Symmetry of Lagrangians of nonholonomic systems of non-Chetaev's type*

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This paper studies the symmetry of Lagrangians of nonholonomic systems of non-Chetaev's type. First, the definition and the criterion of the symmetry of the system are given. Secondly, it obtains the condition under which there exists a conserved quantity and the form of the conserved quantity. Finally, an example is shown to illustrate the application of the result.

Keywords: nonholonomic system, non-Chetaev's type constraint, symmetry of Lagrangians, conserved quantity

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1. Introduction

The researches on the symmetries and the conserved quantities of mechanical systems have not only important mathematical significance, but also profound physical importance. From the point of view of method, the study on symmetries is mostly based on certain invariances under the infinitesimal transformation of time and coordinates, such as the invariance of Hamiltonian action that is called Noether symmetry,[1−5] the invariance of differential equation called Lie symmetry[6−12] and the invariance that the dynamical function still satisfies the original equation after the infinitesimal transformation is called form invariance.[13−15] Reference [16] studied the problem of equivalent Lagrangians for single degree of freedom mechanical systems and pointed out that there exist conserved quantities in this case. Reference [17] extended the result to conservative multi-degree-of-freedom systems. Reference [18] extended the result to nonconservative holonomic systems and called it the symmetry of Lagrangians. Reference [19] discussed the symmetry of Lagrangians of nonholonomic systems of Chetaev's type and obtained the corresponding result. We now attempt to extend the above result to nonholonomic systems of non-Chetaev's type.

2. Equations of motion of the system

Suppose that the position of a mechanical system is determined by the n generalised coordinates $q_s\,(s = 1, \ldots, n)$. Let the motion of the system be subjected to the g nonholonomic constraints of non-Chetaev's type

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \qquad (\beta = 1, \ldots, g), \tag{1}$$

the equations of virtual displacement are

$$f_{\beta s}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\delta q_s = 0. \tag{2}$$

Particularly, if

$$f_{\beta s} = \frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{3}$$

they are reduced to constraints of Chetaev's type. The differential equations of motion of the system can be written in the form[20]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta f_{\beta s}, \quad (s = 1, \ldots, n). \tag{4}$$

Suppose that the system is nonsingular, i.e.

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0. \tag{5}$$

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Using Eqs. (1) and (4), we can express the constraint multipliers λ_β as functions of $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$, as

$$\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{6}$$

Then Eqs. (4) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \Lambda_s, \tag{7}$$

where

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) f_{\beta s}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{8}$$

are called generalised constraint reaction forces, the Eqs. (7) are called the equations of the corresponding holonomic system for the nonholonomic system represented by Eqs. (1) and (4) of non-Chetaev's type. The solution of the nonholonomic system can be obtained from those of the corresponding holonomic system provided the initial conditions satisfy the constraint equations of Eqs. (1), namely

$$f_\beta(t_0, \boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) = 0. \tag{9}$$

Therefore, in order to study the solution of the nonholonomic system, we first discuss the solution of the corresponding holonomic system Eqs. (7), then impose the restriction of the constraints on the initial conditions.

3. Definition and criterion of symmetry of Lagrangians

Let other Lagrangian, generalised forces and generalised constraint reaction forces be respectively $\bar{L}$, $\bar{Q}_s$ and $\bar{\Lambda}_s$ and the differential equations of motion expressed by these dynamical functions have the same solution set as that of the equations expressed by L, Q_s and Λ_s.

Also,

$$\begin{aligned}\mathcal{L}_s &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s - \Lambda_s \\ &= W_{sk}\ddot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_s \partial t} - \frac{\partial L}{\partial q_s} \\ &\quad -Q_s - \Lambda_s, \end{aligned} \tag{10}$$

$$\begin{aligned}\bar{\mathcal{L}}_s &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{q}_s} - \frac{\partial \bar{L}}{\partial q_s} - \bar{Q}_s - \bar{\Lambda}_s \\ &= \bar{W}_{sk}\ddot{q}_k + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial t} - \frac{\partial \bar{L}}{\partial q_s} \\ &\quad -\bar{Q}_s - \bar{\Lambda}_s, \end{aligned} \tag{11}$$

where

$$W_{sk} = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}, \quad \bar{W}_{sk} = \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial \dot{q}_k}. \tag{12}$$

Definition For Eqs. (7), if one can obtain $\bar{\mathcal{L}}_k = 0\,(k = 1, \ldots, n)$ from $\mathcal{L}_s = 0\,(s = 1, \ldots, n)$ and vice versa, then the symmetry is called Lagrangian.

From $\bar{\mathcal{L}}_s = 0$, all of the generalised accelerations $\ddot{q}_s$ can be solved as

$$\ddot{q}_s = \bar{U}^{sk}\left(\bar{Q}_k + \bar{\Lambda}_k + \frac{\partial \bar{L}}{\partial q_k} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial q_r}\dot{q}_r - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial t}\right), \tag{13}$$

where

$$\bar{W}_{sk}\bar{U}^{kr} = \delta_s^r. \tag{14}$$

Substituting Eq. (13) into $\mathcal{L}_k = 0$, one can obtain

$$\begin{aligned}&W_{rs}\bar{U}^{sk}\left(\bar{Q}_k + \bar{\Lambda}_k + \frac{\partial \bar{L}}{\partial q_k} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial q_l}\dot{q}_l - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial t}\right) \\ &= Q_r + \Lambda_r + \frac{\partial L}{\partial q_r} - \frac{\partial^2 L}{\partial \dot{q}_r \partial q_l}\dot{q}_l - \frac{\partial^2 L}{\partial \dot{q}_r \partial t}, \\ &\quad (k, l, r, s = 1, \ldots, n),\end{aligned} \tag{15}$$

Criterion If two groups of dynamical functions L, Q_s, Λ_s and $\bar{L}, \bar{Q}_s, \bar{\Lambda}_s$ satisfy Eqs. (15), then the symmetry is Lagrangian.

4. Conserved quantity deduced by symmetry of Lagrangians

Deducing relation just like Ref. [19], from the symmetry of Lagrangians, we can obtain

$$\begin{aligned}&\frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_k} - \frac{\partial^2 \bar{L}}{\partial q_s \partial \dot{q}_k} - \frac{\partial \bar{Q}_s}{\partial \dot{q}_k} - \frac{\partial \bar{\Lambda}_s}{\partial \dot{q}_k} \\ &= -\frac{\mathrm{d}A_s^r}{\mathrm{d}t}W_{rk} \\ &\quad + A_s^r\left(\frac{\partial^2 L}{\partial \dot{q}_r \partial q_k} - \frac{\partial^2 L}{\partial q_r \partial \dot{q}_k} - \frac{\partial Q_r}{\partial \dot{q}_k} - \frac{\partial \Lambda_r}{\partial \dot{q}_k}\right),\end{aligned} \tag{16}$$

where

$$A_s^r = \bar{W}_{sk}U^{kr},\ W_{sk}U^{kr} = \delta_s^r, \tag{17}$$

$$\bar{\mathcal{L}}_s = A_s^r\mathcal{L}_r,\ \bar{W}_{rs} = A_r^k W_{ks}. \tag{18}$$

Introducing two matrices $\boldsymbol{T}$ and $\bar{\boldsymbol{T}}$ whose elements are respectively

$$\begin{aligned}T_{rk} &= \frac{\partial^2 L}{\partial \dot{q}_r \partial q_k} - \frac{\partial^2 L}{\partial q_r \partial \dot{q}_k}, \\ \bar{T}_{rk} &= \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial q_k} - \frac{\partial^2 \bar{L}}{\partial q_r \partial \dot{q}_k},\end{aligned} \tag{19}$$

and supposing that generalised forces and generalised constraint reaction forces satisfy the following conditions:

$$\frac{\partial}{\partial \dot{q}_k}\left(\bar{Q}_s+\bar{\Lambda}_s\right)=A_s^r\frac{\partial}{\partial \dot{q}_k}(Q_r+\Lambda_r) \quad (s,k,r=1,\ldots,n), \tag{20}$$

we can express Eq. (16) in the matrix form

$$\dot{\boldsymbol{A}}=-\bar{\boldsymbol{T}}\boldsymbol{U}+\boldsymbol{A}\boldsymbol{T}\boldsymbol{U}, \tag{21}$$

where $\boldsymbol{A}=(A_s^r)$ and $\boldsymbol{U}=(U^{sk})$ are $n\times n$ matrices. Then for any positive integer m, considering the fact that $\boldsymbol{T}$ and $\bar{\boldsymbol{T}}$ are antisymmetric, $\boldsymbol{U}$ and $\bar{\boldsymbol{W}}$ are symmetric, and $\boldsymbol{A}=\bar{\boldsymbol{W}}\boldsymbol{U}$, using the properties of the trace of matrices, we obtain

$$\begin{aligned}\dot{\boldsymbol{A}}\boldsymbol{A}^{m-1} &= (-\bar{\boldsymbol{T}}\boldsymbol{U}+\bar{\boldsymbol{W}}\boldsymbol{U}\boldsymbol{T}\boldsymbol{U})\boldsymbol{A}^{m-1}\\ &= -\bar{\boldsymbol{T}}\boldsymbol{U}(\bar{\boldsymbol{W}}\boldsymbol{U})^{m-1}+\bar{\boldsymbol{W}}\boldsymbol{U}\boldsymbol{T}\boldsymbol{U}(\bar{\boldsymbol{W}}\boldsymbol{U})^{m-1},\end{aligned}$$

$$\begin{aligned}&\mathrm{tr}\left[\bar{\boldsymbol{T}}\boldsymbol{U}(\bar{\boldsymbol{W}}\boldsymbol{U})^{m-1}\right]=0,\\ &\mathrm{tr}\left[\bar{\boldsymbol{W}}\boldsymbol{U}\boldsymbol{T}\boldsymbol{U}(\bar{\boldsymbol{W}}\boldsymbol{U})^{m-1}\right]=0,\end{aligned}$$

and

$$\mathrm{tr}\left(\dot{\boldsymbol{A}}\boldsymbol{A}^{m-1}\right)=0.$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{tr}(\boldsymbol{A}^m)\right] &= \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A}^m)\right]\\ &= m\mathrm{tr}\left(\dot{\boldsymbol{A}}\boldsymbol{A}^{m-1}\right)\\ &= 0,\end{aligned}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{tr}(\boldsymbol{A}^m)\right]=0, \tag{22}$$

for any positive integer m, and we have the conserved quantities

$$\mathrm{tr}(\boldsymbol{A}^m)=\mathrm{const.}, \tag{23}$$

namely, the trace of all positive integer powers of $\boldsymbol{A}$ are constants of the motion. Therefore, we have

Proposition For the system Eqs. (7), if the condition Eq. (20) is satisfied, then the symmetry of Lagrangians of the system will lead to the conserved quantity Eq. (23).

The constraint of Chetaev's type is a special case of non-Chetaev's type constraints, therefore, we have

Corollary 1 For the holonomic system corresponding a nonholonomic system of Chetaev's type if the following conditions are satisfied:

$$\frac{\partial}{\partial \dot{q}_k}\left(\bar{Q}_s+\bar{\Lambda}'_s\right)=A_s^r\frac{\partial}{\partial \dot{q}_k}\left(Q_r+\Lambda'_r\right), \tag{24}$$

where

$$\Lambda'_s=\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{25}$$

then the symmetry of Lagrangians of the system will lead to the conserved quantity Eq. (23).

Corollary 2 For a holonomic system, if $Q_r,\bar{Q}_s$ satisfy the following conditions:

$$\frac{\partial \bar{Q}_s}{\partial \dot{q}_k}=A_s^r\frac{\partial Q_r}{\partial \dot{q}_k}, \tag{26}$$

then the symmetry of Lagrangians of the system will lead to the conserved quantity Eq. (23).

Corollary 2 was obtained by Ref. [19].

By using the above proposition, one can seek the conserved quantities of the corresponding holonomic system Eqs. (7). If one can find sufficient conserved quantities, then the solution of the system can be given. For the nonholonomic system represented by Eqs. (1) and (4), imposing the restriction of the nonholonomic constraints on the initial conditions, one can find the solution of the nonholonomic system.

5. Illustrative example

In order to illustrate the above result, we give an example.

Suppose that a nonholonomic system of non-Chetaev's type is as follows:

$$\begin{aligned}&L=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right),\\ &Q_1=Q_2=0,\\ &f=\dot{q}_1+bt\dot{q}_2-bq_2+\frac{1}{2}t^2=0 \quad (b>0),\\ &\delta q_1+t\delta q_2=0.\end{aligned} \tag{27}$$

Equations (7) become

$$\ddot{q}_1=-\frac{t}{1+bt^2},\quad \ddot{q}_2=-\frac{t^2}{1+bt^2}. \tag{28}$$

Then

$$\mathcal{L}_1=\ddot{q}_1+\frac{t}{1+bt^2},\quad \mathcal{L}_2=\ddot{q}_2+\frac{t^2}{1+bt^2}.$$

Choosing other Lagrangian, generalised forces and generalised constraint forces as

$$\begin{aligned}\bar{L} &= \frac{1}{6}\left\{\dot{q}_1+\frac{1}{2b}\ln(1+bt^2)\right\}^3\\ &\quad+\frac{1}{2}\left\{\dot{q}_2+\frac{1}{b}t-\frac{1}{b\sqrt{b}}\arctan\sqrt{b}t\right\}^2,\end{aligned}$$

$$\bar{Q}_1 = \bar{Q}_2 = \bar{\Lambda}_1 = \bar{\Lambda}_2 = 0,$$

the condition Eq. (20) is satisfied and we have

$$\bar{\mathcal{L}}_1 = \left\{\dot{q}_1 + \frac{1}{2b}\ln(1+bt^2)\right\}\mathcal{L}_1, \quad \bar{\mathcal{L}}_2 = \mathcal{L}_2,$$

i.e.,

$$A_1^1 = \dot{q}_1 + \frac{1}{2b}\ln(1+bt^2), \quad A_2^2 = 1.$$

Then conserved quantity Eq. (23) gives

$$I_1 = \dot{q}_1 + \frac{1}{2b}\ln(1+bt^2) = C_1. \tag{29}$$

Similarly, taking

$$\bar{L} = \frac{1}{2}\left\{\dot{q}_1 + \frac{1}{2b}\ln(1+bt^2)\right\}^2 + \frac{1}{6}\left\{\dot{q}_2 + \frac{1}{b}t - \frac{1}{b\sqrt{b}}\arctan\sqrt{b}t\right\}^3,$$

we obtain the conserved quantity

$$I_2 = \dot{q}_2 + \frac{1}{b}t - \frac{1}{b\sqrt{b}}\arctan\sqrt{b}t = C_2. \tag{30}$$

Integrating Eqs. (29) and (30), we obtain

$$q_1 + \frac{1}{2b}\int \ln(1+bt^2)\mathrm{d}t - C_1 t = C_3, \tag{31}$$

$$q_2 + \frac{1}{2b}t^2 - \frac{1}{b\sqrt{b}}\int \arctan\sqrt{b}t\,\mathrm{d}t - C_2 t = C_4. \tag{32}$$

Suppose that the initial condition is

$$t = 0,\ q_1 = q_{10},\ q_2 = q_{20},\ \dot{q}_1 = \dot{q}_{10},\ \dot{q}_2 = \dot{q}_{20},$$

then we have

$$C_1 = \dot{q}_{10}, \quad C_2 = \dot{q}_{20}, \quad C_3 = q_{10}, \quad C_4 = q_{20}.$$

The solution of the corresponding holonomic system Eqs. (28) is

$$q_1 = -\frac{1}{2b}\int \ln(1+bt^2)\mathrm{d}t + \dot{q}_{10}t + q_{10},$$
$$q_2 = -\frac{1}{2b}t^2 + \frac{1}{b\sqrt{b}}\int \arctan\sqrt{b}t\,\mathrm{d}t + \dot{q}_{20}t + q_{20}. \tag{33}$$

The restriction of nonholonomic constraint on the initial conditions is

$$\dot{q}_{10} - bq_{20} = 0. \tag{34}$$

The solution of the nonholonomic system is composed of Eqs. (33) and (34), which has 3 arbitrary constants.

6. Conclusion

The symmetry of Lagrangians is different from the Noether symmetry, the Lie symmetry and the form invariance. This paper extends the result of Refs. [16]–[19] to the nonholonomic system of non-Chetaev's type, and obtains the corresponding result.

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