Mei symmetry and Mei conserved quantity of Appell equations for a variable mass holonomic system*

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Mei symmetry and Mei conserved quantity of Appell equations for a variable mass holonomic system are investigated. Appell equations and differential equations of motion for a variable mass holonomic system are established. A new expression of the total first derivative of the function with respect of time t along the systematic motional track curve, and the definition and the criterion of Mei symmetry for Appell equations under the infinitesimal transformations of groups are given. The expressions of the structural equation and Mei conserved quantity for Mei symmetry in Appell are obtained. An example is given to illustrate the application of the results.

Keywords: variable mass holonomic system, Appell equation, Mei symmetry, Mei conserved quantity

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1. Introduction

The Lagrange system, Nielsen system and Appell system are three important mechanic systems in analytical mechanics.[1] Since the potential relations between symmetry and conserved quantity were revealed by Noether, a German female scientist in 1918,[2] the study regarding the three main symmetries[3] relative to the Lagrange system[4,5] and the Nielsen system[6−11] has been perfected relatively to the Appell system.

In order to seek the methods of solving Appell equations, Mei[12] first gave the Noether conserved quantity deduced indirectly from the Noether symmetry according to the form invariance. The conserved quantity of Appell equations for a rotational relativistic holonomic system was gained indirectly by means of the Noether symmetry and the Lie symmetry separately from the form invariance, and Luo[13,14] also acquired the knowledge of the conserved quantity for a general holonomic system through the Lie symmetry. In Ref. [15], the structural equations of Mei symmetry and the Mei conserved quantity expressed directly by Appell function were obtained.

In recent years, a great progress has been made about the study of symmetry and conserved quantity for variable mass mechanical systems.[16−21] In order to further perfect the research of Mei symmetry and Mei conserved quantity of Appell equations,[22,23] in the present paper, on the basis of the above-mentioned research achievements, the Mei symmetry and the Mei conserved quantity of Appell equations for a variable mass holonomic system are investigated. The differential equations of motion of Appell equations for the system are established. The definition and the criterion of Mei symmetry of Appell equations for the system under the infinitesimal transformations of groups are also given. The expressions of the structural equation and Mei conserved quantity of Mei symmetry of Appell equations for the variable mass holonomic system expressed by the Appell functions are obtained. Finally, an example is given to illustrate the application of the results.

2. Appell equations and differential equations of motion for variable mass holonomic system

Assume that a variable mass holonomic mechanical system consists of N particles each with mass m_i and position vector $\boldsymbol{r}_i$ respectively at the mo-

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ment t, and at the moment $t+\mathrm{d}t$, $\mathrm{d}m_i$ is the infinitesimal element of mass, the mass that is released from or merged into a particle. The configuration of the system is determined by n generalised coordinates $q_s(s=1,\ldots,n)$, and $Q_s=Q_s\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right)$ are the generalised forces corresponding to the s generalised coordinate q_s, on the assumption that the mass of a particle is dependent on time and generalised coordinates, i.e.

$$m_i=m_i(t,\boldsymbol{q}),\quad (i=1,\ldots,N). \tag{1}$$

The energy of acceleration for the system is expressed as

$$S=S(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})=\frac{1}{2}m_i\ddot{\boldsymbol{r}}_i^2, \tag{2}$$

and Appell equations are written as follows:

$$\frac{\partial S}{\partial\ddot{q}_s}=Q_s+P_s,\quad (s=1,\ldots,n), \tag{3}$$

where P_s is the generalised counter-thrust and given as

$$P_s=\dot{m}_i\left(\boldsymbol{u}_i+\dot{\boldsymbol{r}}_i\right)\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_s}-\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial q_s}, \tag{4}$$

where $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ are the position-vector and the velocity of the ith particle respectively, and $\boldsymbol{u}_i$ is the relative velocity for the infinitesimal element of mass relative to the ith particle.

By virtue of Eq. (3), all generalised accelerations can be obtained as follows:

$$\ddot{q}_s=\alpha_s\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right),\quad (s=1,\ldots,n). \tag{5}$$

3. Mei symmetry of Appell equations for a variable mass holonomic system

Introducing the infinitesimal transformations of time and generalised coordinates

$$t^*=t+\Delta t,\quad q_s^*\left(t^*\right)=q_s\left(t\right)+\Delta q_s,\quad (s=1,\ldots,n), \tag{6}$$

and its expansion is

$$\begin{aligned}&t^*=t+\varepsilon\xi_0\left(t,\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}\right),\\&q_s^*\left(t^*\right)=q_s\left(t\right)+\varepsilon\xi_s\left(t,\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}\right),\quad (s=1,\ldots,n),\end{aligned} \tag{7}$$

where ε is an infinitesimal parameter; ξ_0 and ξ_s are infinitesimal generators.

Introducing a vector $X^{(0)}$ of the infinitesimal generator

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{8}$$

and its first expansion

$$\tilde{X}^{(1)}=X^{(0)}+\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial\dot{q}_s}, \tag{9}$$

and its second expansion

$$\tilde{X}^{(2)}=\tilde{X}^{(1)}+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)-\ddot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right]\frac{\partial}{\partial\ddot{q}_s}, \tag{10}$$

where the total derivative along the trajectory of the system with respect to t is

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+\alpha_s\frac{\partial}{\partial\dot{q}_s}+\dot{\alpha}_s\frac{\partial}{\partial\ddot{q}_s}. \tag{11}$$

From Eq. (7), we obtain

$$\begin{aligned}&\frac{\mathrm{d}q_s^*}{\mathrm{d}t^*}=\frac{\mathrm{d}q_s+\varepsilon\mathrm{d}\xi_s}{\mathrm{d}t+\varepsilon\mathrm{d}\xi_0}=\dot{q}_s+\varepsilon\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)+O\left(\varepsilon^2\right),\\&\frac{\mathrm{d}^2q_s^*}{\mathrm{d}t^{*2}}=\ddot{q}_s+\varepsilon\left[\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)^{\cdot}-\ddot{q}_s\dot{\xi}_0\right]+O\left(\varepsilon^2\right).\end{aligned} \tag{12}$$

After transformations (7), the functions S, Q_s and P_s respectively become S^*, Q_s^* and P_s^*. Taking the Taylor expansions of S^*, Q_s^* and P_s^* at point $(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ along the trajectory of motion of the system, we have

$$S^*=S\left(t^*,\boldsymbol{q}^*,\frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*},\frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}\right)=S(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})+\varepsilon\left\{\frac{\partial S}{\partial t}\xi_0+\frac{\partial S}{\partial q_s}\xi_s+\frac{\partial S}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)+\frac{\partial S}{\partial\ddot{q}_s}[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^{\cdot}-\ddot{q}_s\dot{\xi}_0]\right\}+O(\varepsilon^2),$$

namely

$$S^*=S(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(2)}(S)+O(\varepsilon^2), \tag{13}$$

$$Q_s^*=Q_s^*\left(t^*,\boldsymbol{q}^*,\frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}(Q_s)+O(\varepsilon^2),\quad (s=1,\ldots,n), \tag{14}$$

$$P_s^*=P_s^*\left(t^*,\boldsymbol{q}^*,\frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)=P_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}(P_s)+O(\varepsilon^2),\quad (s=1,\ldots,n). \tag{15}$$

Definition If the dynamic functions S^*, Q_s^* and P_s^* which are transformed by Eq. (7) are used to replace respectively the original dynamic functions S, Q_s and P_s, Eq. (3) remains unchanged in form, namely,

$$\frac{\partial S^*}{\partial \ddot{q}_s} = Q_s^* + P_s^*, \quad (s = 1, \ldots, n). \tag{16}$$

Hence, this symmetry is called the Mei symmetry of Appell equations for a variable mass holonomic system.

4. Criterion of Mei symmetry of Appell equations

Substituting Eqs. (13), (14) and (15) into Eq. (16) and neglecting terms that are of the order of $O(\varepsilon^2)$ or of higher orders, and using Eq. (3), we obtain

$$\frac{\partial}{\partial \ddot{q}_s}\left[\tilde{X}^{(2)}(S)\right] - \tilde{X}^{(1)}(Q_s) - \tilde{X}^{(1)}(P_s) = 0. \tag{17}$$

Equation (17) is called the criterion equation of the Mei symmetry for Appell equations in holonomic system with variable mass.

Hence, we have

Criteria As to Appell equation (3) for a variable mass holonomic system, if the infinitesimal generators ξ_0 and ξ_s make the criterion Eq. (17) tenable, then the invariance of Eq. (3) under the infinitesimal transformations (7) is called Mei symmetry.

5. Structure equation and Mei conserved quantity

Proposition If the infinitesimal generators ξ_0 and ξ_s, along with the gauge function $G_{\rm M} = G_{\rm M}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Mei symmetry satisfy the following structural equation:

$$\tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right] + (\xi_s - \dot{q}_s\xi_0)\tilde{E}_s\left[\tilde{X}^{(2)}(S)\right] + \xi_0\left[\tilde{X}^{(1)}(Q_s + P_s)\right]\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} + \frac{\bar{\mathrm{d}}G_M}{\mathrm{d}t} = 0, \tag{18}$$

then the Mei conserved quantities deduced from the Mei symmetry of the Appell Eq. (3) are expressed as

$$I_{\rm M} = \xi_0\tilde{X}^{(2)}(S) + \frac{\partial \tilde{X}^{(2)}(S)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_M = \text{const.} \tag{19}$$

Proof Using Eq. (11), we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_M}{\mathrm{d}t} = {} & \left[\frac{\partial \tilde{X}^{(2)}(S)}{\partial t} + \dot{q}_s\frac{\partial \tilde{X}^{(2)}(S)}{\partial q_s} + a_s\frac{\partial \tilde{X}^{(2)}(S)}{\partial \dot{q}_s} + \frac{\partial \tilde{X}^{(2)}(S)}{\partial \ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{dt}\right]\xi_0 + \tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} \\ & + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \tilde{X}^{(2)}(S)}{\partial \dot{q}_s}\right](\xi_s - \dot{q}_s\xi_0) + \frac{\partial \tilde{X}^{(2)}(S)}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - a_s\xi_0 - \dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) + \frac{\bar{\mathrm{d}}G_M}{\mathrm{d}t}.\end{aligned} \tag{20}$$

Note that

$$\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right] = \xi_0\frac{\partial \tilde{X}^{(2)}(S)}{\partial t} + \xi_s\frac{\partial \tilde{X}^{(2)}(S)}{\partial q_s} + \left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial \tilde{X}^{(2)}(S)}{\partial \dot{q}_s},$$

then Eq. (20) becomes

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_M}{\mathrm{d}t} = {} & \tilde{X}^{(1)}[\tilde{X}^{(2)}(S)] - \xi_s\frac{\partial \tilde{X}^{(2)}(S)}{\partial q_s} + \xi_0\dot{q}_s\frac{\partial \tilde{X}^{(2)}(S)}{\partial q_s} + \xi_0\frac{\partial \tilde{X}^{(2)}(S)}{\partial \ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} + \tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} \\ & + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \tilde{X}^{(2)}(S)}{\partial \dot{q}_s}\right](\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}G_{\rm M}}{\mathrm{d}t} \\ = {} & \tilde{X}^{(1)}[\tilde{X}^{(2)}(S)] - (\xi_s - \dot{q}_s\xi_0)\frac{\partial \tilde{X}^{(2)}(S)}{\partial q_s} + \xi_0\frac{\partial \tilde{X}^{(2)}(S)}{\partial \ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} + \tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} \\ & + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \tilde{X}^{(2)}(S)}{\partial \dot{q}_s}\right](\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}G_{\rm M}}{\mathrm{d}t}.\end{aligned}$$

Pay attention to structural Eq. (18) and criterion Eq. (19), we obtain

$$\frac{\bar{\mathrm{d}}I_M}{\mathrm{d}t} = \tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right] + (\xi_s - \dot{q}_s\xi_0)\tilde{E}_s\left[\tilde{X}^{(2)}(S)\right] + \xi_0\frac{\partial\tilde{X}^{(2)}(S)}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} + \tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}$$
$$= \xi_0\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}\left[\frac{\partial\tilde{X}^{(2)}(S)}{\partial\ddot{q}_s} - \tilde{X}^{(1)}(Q_s + P_s)\right] = 0.$$

6. An illustrative example

Consider an Appell function for a variable mass holonomic system

$$S = \frac{1}{2}m(t)\ddot{q}^2, \tag{21}$$

where

$$m(t) = at, (a = \text{const.}), \tag{22}$$

and assume the absolute velocity to equal zero when the infinitesimal element of mass is released from a particle, that is

$$\boldsymbol{u} = -\dot{\boldsymbol{r}} = -\dot{q}\boldsymbol{k}. \tag{23}$$

The generalised force is

$$Q = -a\dot{q}. \tag{24}$$

Now, we try to study the Mei conserved quantity and the Mei symmetry for the system.

First, we study the Mei symmetry for the system.

On the assumption that the absolute velocity is equal to zero when the infinitesimal element of mass is released from a particle, and paying attention to Eq. (4), we have

$$P = 0. \tag{25}$$

From the criterion Eq. (17), we have

$$\frac{\partial}{\partial\ddot{q}}\left[\frac{1}{2}a\xi_0\ddot{q}^2 + \left(\frac{\bar{\mathrm{d}}^2\xi}{\mathrm{d}t^2} - 2\ddot{q}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \dot{q}\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2}\right)at\ddot{q}\right]$$
$$+ \frac{\bar{\mathrm{d}}\xi}{\mathrm{d}t} - \dot{q}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = 0. \tag{26}$$

The above Eq. (26) has the following solutions:

$$\xi_0 = 0, \quad \xi = 1, \tag{27}$$
$$\xi_0 = 1, \quad \xi = 0, \tag{28}$$
$$\xi_0 = 0, \quad \xi = \ln t, \tag{29}$$

then the system has the Mie symmetry.

Secondly, we study the Mei conserved quantity deduced from the Mei symmetry for the syetem.

Substituting Eq. (27) into Eq. (18) yields

$$\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t} = 0, \tag{30}$$

and substituting Eqs. (21), (24) and (25) into Eq. (3), we have

$$t\ddot{q} + \dot{q} = 0. \tag{31}$$

Hence, by virtue of Eqs. (30) and (31), the gauge function is obtained as follows:

$$G_{\mathrm{M}} = t\dot{q}, \tag{32}$$

then, from Eq. (19), we obtain

$$I_{\mathrm{M}} = G_M = t\dot{q} = \text{const.} \tag{33}$$

In the same way, we can have the Mei conserved quantity corresponding to Eqs. (28) and (29), and its expression is the same as Eq. (33).

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