Mei symmetry and Mei conserved quantity of Nielsen equations for a non-holonomic system of Chetaev's type with variable mass*

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Mei symmetry and Mei conserved quantity of Nielsen equations for a non-holonomic, non-conservative system of Chetaev's type with variable mass are studied. The differential equations of motion of the Nielsen equation for the system, the definition and criterion of Mei symmetry, and the condition and the form of Mei conserved quantity deduced directly by Mei symmetry for the system are obtained. An example is given to illustrate the application of the results.

Keywords: variable mass, Nielsen equation, Mei symmetry, Mei conserved quantity

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1. Introduction

In 1918, Noether A E, a German scientist, revealed the internal relation between the symmetry and the conserved quantity.[1] However, the scientific value of Noether theory was not found until the 1970s. From then on, the study on both symmetry and conserved quantity has flourished and is therefore producing plentiful and substantial outcomes.[2−8] The research on both symmetries and conserved quantities of a constrained mechanical system is one of the very important studying directions in analytical mechanical fields. In recent years, the studies regarding both symmetries and conserved quantities of constraint mechanical systems have also flourished.[9−20] Nielsen style is one of the three important mechanical systems and occupies an important position in analytical mechanics.[21] At present, the studies regarding symmetry and conserved quantity of Nielsen equations have also made some progress. In Ref. [22], Mei symmetry of Nielsen equations are studied, the results were extended to the non-conserved systems in Ref. [23] and continued to extend to the variable mass non-holonomic systems in Refs. [24] and [25]. In Refs. [26] and [27], Mei symmetry and Mei conserved quantity of Nielsen equations or non-holonomic systems of unilateral Chetaev's type and non-holonomic systems of non-Chetaev's type are studied respectively.

This paper adopts total derivatives of the function along the trajectory of the system with respect to t. The definition and criterion of Mei symmetry of Nielsen equations for a non-holonomic system of Chetaev's type with variable mass are given. In addition, the existence condition and the expression of the Mei conserved quantity of Nielsen equations for a non-holonomic system of Chetaev's type with variable mass are deduced directly by Mei symmetry by means of the study of Mei symmetry. Finally, an example is given to illustrate the application of the results.

2. Differential equations of motion for the system

Assume that a mechanical system consists of N particles with masses m_i at the moment t, and at the moment $t + \mathrm{d}t$, the infinitesimal element of mass to which a particle is separating to or merging into is $\mathrm{d}m_i$. Suppose that the configuration of the mechanical system is determined by n generalized coordinates $q_s (s = 1, \ldots, n)$, and it is subjected to the g ideal double-faced non-holonomic constraints of Chetaev's

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type.

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0,\quad(\beta=1,\ldots,g),\tag{1}$$

and consider the constraints are similar and independent each other. The Chetaev condition under which the constraints (1) impose on virtual displacement is

$$\frac{\partial f_\beta}{\partial\dot{q}_s}\delta q_s=0,\quad(\beta=1,\ldots,g).\tag{2}$$

The differential equations of the system is

$$\frac{\partial}{\partial\dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}-2\frac{\partial L}{\partial q_s}=Q_s+P_s+\lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_s}\quad(s=1,\ldots,n).\tag{3}$$

Where

$$P_s=\dot{m}_i(\boldsymbol{u}_i+\dot{\boldsymbol{r}}_i)\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_s}-\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial q_s}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial\dot{q}_s}\right)\tag{4}$$

are generalized thrust. $\boldsymbol{u}_i$ is the relative velocity for the infinitesimal element of mass relatively to the ith particle, $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ are position vector and speed of the ith particle respectively. $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is a Lagrangian function of the system, $Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is a non-potential generalized force, λ_β is a Lagrange multiplier. Equation (3) is also called Nielsen equation for a non-holonomic non-conservative system of Chetaev's type with variable mass. Introduce Mei arithmetic operators

$$N_s=\frac{\partial}{\partial\dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}-2\frac{\partial}{\partial q_s},\quad(s=1,\ldots,n).\tag{5}$$

Then Eq. (3) can be written as

$$N_s(L)=Q_s+P_s+\lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_s},\quad(s=1,\ldots,n).\tag{6}$$

Let the system be nonsingular, we can derive all multipliers $\lambda_\beta=\lambda_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ from Eqs. (1) and (6). So Eq. (6) can be rewritten as

$$N_s(L)=Q_s+P_s+\Lambda_s,\quad(s=1,\ldots,n).\tag{7}$$

Where

$$\Lambda_s{=}\Lambda_s(t,q_s,\dot{q}_s)=\lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_s},\quad(s=1,\ldots,n)\tag{8}$$

is called the generalized non-holonomic constrained counterforce. Equation (7) is the equation of holonomic non-conservative system of Chetaev's type corresponding to Eqs. (1) and (3). The motion of non-holonomic non-conservative system of Chetaev's type with variable mass can be found from the solution of Eq. (7). As long as original condition of the motion satisfies the condition Eq. (1), then the solution of Eq. (7) would provide the motion of the system Eqs. (1) and (3). From Eq. (7), all generalized accelerations can be solved:

$$\ddot{q}_s=\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad(s=1,\ldots,n).\tag{9}$$

3. The definition of Mei symmetry

Introducing the infinitesimal transformation functions of time and generalized coordinates

$$t^*=t+\Delta t,\ q_s^*(t^*)=q_s(t)+\Delta q_s,\quad(s=1,\ldots,n),\tag{10}$$

the transformation (10) can be extended to

$$\begin{aligned}&t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\&q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad(s=1,\ldots,n),\end{aligned}\tag{11}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are infinitesimal generators.

Introducing the infinitesimal generators vector $X^{(0)}$ and its extension along the trajectory of a curve of the system as

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s},\tag{12}$$

$$\tilde{X}^{(1)}=X^{(0)}+\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-\dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial\dot{q}_s}.\tag{13}$$

We adopt the total derivative along the trajectory of the system with respect to t in Eq. (13)

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+\alpha_s\frac{\partial}{\partial\dot{q}_s}.\tag{14}$$

After transformations (11), the functions L, Q_s, P_s, Λ_s and f_β become L^*, Q_s^*, P_s^*, Λ_s^* and f_β^* respectively. Taking the Taylor expansions of L^*, Q_s^*, P_s^*, Λ_s^* and f_β^* at a point of $(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ along the trajectory of motion, we have

$$L^*=L\left(t^*,\boldsymbol{q}^*,\frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}(L)+O(\varepsilon^2),$$

$$Q_s^* = Q_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{X}^{(1)}(Q_s) + O(\varepsilon^2), \quad (s = 1, \ldots, n),$$
$$P_s^* = P_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = P_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{X}^{(1)}(P_s) + O(\varepsilon^2), \quad (s = 1, \ldots, n),$$
$$\Lambda_s^* = \Lambda_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{X}^{(1)}(\Lambda_s) + O(\varepsilon^2), \quad (s = 1, \ldots, n),$$
$$f_\beta^* = f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{X}^{(1)}(f_\beta) + O(\varepsilon^2), \quad (\beta = 1, \ldots, g). \tag{15}$$

Definition 1 If Eq. (7) keeps its form invariant when dynamic functions L, Q_s, P_s and Λ_s are substituted by the functions L^*, Q_s^*, P_s^* and Λ_s^* under the infinitesimal transformations Eq. (11), that is

$$N_s(L^*) = Q_s^* + P_s^* + \Lambda_s^*, \quad (s = 1, \ldots, n). \tag{16}$$

Then the invariance is called Mei symmetry of Nielsen equations for holonomic non-conservative systems of Chetaev's type with variable mass corresponding to the same non-holonomic systems.

Definition 2 If Eqs. (1) and (7) keep their forms invariant when dynamic functions L, Q_s, P_s and Λ_s are substituted by the functions L^*, Q_s^*, P_s^* and Λ_s^* under the infinitesimal transformations Eq. (11), that is

$$f_\beta^* = f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = 0, \quad (\beta = 1, \ldots, g), \tag{17}$$

and Eq. (16) is also satisfied, then the invariance is called the loose Mei symmetry of the system (1) and (3).

Definition 3 If Eqs. (1) and (7) keep their forms invariant when dynamic functions L, Q_s, P_s and Λ_s are substituted by the functions L^*, Q_s^*, P_s^* and Λ_s^* under the infinitesimal transformations Eq. (11), and the additional restriction equation of the infinitesimal generators ξ_0, ξ_s

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0, \quad (\beta = 1, \ldots, g) \tag{18}$$

is satisfied, then the invariance is called strictly Mei symmetry of the systems (1) and (3).

4. Criterion of Mei symmetry

Substituting the first to fourth equations of Eq. (15) into Eq. (16) and neglecting terms that are $O(\varepsilon^2)$ or higher, noting Eq. (7), we have

$$N_s[\tilde{X}^{(1)}(L)] = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(P_s) + \tilde{X}^{(1)}(\Lambda_s), \quad (s = 1, \ldots, n). \tag{19}$$

Finding the total derivative of the dynamic function along the trajectory of the system with respect to t by means of the Mei operators, hence the Mei operators of the system may be changed by Eq. (5) as follows.

$$\tilde{N}_s = \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s}, \quad (s = 1, \ldots, n), \tag{20}$$

where $\tilde{N}_s$ is called the generalized Mei operator. Hence, we have

$$\tilde{N}_s[\tilde{X}^{(1)}(L)] = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(P_s) + \tilde{X}^{(1)}(\Lambda_s), \quad (s = 1, \ldots, n). \tag{21}$$

Substituting the fifth equation of Eq. (15) into Eq. (17), neglecting terms that are of the order of $O(\varepsilon^2)$ or higher orders and using Eq. (1), we have

$$\tilde{X}^{(1)}(f_\beta) = 0, \quad (\beta = 1, \ldots, g). \tag{22}$$

Hence, we have:

Criterion 1 For a holonomic nonconservative system (7) with variable mass corresponding to Nielsen equations (1) and (3) for a non-holonomic, non-conservative system of Chetaev's type with variable mass, if the infinitesimal generators ξ_0, ξ_s under the infinitesimal transformations (11) satisfy Eq. (21), then corresponding symmetry is called Mei symmetry of the system.

Equation (21) is called the criterion equations of the Mei symmetry for a holonomic non-conservative with variable mass corresponding to Nielsen equation for a non-holonomic, non-conservative system of Chetaev's type with variable mass.

Criterion 2 For Nielsen equations (1) and (3) for a non-holonomic, non-conservative system of Chetaev's type with variable mass, if the infinitesimal generators ξ_0, ξ_s under the infinitesimal transformations (11) satisfy Eqs. (21) and (22), then the relative symmetry is called the loose Mei symmetry of the system.

Equations (21) and (22) are called criterion equations and constraint equations of the Mei symmetry of Nielsen equation for a non-holonomic, non-conservative system of Chetaev's type with variable mass respectively.

Criterion 3 For Nielsen equations (1), (3) for a non-holonomic, non-conservative system of Chetaev's type with variable mass, if the infinitesimal generators ξ_0, ξ_s under the infinitesimal transformations (11) satisfy Eqs. (18), (21) and (22), then the relative symmetry is called the strict Mei symmetry of the system.

Equation (18) is called the additional restrictive equation of the Mei symmetry of Nielsen equations (1), (3) for a non-holonomic, non-conservative system of Chetaev type with variable mass.

5. Mei conserved quantity deduced from Mei symmetry

The condition and the expression of Mei conserved quantity can be directly deduced from Mei symmetry of Nielsen equations (1), (3) for a non-singular non-holonomic non-conservative system of Chetaev type with variable mass under following propositions.

Proposition 1 If the infinitesimal generators ξ_0, ξ_s and the gauge function $G_{\rm M} = G_{\rm M}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Mei symmetry i.e. the loose Mei symmetry and the strict Mei symmetry of the Nielsen equations for a non-singular non-holonomic non-conservative system of Chetaev's type with variable mass satisfy the following structural equation:

$$\tilde{X}^{(1)}(L)\frac{\bar{\rm d}}{{\rm d}t}\xi_0 + \tilde{X}^{(1)}[\tilde{X}^{(1)}(L)] + \tilde{X}^{(1)}(Q_s + P_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\rm d}}{{\rm d}t}G_{\rm M} = 0, \tag{23}$$

then the Mei conserved quantities deduced from the Mei symmetries are expressed as follows:

$$I_{\rm M} = \tilde{X}^{(1)}(L)\xi_0 + \frac{\partial\tilde{X}^{(1)}(L)}{\partial\dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\rm M} = \text{const.} \tag{24}$$

Proof

$$\begin{aligned}
\frac{\bar{\rm d}}{{\rm d}t}I_{\rm M} =& \left[\frac{\partial}{\partial t}\tilde{X}^{(1)}(L) + \dot{q}_s\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L) + \alpha_s\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\right]\xi_0 + \tilde{X}^{(1)}(L)\frac{\bar{\rm d}}{{\rm d}t}\xi_0 \\
&+\left[\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\right](\xi_s - \dot{q}_s\xi_0) + \frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\left[\frac{\bar{\rm d}}{{\rm d}t}\xi_s - \alpha_s\xi_0 - \dot{q}_s\frac{\bar{\rm d}}{{\rm d}t}\xi_0\right] + \frac{\bar{\rm d}}{{\rm d}t}G_{\rm M} \\
=& \left[\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L) - \frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L) - \tilde{X}^{(1)}(Q_s + P_s + \Lambda_s)\right](\xi_s - \dot{q}_s\xi_0) \\
=& \left[\frac{\partial}{\partial\dot{q}_s}\frac{\bar{\rm d}}{{\rm d}t}\tilde{X}^{(1)}(L) - 2\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L) - \tilde{X}^{(1)}(Q_s + P_s + \Lambda_s)\right](\xi_s - \dot{q}_s\xi_0).
\end{aligned}$$

Substituting criterion equation (21) into the equation above, we obtain

$$\frac{\bar{\rm d}}{{\rm d}t}I_{\rm M} = 0.$$

6. Illustrative example

Consider a non-holonomic constraint system with variable mass

$$L = \frac{1}{2}m(t)(\dot{q}_1^2 + \dot{q}_2^2), \tag{25}$$

where

$$m(t) = m_0 t. \tag{26}$$

The generalized non-potential forces are

$$Q_1 = m_0\dot{q}_1, \quad Q_2 = m_0\dot{q}_2. \tag{27}$$

The non-holonomic constraint forces of the system are

$$f = t\dot{q}_1 - \dot{q}_2 - q_1 + t. \tag{28}$$

The absolute velocity at which the infinitesimal element of mass is separating

$$\boldsymbol{u} = -\dot{\boldsymbol{r}} = -(\dot{q}_1\boldsymbol{i} + \dot{q}_2\boldsymbol{j}). \tag{29}$$

Try to study the Mei conserved quantity deduced directly by Mei symmetry of Nielsen equation for a

non-holonomic system of Chetaev's type with variable mass. By using Eq. (4), we have

$$P_1 = P_2 = 0. \tag{30}$$

From Eq. (3), we obtain

$$\ddot{q}_1 = \frac{\lambda}{m_0}, \quad \ddot{q}_2 = -\frac{\lambda}{m_0 t}. \tag{31}$$

Combined Eqs. (28) and (31), we achieve

$$\lambda = -\frac{m_0 t}{1+t^2}. \tag{32}$$

Substituting Eq. (32) into Eq. (31), the differential equations of motion can be obtained

$$\ddot{q}_1 = \alpha_1 = -\frac{t}{1+t^2}, \quad \ddot{q}_2 = \alpha_2 = \frac{1}{1+t^2}. \tag{33}$$

Taking the infinitesimal generators

$$\xi_0 = 0, \quad \xi_1 = t, \quad \xi_2 = 1. \tag{34}$$

Then we achieve

$$\begin{aligned} &\tilde{X}^{(1)}(L) = m_0 t\dot{q}_1, \\ &N_1[\tilde{X}^{(1)}(L)] = m_0, \ N_2[\tilde{X}^{(1)}(L)] = 0, \\ &\tilde{X}^{(1)}(Q_1 + P_1 + \Lambda_1) = m_0, \\ &\tilde{X}^{(1)}(Q_2 + P_2 + \Lambda_2) = 0, \\ &\tilde{X}^{(1)}[\tilde{X}^{(1)}(L)] = m_0 t. \end{aligned} \tag{35}$$

Using Eq. (35), we can easily verify that the criterion equation (21) and the constraint equation (22) are tenable. But the additional restrictive equation (18) is not existent. By virtue of Criterion 2, we know that the infinitesimal generators in Eq. (34) are of the loose Mei symmetry of Nielsen equation for a non-holonomic system of Chetaev's type with variable mass. From the structural equation (23), we achieve

$$G_{\rm M} = \dot{q}_1 + t\dot{q}_2 - q_2 - m_0 t^2. \tag{36}$$

By using Eq. (24), we obtain the Mei conserved quantity deduced directly by Mei symmetry of Nielsen equation for a non-holonomic system of Chetaev's type with variable mass:

$$I_{\rm M} = \dot{q}_1 + t\dot{q}_2 - q_2 = \text{const.} \tag{37}$$

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