Decomposition of almost-Poisson structure of generalised Chaplygin's nonholonomic systems*

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This paper constructs an almost-Poisson structure for the non-self-adjoint dynamical systems, which can be decomposed into a sum of a Poisson bracket and the other almost-Poisson bracket. The necessary and sufficient condition for the decomposition of the almost-Poisson bracket to be two Poisson ones is obtained. As an application, the almost-Poisson structure for generalised Chaplygin's systems is discussed in the framework of the decomposition theory. It proves that the almost-Poisson bracket for the systems can be decomposed into the sum of a canonical Poisson bracket and another two noncanonical Poisson brackets in some special cases, which is useful for integrating the equations of motion.

Keywords: almost-Poisson structure, non-self-adjointness, Jacobi identity, generalised Chaplygin's nonholonomic systems

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1. Introduction

Self-adjoint dynamical systems such as conservative mechanical systems can admit the Lagrangian or Hamiltonian formulations which possess a symplectic strucure or Poisson structure.[1,2] However, there exist many non-self-adjoint systems in nature, e.g., nonholonomic systems,[3−9] some electro-mechanical systems,[10] elementary particle systems in Riemann–Cartan space–time,[11−17] the motion of rigid body in body-fixed coordinate systems,[18] etc., whose phase space is not symplectic and their Poisson structure does not exist since the fundamental 2-form on the phase space is not closed or the Jacobi identity for Poisson bracket is not satisfied. Fortunately there exist almost-Poisson structures for these non-self-adjoint systems, represented by the bracket operation consisting of anticommutation, bilinearity and Leibniz's rule except for the Jacobi identity.[4,19,20] In general, the almost-Poisson structure is more complicated than the Poisson one, which makes the non-self-adjoint systems difficult to integrate.[21] Nevertheless, the almost-Poisson structure can be decomposed into the sum of a Poisson one and a new almost-Poisson one which may become a Poisson structure in some instances, such as Lie–Poisson one.[22,23] Such decomposition of almost-Poisson bracket is very useful for formulating the algebraic structure for the equations of motion for the non-self-adjoint systems in order to integrate it easily. In this paper we discuss the decomposition of a general almost-Poisson structure for non-self-adjoint systems at first and then apply it to a kind of generalised Chaplygin's nonholonomic systems. The almost-Poisson structure for generalised Chaplygin's systems can be split into the sum of a canonical Poisson bracket and other two noncanonical Poisson brackets in some special cases, which is useful for integrating the equations of motion.

2. Decomposition of almost-Poisson structure

In this paper we first formulate the general decomposition theory of the almost-Poisson bracket on manifold M. We define a contravariant tensor $\boldsymbol{\Lambda}$ to be skew-symmetric $(2,0)$-tensor on manifold M as follows

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$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}. \tag{1}$$

A bracket is defined on M by

$$[A, B]^* = \boldsymbol{\Lambda}(\mathrm{d}A, \mathrm{d}B) = \boldsymbol{\Lambda}^{ij} \frac{\partial A}{\partial x^i} \wedge \frac{\partial B}{\partial x^j}, \tag{2}$$

for $A, B \in F(M)$. If smooth functions A, B, C satisfy the following properties:

$$[A, B]^* = -[B, A]^*, \tag{3a}$$

$$[A, c_1 B + c_2 C]^* = c_1 [A, B]^* + c_2 [A, C]^*, \tag{3b}$$

$$[A, BC]^* = [A, B]^* C + B [A, C]^*, \tag{3c}$$

where c_1 and c_2 are constants. Then the bracket in Eq. (2) is an almost-Poisson bracket, and $\boldsymbol{\Lambda}$ is an almost-Poisson constravariant tensor. The manifold M is called almost-Poisson manifold. However, it is not necessary to satisfy the Jacobi identity, that is

$$\left[A, [B, C]^*\right]^* + \left[B, [C, A]^*\right]^* + \left[C, [A, B]^*\right]^* \neq 0 \tag{3d}$$

for $A, B, C \in F(M)$. Then, there exists a theorem about the decomposition of almost-Poisson bracket.

Theorem 2.1 Any almost-Poisson bracket can always be decomposed into a sum of a Poisson one and a new almost-Poisson one

$$[A, B]^* = [A, B] + [A, B]_1^*; \tag{4}$$

or on the contrary, the summation of an almost-Poisson bracket and a Poisson bracket must be another almost-Poisson one

$$[A, B]_1^* + [A, B] = [A, B]^*. \tag{5}$$

The proof of the theorem is evident and we can consult Ref. [23]. If we define

$$[A, B] \doteq [A, B]_1 + [A, B]_2, \tag{6}$$

from Ref. [23], we can easily obtain the following theorem.

Theorem 2.2. A Poisson bracket $[\cdot, \cdot]$ can be decomposed into a sum of the other two Poisson brackets $[\cdot, \cdot]_1 + [\cdot, \cdot]_2$, or on the contrary, the sum of two Poisson brackets is also a Poisson bracket, namely, $[\cdot, \cdot]_1 + [\cdot, \cdot]_2 = [\cdot, \cdot]$, if and only if the coupling of the two Poisson brackets in the cyclical summation on A, B and C is zero. That is

$$\begin{aligned} &[A, [B, C]_2]_1 + [B, [C, A]_2]_1 + [C, [A, B]_2]_1 \\ &+ [A, [B, C]_1]_2 + [B, [C, A]_1]_2 + [C, [A, B]_1]_2 = 0. \end{aligned} \tag{7}$$

Due to Theorem 2.1, Eq. (2) can be divided into the following

$$\begin{aligned} [A, B]^* &= \boldsymbol{\Lambda}^{ij} \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^j} \\ &= [A, B] + [A, B]_1^* \\ &= \boldsymbol{\pi}^{ij} \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^j} + \boldsymbol{\Lambda}_1^{ij} \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^j}, \end{aligned} \tag{8}$$

in which $\boldsymbol{\pi}^{ij}$ and $\boldsymbol{\Lambda}_1^{ij}$ are Poisson contravariant tensor and almost Poisson contravariant tensor, respectively. From Eqs. (3a)–(3c), we can obtain

$$\boldsymbol{\Lambda}_1^{ij} = -\boldsymbol{\Lambda}_1^{ji}, \tag{9a}$$

$$\begin{aligned} &\boldsymbol{\Lambda}_1^{ij} \frac{\partial A}{\partial x^i} \frac{\partial (c_1 B + c_2 C)}{\partial x^j} \\ =&c_1 \boldsymbol{\Lambda}_1^{ij} \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^j} + c_2 \boldsymbol{\Lambda}_1^{ij} \frac{\partial A}{\partial x^i} \frac{\partial C}{\partial x^j}, \end{aligned} \tag{9b}$$

$$\begin{aligned} &\boldsymbol{\Lambda}_1^{ij} \frac{\partial A}{\partial x^i} \frac{\partial (BC)}{\partial x^j} \\ =&\boldsymbol{\Lambda}_1^{ij} \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^j} C + B \boldsymbol{\Lambda}_1^{ij} \frac{\partial A}{\partial x^i} \frac{\partial C}{\partial x^j}. \end{aligned} \tag{9c}$$

Because of

$$\boldsymbol{\Lambda}^{kl} \frac{\partial \Lambda^{ij}}{\partial x^l} + \boldsymbol{\Lambda}^{il} \frac{\partial \Lambda^{jk}}{\partial x^l} + \boldsymbol{\Lambda}^{jl} \frac{\partial \Lambda^{ki}}{\partial x^l} \neq 0, \tag{10}$$

the following inequality is easily obtained

$$\boldsymbol{\Lambda}_1^{kl} \frac{\partial \Lambda_1^{ij}}{\partial x^l} + \boldsymbol{\Lambda}_1^{il} \frac{\partial \Lambda_1^{jk}}{\partial x^l} + \boldsymbol{\Lambda}_1^{jl} \frac{\partial \Lambda_1^{ki}}{\partial x^l} \neq 0. \tag{11}$$

Thus, from Eqs. (9a)–(9c) and Eq. (11), the same conclusion as Theorem 2.1 is obtained by the $(2, 0)$-tensor $\boldsymbol{\Lambda}$.

Corollary 2.1. The almost-Poisson contravariant tensor $\boldsymbol{\Lambda}$ can always be decomposed by the sum of a Poisson contratensor $\boldsymbol{\pi}$ and a new almost-Poisson contravariant tensor $\boldsymbol{\Lambda}_1$, that is

$$\begin{aligned} &\boldsymbol{\Lambda}^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \\ =& \boldsymbol{\pi}^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \boldsymbol{\Lambda}_1^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \end{aligned} \tag{12}$$

or on the contrary, the summation of a Poisson contravariant tensor $\boldsymbol{\pi}$ and an almost-Poisson contravariant tensor $\boldsymbol{\Lambda}_1$ is a new almost-Poisson contravariant tensor $\boldsymbol{\Lambda}$.

We define the summation of two Poisson brackets

$$\begin{aligned} &[A, B]_1 + [A, B]_2 \\ =& \boldsymbol{\pi}_1^{ij} \frac{\partial A}{\partial x^i} \wedge \frac{\partial B}{\partial x^j} + \boldsymbol{\pi}_2^{ij} \frac{\partial A}{\partial x^i} \wedge \frac{\partial B}{\partial x^j} \\ \doteq& \boldsymbol{K}^{ij} \frac{\partial A}{\partial x^i} \wedge \frac{\partial B}{\partial x^j}. \end{aligned} \tag{13}$$

The following relations can be obtained evidently

$$\boldsymbol{K}^{ij} = -\boldsymbol{K}^{ji}, \tag{14a}$$

$$\begin{aligned}
&\boldsymbol{K}^{ij}\frac{\partial A}{\partial x^i}\wedge\frac{\partial\left(c_1B+c_2C\right)}{\partial x^j}\\
&=c_1\boldsymbol{K}^{ij}\frac{\partial A}{\partial x^i}\wedge\frac{\partial B}{\partial x^j}+c_2\boldsymbol{K}^{ij}\frac{\partial A}{\partial x^i}\wedge\frac{\partial C}{\partial x^j}, \qquad (14b)\\
&\boldsymbol{K}^{ij}\frac{\partial A}{\partial x^i}\wedge\frac{\partial\left(BC\right)}{\partial x^j}\\
&=\boldsymbol{K}^{ij}\frac{\partial A}{\partial x^i}\wedge\frac{\partial B}{\partial x^j}C+B\boldsymbol{K}^{ij}\frac{\partial A}{\partial x^i}\wedge\frac{\partial C}{\partial x^j}, \qquad (14c)\\
&\boldsymbol{K}^{kl}\frac{\partial\boldsymbol{K}^{ij}}{\partial x^l}+\boldsymbol{K}^{il}\frac{\partial\boldsymbol{K}^{jk}}{\partial x^l}+\boldsymbol{K}^{jl}\frac{\partial\boldsymbol{K}^{ki}}{\partial x^l}\\
&=\left(\boldsymbol{\pi}_1^{kl}\frac{\partial\boldsymbol{\pi}_2^{ij}}{\partial x^l}+\boldsymbol{\pi}_1^{il}\frac{\partial\boldsymbol{\pi}_2^{jk}}{\partial x^l}+\boldsymbol{\pi}_1^{jl}\frac{\partial\boldsymbol{\pi}_2^{ki}}{\partial x^l}\right)\\
&\quad+\left(\boldsymbol{\pi}_2^{kl}\frac{\partial\boldsymbol{\pi}_1^{ij}}{\partial x^l}+\boldsymbol{\pi}_2^{il}\frac{\partial\boldsymbol{\pi}_1^{jk}}{\partial x^l}+\boldsymbol{\pi}_2^{jl}\frac{\partial\boldsymbol{\pi}_1^{ki}}{\partial x^l}\right). \qquad (14d)
\end{aligned}$$

The right hand side of Eq. (14d) is always called Jacobiizer $\boldsymbol{J}^{ijk}$.[24] From Eqs. (14a)–(14d), we can also get another corollary.

Corollary 2.2. The sum of two Poisson contravariant tensors is also a Poisson contravariant tensor, or on the contrary, a Poisson contravariant tensor can be decomposed into the summation of two Poisson contravariant tensors, if and only if when the corresponding Jacobiizer (14d) is zero.

3. Application of the decomposition theory of almost-Poisson structure to the generalised Chaplygin's systems

In the following, we will specialise our discussion on the decomposition of an almost-Poisson structure for a kind of generalised Chaplygin's nonholonomic system. For the systems if the coordinates $q^i=(q^a,q^\mu)$ can be founded on the configuration manifold, with $a=1,\ldots,k$ and $\mu=k+1,\ldots,n$, such that the nonholonomic constraints can be written in the form

$$\dot{q}^a=\boldsymbol{B}_\mu^a\left(q^1,\ldots,q^n\right)\dot{q}^\mu. \qquad (15)$$

The kinematic equation of the generalised Chaplygin's nonholonomic systems[25,26] are as follows:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}^\mu}-\frac{\partial\tilde{L}}{\partial q^\mu}-\boldsymbol{B}_\mu^a\frac{\partial\tilde{L}}{\partial q^a}-\sum_{a=1}^{k}\frac{\partial L}{\partial\dot{q}^a}\sum_{\nu=k+1}^{n}\boldsymbol{R}_{\mu\nu}^a\dot{q}^\nu=0,\\
&\left(\mu=k+1,\ldots,n\right), \qquad (16)
\end{aligned}$$

where

$$\boldsymbol{R}_{\mu\nu}^a=\left(\frac{\partial\boldsymbol{B}_\mu^a}{\partial q^\nu}-\frac{\partial\boldsymbol{B}_\nu^a}{\partial q^\mu}+\boldsymbol{B}_\mu^a\frac{\partial\boldsymbol{B}_\nu^a}{\partial q^a}-\boldsymbol{B}_\nu^a\frac{\partial\boldsymbol{B}_\mu^a}{\partial q^a}\right). \qquad (17)$$

The Legendre transformation is used to define the new Hamiltonian $\tilde{H}=\sum_{\mu=k+1}^{n}\tilde{p}_\mu\dot{q}^\mu-\tilde{L}$ and the general momenta $\tilde{p}_\mu=\partial\tilde{L}/\partial\dot{q}^\mu$, then the generalised Chaplygin's equation (16) can be described by an equivalent Hamiltonian type

$$\begin{pmatrix}\dot{q}^a\\ \dot{q}^\mu\\ \dot{\tilde{p}}_\mu\end{pmatrix}=\begin{pmatrix}0&0&\boldsymbol{B}_\nu^a\\ 0&0&\boldsymbol{\delta}_\nu^\mu\\ -\boldsymbol{B}_\nu^a&-\boldsymbol{\delta}_\nu^\mu&p_b\boldsymbol{R}_{\mu\nu}^b\end{pmatrix}\begin{pmatrix}\dfrac{\partial\tilde{H}}{\partial q^b}\\ \dfrac{\partial\tilde{H}}{\partial q^\nu}\\ \dfrac{\partial\tilde{H}}{\partial\tilde{p}_\nu}\end{pmatrix}, \qquad (18)$$

where $p_b=\partial L/\partial\dot{q}^b$, $\boldsymbol{R}_{\mu\nu}^a=(\partial\boldsymbol{B}_\mu^a/\partial q^\nu-\partial\boldsymbol{B}_\nu^a/\partial q^\mu+\boldsymbol{B}_\mu^a\partial\boldsymbol{B}_\nu^a/\partial q^a-\boldsymbol{B}_\nu^a\partial\boldsymbol{B}_\mu^a/\partial q^a)$. Evidently it can be proved that the skew-symmetric matrix

$$\boldsymbol{\Lambda}=\begin{pmatrix}0&0&\boldsymbol{B}_\nu^a\\ 0&0&\boldsymbol{\delta}_\nu^\mu\\ -\boldsymbol{B}_\nu^a&-\boldsymbol{\delta}_\nu^\mu&p_b\boldsymbol{R}_{\mu\nu}^b\end{pmatrix} \qquad (19)$$

is an almost-Poisson one.

We denote the elements of $\boldsymbol{\Lambda}$ by $\boldsymbol{\Lambda}^{ij}$ $(i,j=1,\ldots,n)$, then the matrix $\boldsymbol{\Lambda}$ satisfies the following conditions:

$$\begin{aligned}
&\boldsymbol{\Lambda}^{ij}=-\boldsymbol{\Lambda}^{ji}, \qquad (20a)\\
&\boldsymbol{\Lambda}^{kl}\frac{\partial\boldsymbol{\Lambda}^{ij}}{\partial x^l}+\boldsymbol{\Lambda}^{il}\frac{\partial\boldsymbol{\Lambda}^{jk}}{\partial x^l}+\boldsymbol{\Lambda}^{jl}\frac{\partial\boldsymbol{\Lambda}^{ki}}{\partial x^l}\neq 0. \qquad (20b)
\end{aligned}$$

By the way, it can be verified that the right-hand side of Eq. (20b) occasionally vanishes for the special case of Eq. (18), such as the constraint are holonomic.[4] Utilizing corollary 2.1, we can decompose the almost-Poisson matrix (19) into

$$\boldsymbol{\Lambda}=\boldsymbol{\Lambda}_1+\boldsymbol{\Lambda}_2, \qquad (21)$$

where

$$\boldsymbol{\Lambda}_1=\begin{pmatrix}0&0&0\\ 0&0&\boldsymbol{\delta}_\nu^\mu\\ 0&-\boldsymbol{\delta}_\nu^\mu&0\end{pmatrix},\quad \boldsymbol{\Lambda}_2=\begin{pmatrix}0&0&\boldsymbol{B}_\nu^a\\ 0&0&0\\ -\boldsymbol{B}_\nu^a&0&p_b\boldsymbol{R}_{\mu\nu}^b\end{pmatrix}.$$

It is evident that the second almost-Poisson tensor $\boldsymbol{\Lambda}_2$ can also be split by the sum of another two almost-Poisson tensors $\boldsymbol{\Lambda}_3$ and $\boldsymbol{\Lambda}_4$, i.e.

$$\boldsymbol{\Lambda}_2=\boldsymbol{\Lambda}_3+\boldsymbol{\Lambda}_4, \qquad (22)$$

where

$$\boldsymbol{\Lambda}_3 = \begin{pmatrix} 0 & 0 & \boldsymbol{B}^a_\nu \\ 0 & 0 & 0 \\ -\boldsymbol{B}^a_\nu & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\Lambda}_4 = \begin{pmatrix} 0\ 0 & 0 \\ 0\ 0 & 0 \\ 0\ 0 & p_b \boldsymbol{R}^b_{\mu\nu} \end{pmatrix}.$$

We can define a bracket operation of two smooth functions on the almost-Poisson manifold, e.g., any two smooth functions A and B of the systems

$$\begin{aligned}[A,B]^* \doteq & \frac{\partial A}{\partial x^i}\boldsymbol{\Lambda}^{ij}\frac{\partial B}{\partial x^j} \\ = & \boldsymbol{\pi}^{\mu\nu}\frac{\partial A}{\partial q^\mu}\frac{\partial B}{\partial \tilde{p}_\nu} + \boldsymbol{B}^b_\mu \frac{\partial A}{\partial q^b}\frac{\partial B}{\partial \tilde{p}_\mu} \\ & -\boldsymbol{B}^b_\mu \frac{\partial B}{\partial q^b}\frac{\partial A}{\partial \tilde{p}_\mu} + \boldsymbol{R}^a_{\mu\nu}p_a \frac{\partial A}{\partial \tilde{p}_\mu}\frac{\partial B}{\partial \tilde{p}_\nu} \\ = & [A,B] + [A,B]_1 + [A,B]_2^*, \end{aligned} \tag{23}$$

where

$$\begin{aligned}&[A,B] = \{q^\mu, \tilde{p}_\nu\}\left(\frac{\partial A}{\partial q^\mu}\frac{\partial B}{\partial \tilde{p}_\nu} - \frac{\partial B}{\partial q^\mu}\frac{\partial A}{\partial \tilde{p}_\nu}\right), \\ &[A,B]_1 = \left\{q^b, \tilde{p}_\nu\right\}\left(\frac{\partial A}{\partial q^b}\frac{\partial B}{\partial \tilde{p}_\nu} - \frac{\partial B}{\partial q^b}\frac{\partial A}{\partial \tilde{p}_\nu}\right), \\ &[A,B]_2^* = \{\tilde{p}_\mu, \tilde{p}_\nu\}\frac{\partial A}{\partial \tilde{p}_\mu}\frac{\partial B}{\partial \tilde{p}_\nu}, \\ &\tilde{p}_\mu = p_\mu + \boldsymbol{B}^a_\mu p_a, \end{aligned} \tag{24}$$

in which $\boldsymbol{\pi}^{\mu\nu} = \{q^\mu, \tilde{p}_\nu\}$, $\boldsymbol{\Lambda}_3^{b\nu} = \left\{q^b, \tilde{p}_\nu\right\}$ and $\boldsymbol{\Lambda}_4^{\mu\nu} = \{\tilde{p}_\mu, \tilde{p}_\nu\}$. Utilising Eq. (24), we can obtain

$$\begin{aligned}&[A,[B,C]_1]_1 \\ = & \boldsymbol{B}^b_\nu \boldsymbol{B}^a_\mu \left[\frac{\partial A}{\partial q^b}\frac{\partial}{\partial \tilde{p}_\nu}\left(\frac{\partial B}{\partial q^a}\frac{\partial C}{\partial \tilde{p}_\mu} - \frac{\partial B}{\partial \tilde{p}_\mu}\frac{\partial C}{\partial q^a}\right)\right. \\ & \left. -\frac{\partial A}{\partial \tilde{p}_\nu}\frac{\partial}{\partial q^b}\left(\frac{\partial B}{\partial q^a}\frac{\partial C}{\partial \tilde{p}_\mu} - \frac{\partial B}{\partial \tilde{p}_\mu}\frac{\partial C}{\partial q^a}\right)\right] \\ & -\boldsymbol{B}^b_\nu \frac{\partial \boldsymbol{B}^a_\mu}{\partial q^b}\frac{\partial A}{\partial \tilde{p}_\nu}\left(\frac{\partial B}{\partial q^a}\frac{\partial C}{\partial \tilde{p}_\mu} - \frac{\partial B}{\partial \tilde{p}_\mu}\frac{\partial C}{\partial q^a}\right). \end{aligned}$$

Summing cyclically on A, B and C, we obtain

$$[A,[B,C]_1]_1 + [B,[C,A]_1]_1 + [C,[A,B]_1]_1 = 0. \tag{25}$$

Clearly, the almost-Poissson bracket $[\cdot,\cdot]^*$ can be decomposed into a sum of a canonical Poisson bracket $[\cdot,\cdot]$, a Poisson bracket $[\cdot,\cdot]_1$ and another almost-Poisson bracket $[\cdot,\cdot]_2^*$. If the almost-Poisson bracket $[A,B]_2^*$ is also a Poisson one, the following condition is obtained from Jacobi identity

$$\left(\boldsymbol{R}^a_{\sigma\lambda}\boldsymbol{R}^b_{\mu\nu} + \boldsymbol{R}^a_{\mu\lambda}\boldsymbol{R}^b_{\nu\sigma} + \boldsymbol{R}^a_{\nu\lambda}\boldsymbol{R}^b_{\sigma\mu}\right)p_a\frac{\partial p_b}{\partial \tilde{p}_\lambda} = 0. \tag{26}$$

So we have the following theorem.

Theorem 3.1. When Eq. (26) is satisfied, the almost-Poisson bracket of generalised Chaplygin's nonholonomic systems can be decomposed into the summation of a canonical Poisson bracket and other two noncanonical Poisson brackets.

However, if the almost-Poisson bracket $[\cdot,\cdot]^*$ defined on the constrained submanifold is a Poisson bracket in some special case, that is

$$\left[A,[B,C]^*\right]^* + \left[B,[C,A]^*\right]^* + \left[C,[A,B]^*\right]^* = 0, \tag{27}$$

namely, the Jacobiizer

$$\boldsymbol{J}^{ijk} = \boldsymbol{\Lambda}^{il}\frac{\partial \boldsymbol{\Lambda}^{jk}}{\partial x^l} + \boldsymbol{\Lambda}^{jl}\frac{\partial \boldsymbol{\Lambda}^{ki}}{\partial x^l} + \boldsymbol{\Lambda}^{kl}\frac{\partial \boldsymbol{\Lambda}^{ij}}{\partial x^l} = 0. \tag{28}$$

So the necessary and sufficient conditions for the generalised Chaplygin's nonholonomic systems to be of a Poisson structure are

$$\boldsymbol{R}^a_{\mu\nu}\frac{\partial p_a}{\partial \tilde{p}_\lambda} = 0, \tag{29a}$$

$$\frac{\partial \boldsymbol{R}^a_{\mu\nu}}{\partial q^b} = 0, \tag{29b}$$

$$\boldsymbol{R}^a_{\mu\nu}\frac{\partial p_a}{\partial \tilde{p}_\sigma} + \boldsymbol{R}^a_{\nu\sigma}\frac{\partial p_a}{\partial \tilde{p}_\mu} + \boldsymbol{R}^a_{\sigma\mu}\frac{\partial p_a}{\partial \tilde{p}_\nu} = 0, \tag{29c}$$

$$\frac{\partial \boldsymbol{R}^a_{\mu\nu}}{\partial q^\sigma} + \frac{\partial \boldsymbol{R}^a_{\nu\sigma}}{\partial q^\mu} + \frac{\partial \boldsymbol{R}^a_{\sigma\mu}}{\partial q^\nu} = 0. \tag{29d}$$

Equation (26) is certainly a direct result of Eq. (29a). Thus we have:

Corollary 3.1 The Poisson bracket of generalised Chaplygin's nonoholonomic systems with a symplectic structure can always be decomposed into the sum of a canonical Poisson bracket and other two noncanonical Poisson brackets.

4. Illustrative example

We discuss a special case of Chaplygin sleigh. Let us consider the free motion of a sleigh on a horizontal plane in the case when the projection of the centre of mass coincides with the point of contact of a sharp wheel and the plane. We suppose that the simplified sleigh has unit mass and unit moment of inertia in $R^2 \times T^1$ with coordinates (x, y, φ), subjected to the nonholonomic constraint $\dot{y} = \dot{x}\tan\varphi$. Then the regular Lagrangian is given by $L = \left(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2\right)/2$.

For the nonholonomic constraint $\dot{y} = \dot{x}\tan\varphi$, let $q^a = q^1 = y$, $q^\mu = (x, \varphi)$, $\mu = 2, 3$, $B^1_2 = \tan\varphi, B^1_3 = 0$ and $\tilde{L} = \left(\dot{x}^2 + \dot{x}^2\tan^2\varphi + \dot{\varphi}^2\right)/2$. We define the generalised momentum by

$$p_2 = (1 + \tan\varphi)\,\dot{x}, \quad p_3 = \dot{\varphi}$$

and assume the Hamiltonian as

$$\begin{aligned}\tilde{H} = & p_2\dot{q}^2 + p_3\dot{q}^3 - \tilde{L} \\ = & \frac{1}{2}\left(\dot{x}^2 + \dot{x}^2\tan^2\varphi + \dot{\varphi}^2\right). \end{aligned}$$

So the Hamiltonian formation of the Chaplygin's system is as follows:

$$\begin{pmatrix} \dot{q}^2 \\ \dot{q}^3 \\ \dot{p}_2 \\ \dot{p}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \dot{x}\tan\varphi\sec^2\varphi \\ 0 & -1 & -\dot{x}\tan\varphi\sec^2\varphi & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial\tilde{H}}{\partial q^2} \\ \dfrac{\partial\tilde{H}}{\partial q^3} \\ \dfrac{\partial\tilde{H}}{\partial p_2} \\ \dfrac{\partial\tilde{H}}{\partial p_3} \end{pmatrix}. \tag{30}$$

And the (0, 2)-tensor matrix is

$$\boldsymbol{\Lambda}^{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \dot{x}\tan\varphi\sec^2\varphi \\ 0 & -1 & -\dot{x}\tan\varphi\sec^2\varphi & 0 \end{pmatrix}, \tag{31}$$

which defines an almost-Poisson bracket as

$$[A, B]^* = \boldsymbol{\Lambda}^{ij}\frac{\partial A}{\partial x^i}\frac{\partial B}{\partial x^j}$$

for smooth function A and B. By simple calculation, we can easily verify that the Jacobiizer

$$\boldsymbol{J}^{ijk} \neq 0,$$

then the 2-form $\boldsymbol{\Omega}$ determined by the almost-Poisson bracket is pseudo-symplectic, namely the symplectiser

$$\boldsymbol{\Omega}_{ijk} \neq 0.$$

However the almost-Poisson matrix $\boldsymbol{\Lambda}^{ij}$ can be decomposed by the sum of a canonical Poisson matrix $\boldsymbol{\pi}^{\mu\nu}$ and another almost-Poisson matrix $\boldsymbol{J}^{\mu\nu}$ as follows:

$$\boldsymbol{\Lambda}^{ij} = \boldsymbol{\pi}^{\mu\nu} + \boldsymbol{J}^{\mu\nu}, \tag{32}$$

where

$$\boldsymbol{\pi}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{J}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{x}\tan\varphi\sec^2\varphi \\ 0 & 0 & -\dot{x}\tan\varphi\sec^2\varphi & 0 \end{pmatrix}.$$

Similarly the almost-Poisson bracket of the Chaplygin's nonholonomic systems on the constrained submanifold can be represented by a canonical Poissson bracket and another almost-Poisson bracket,

$$\begin{aligned} \left[A, \tilde{H}\right]^* &= \left[A, \tilde{H}\right] - \boldsymbol{C}^a_{\mu\nu}p_a\frac{\partial A}{\partial p_\mu}\frac{\partial\tilde{H}}{\partial p_\nu} \\ &= \left[A, \tilde{H}\right] + \left[A, \tilde{H}\right]^*_1, \end{aligned} \tag{33}$$

where

$$p_a = \dot{x}\tan\varphi, \quad C^a_{\mu\nu} = \sec^2\varphi. \tag{34}$$

It can be easily verified that the new almost-Poisson bracket $[\cdot,\cdot]^*$ satisfies the Eq. (27), thus in fact the new bracket $[\cdot,\cdot]^*$ is a Poisson bracket on the momentum space.

Therefore, the generalised Chaplygin's nonholonomic systems can be decomposed into a canonical Hamiltonian system and the other two noncanonical Hamiltonian systems, which will make the equation of generalised Chaplygin's nonholonomic systems to be integrated easily. The geometric algorithm of nonholonomic systems can be transformed to study the geometric integration of Hamilton systems and the Lie-Poisson integration on the momentum space in some special cases. So the idea of decomposition of the almost-Poisson structure is very useful to the geometric algorithm of non-self-adjoint dynamical systems.

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