wave solutions of the mKdV equation (17) as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_{1,2}(\xi) = \pm\frac{1}{2}\sqrt{6\delta}\Big[(c_1^2k^2 - c_2^2k^2 - c_1^2\lambda^2 + c_1c_2\lambda k) + (c_1^2\lambda k + c_2^2\lambda k - 2c_1c_2\lambda^2)\tanh\left(\frac{1}{2}k\xi\right)$$

$$-(c_1^2k^2-c_2^2k^2+c_2^2\lambda^2-c_1c_2\lambda k)\tanh^2\left(\frac{1}{2}k\xi\right)\Big]\Big/\Big[(c_1^2\lambda-c_1c_2k)$$
$$-(c_1^2k+c_2^2k-2c_1c_2\lambda)\tanh\left(\frac{1}{2}k\xi\right)+(c_2^2\lambda-c_1c_2k)\tanh^2\left(\frac{1}{2}k\xi\right)\Big], \quad (26)$$

where $\xi = x+(\delta\lambda^2/2+4\delta\mu)t$, $k=\sqrt{\lambda^2-4\mu}$, c_1 and c_2 are arbitrary constants.

When $\lambda^2-4\mu<0$,

$$u_{3,4}(\xi) = \pm\frac{1}{2}\sqrt{6\delta}\Big[(c_1^2k^2+c_2^2k^2+c_2^2\lambda^2+c_1c_2\lambda k)-(c_1^2\lambda k-c_2^2\lambda k-2c_1c_2\lambda^2)\tan\left(\frac{1}{2}k\xi\right)$$
$$+(c_1^2k^2+c_2^2k^2+c_1^2\lambda^2+c_1c_2\lambda k)\tan^2\left(\frac{1}{2}k\xi\right)\Big]\Big/\Big[(c_2^2\lambda-c_1c_2k)$$
$$+(c_2^2k-c_1^2k+2c_1c_2\lambda)\tan\left(\frac{1}{2}k\xi\right)+(c_1^2\lambda+c_1c_2k)\tan^2\left(\frac{1}{2}k\xi\right)\Big], \quad (27)$$

where $\xi = x+(\delta\lambda^2/2+4\delta\mu)t$, $k=\sqrt{4\mu-\lambda^2}$, c_1 and c_2 are arbitrary constants.

When $\lambda^2-4\mu=0$,

$$u_{5,6}(\xi) = \pm\frac{1}{2}\sqrt{6\delta}\left[\lambda+\frac{4c_2^2}{(c_1^2\lambda-2c_1c_2)+(2c_1c_2\lambda-2c_2^2)\xi+c_2^2\lambda\xi^2}\right], \quad (28)$$

where $\xi = x+3/2\delta\lambda^2t$, c_1 and c_2 are arbitrary constants.

We point out that these exact solutions cannot be obtained by using the (G'/G)-expansion method. Hence, the (G'/G)–(G/G') expansion method indeed extends the range of exact solutions which we are able to obtain by using the (G'/G)-expansion method; meanwhile, it inherits the advantages of the (G'/G)-expansion method: direct, concise, elementary and effective.

5. Conclusion

In our philosophy of finding traveling wave solutions of nonlinear evolution equations, some of the function expansion methods can be considered as a special form of the truncated Painlevé expansion method (standard and nonstandard). In this paper, we show that the recently proposed (G'/G)-expansion method is a special form of the Painlevé truncated expansion method. Then we naturally present the (G'/G)–(G/G') expansion method from the standpoint of the nonstandard truncated Painlevé expansion. An application to the mKdV equation shows that this generalized method extends the range of exact solutions which can be obtained by using the (G'/G)-expansion method.

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