Synchronisation and general dynamic symmetry of a vibrating system with two exciters rotating in opposite directions*

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We derive the non-dimensional coupling equation of two exciters, including inertia coupling, stiffness coupling and load coupling. The concept of general dynamic symmetry is proposed to physically explain the synchronisation of the two exciters, which stems from the load coupling that produces the torque of general dynamic symmetry to force the phase difference between the two exciters close to the angle of general dynamic symmetry. The condition of implementing synchronisation is that the torque of general dynamic symmetry is greater than the asymmetric torque of the two motors. A general Lyapunov function is constructed to derive the stability condition of synchronisation that the non-dimensional inertia coupling matrix is positive definite and all its elements are positive. Numeric results show that the structure of the vibrating system can guarantee the stability of synchronisation of the two exciters, and that the greater the distances between the installation positions of the two exciters and the mass centre of the vibrating system are, the stronger the ability of general dynamic symmetry is.

Keywords: synchronisation, vibrating system, symmetry, stability

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1. Introduction

The synchronisation of two or more coupled, similarly sized, nonlinear, autonomous oscillating systems occurs frequently, with many cases now being actively investigated.[1] One of the central problems in these investigations is to understand how the underlying topological structure affects dynamical process.[2] The earliest detailed accounts on synchronised motion were made by Huygens,[3] who observed that two clock pendulums suspended from stiff wooden beams could run in a steady-state and move in opposition to each other at the same angular velocity. Subsequently, van der Pol observed the synchronisation of certain electrical-mechanical system[4] and Rayleigh found that two organ tubes could produce a synchronised sound when the outlets are close to each other.[5] Now such synchronised behaviour has been widely explored via various systems of coupled oscillators.[6−12] Actual rotating bodies, or more general rotor mechanical structures form a very important and special class of systems that with or without the interaction through some coupling, exhibit synchronised motion.[13] The self-synchronisation of unbalanced rotors is one of the most remarkable phenomena, caused by vibration,[14−16] which is explained to be the principle of minimum of the average kinetic energy. In fact, mechanical systems have dynamic symmetries, which lead to various types of conserved quantity.[17−21] It is prognosticated that these dynamic symmetries have an effect on the dynamic characteristics of mechanical systems.

This paper takes the self-synchronisation of the two coupled exciters in a vibrating system for example to explain the mechanism of synchronisation. We deduce the non-dimensional coupling equation of the two exciters and propose the concept of general dynamic symmetry to physically explain the particularity of synchronisation of the two exciters.

2. Synchronisation of two coupled exciters in a vibrating system

Figure 1 shows the mathematical model of a vibrating system with dual-motor drivers rotating in op-

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posite directions, which consists of a rigid frame and two unbalanced rotors (exciters). The rigid frame is supported by an elastic foundation that consists of four springs installed symmetrically. The point o'' is the mass centre of the rigid frame, the points o_1 and o_2 are the rotating centres of the exciters 1 and 2, respectively. The symbol oxy is the fixed frame and the symbol $o'x'y'$ is the moving frame. According to Lagrange's equations, the equations of motion of the system can be set up as follows:[16]

$$\begin{aligned}
&\Big(m+\sum_{i=1}^{2} m_i\Big)\ddot{x} + f_x\dot{x} + k_x x = \sum_{i=1}^{2} m_i r_i(\dot{\varphi}_i^2\cos\varphi_i + \ddot{\varphi}_i\sin\varphi_i),\\
&\Big(m+\sum_{i=1}^{2} m_i\Big)\ddot{y} + f_y\dot{y} + k_y y = \sum_{i=1}^{2} m_i r_i(\dot{\varphi}_i^2\sin\varphi_i - \ddot{\varphi}_i\cos\varphi_i),\\
&J\ddot{\psi} + f_\psi\dot{\psi} + + k_\psi\psi = \sum_{i=1}^{2} m_i r_i l_i[\dot{\varphi}_i^2\sin(\varphi_i-\beta_i) - \ddot{\varphi}_i\cos(\varphi_i-\beta_i)], \qquad (1)\\
&J_{0j}\ddot{\varphi}_j + f_j\dot{\varphi}_j = T_{ej} - m_{0j}r_j[(\ddot{x} - l_j\ddot{\psi}\sin\beta_j - l_j\dot{\psi}^2\cos\beta_j)\sin\varphi_j\\
&\qquad - (\ddot{y} + l_j\ddot{\psi}\cos\beta_j - l_j\dot{\psi}_j^2\sin\beta_j)\cos\varphi_j], \quad (j=1,2),
\end{aligned}$$

where m is the mass of the rigid frame; m_i is the mass of the exciter i; r_i is the eccentric radius; k_x, k_y, and k_ψ are the spring constants in x-, y-, and ψ-directions, respectively; f_x, f_y, and f_ψ are the damping constants in x-, y-, and ψ- directions, respectively; l_i is the distance between the installed position of the exciter i and the mass centre of the rigid frame; β_i is the angle between $o''o_i$ and x-axis; J is the moment of inertia of the vibrating system about the mass centre of the rigid frame; J_{0j} is the moment of inertia of the exciter j; T_{ej} is the electromagnetic torque of the motor j; f_j is the damping constant of the motor's rotor j; $(\dot{\bullet})$ and $(\ddot{\bullet})$ denote $\mathrm{d}\bullet/\mathrm{d}t$ and $\mathrm{d}^2\bullet/\mathrm{d}t^2$, respectively.

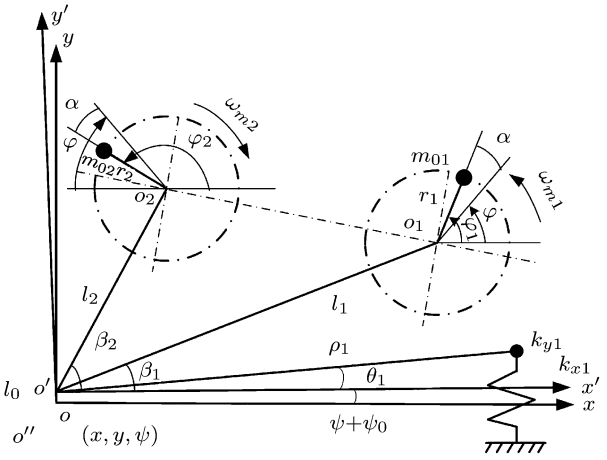


Fig. 1. Dynamic model of the two coupled exciters in a vibrating system.

As shown in Fig. 1, the two new angles are defined as follows:

$$\varphi = (\varphi_1 + \pi - \varphi_2)/2, \quad \alpha = (\varphi_1 - \pi + \varphi_2)/2. \qquad (2)$$

Assuming that the average value of the angular velocity of the two exciters over time is ω_{m0},[22] and the instantaneous change coefficients of $\dot{\varphi}$ and $\dot{\alpha}$ are ε_1 and ε_2, respectively(i.e. $\dot{\varphi} = (1+\varepsilon_1)\omega_{m0}$ and $\dot{\alpha} = \varepsilon_1\omega_{m0}$) when the system operates in the steady-state, we have

$$\begin{aligned}
\dot{\varphi}_1 &= (1+\varepsilon_1+\varepsilon_2)\omega_{m0} = (1+\nu_1)\omega_{m0},\\
\dot{\varphi}_2 &= -(1+\varepsilon_1-\varepsilon_2)\omega_{m0} = -(1+\nu_2)\omega_{m0}, \qquad (3)
\end{aligned}$$

where $\nu_1 = \varepsilon_1 + \varepsilon_2$ and $\nu_2 = \varepsilon_1 - \varepsilon_2$ stand for the instantaneous change coefficients of angular velocities of the two exciters around ω_{m0}.

Because the motion of the vibrating system is periodical, the load torques of the two motors stemming from vibration of the system are also periodical. Hence, the angular velocities of the two exciters vary periodically. If the two exciters operate synchronously, the average values of their instantaneous change coefficients of angular velocities over one period must be zero, i.e., $\bar{\nu}_1 = 0$ and $\bar{\nu}_2 = 0$ ($\bar{\varepsilon}_1 = 0$ and $\bar{\varepsilon}_2 = 0$). Here we study the case that the vibrating system operates in the state of far super-resonance. In this case, the change of angular velocities of the two exciters almost has no effect on the response of the vibrating system,[22] which can be expressed as follows:

$$
\begin{aligned}
x &= -r_m\mu_x[\cos(\varphi+\alpha+\gamma_x)-\eta\cos(\varphi-\alpha+\gamma_x)],\\
y &= -r_0 r_m\mu_y[\sin(\varphi+\alpha+\gamma_y)+\eta\sin(\varphi-\alpha+\gamma_y],\\
\psi &= -r_0 r_m\mu_\psi[r_{l1}\sin(\varphi+\alpha-\beta_1+\gamma_\psi)/l_{\rm e}-r_{l2}\sin(\varphi-\alpha+\beta_2+\gamma_\psi)/l_{\rm e}],
\end{aligned}
\tag{4}
$$

where $\mu_x = 1/(1-\omega_{nx}^2/\omega_{m0}^2)$, $\mu_y = 1/(1-\omega_{ny}^2/\omega_{m0}^2)$, $\mu_\psi = 1/(1-\omega_{n\psi}^2/\omega_{m0}^2)$, $r_m = m_1/M$, $\eta = m_2/m_1$, $l_{\rm e} = \sqrt{J/M}$, $r_{l1} = l_1/l_{\rm e}$, $r_{l2} = l_2/l_{\rm e}$; $r_1 = r_2 = r_0$; $\pi-\gamma_x$, $\pi-\gamma_y$ and $\pi-\gamma_\psi$ are the phase angles in x-, y- and ψ-directions, respectively; ω_{nx}, ω_{ny}, and $\omega_{n\psi}$ are the natural frequencies of the vibrating system in x-, y-, and ψ-directions, respectively.

Differentiating the three formulae in Eq. (4) with respect to time t by the chain rule (applied to each component of α and φ) to obtain $\ddot{x}$, $\ddot{y}$, $\dot{\psi}$ and $\ddot{\psi}$, neglecting the high order terms of ε_1 and ε_2, substituting them into the last two formulae in Eq. (1), and integrating them over $\varphi = 0-2\pi$, respectively, we obtain

$$
\begin{aligned}
J_{01}\omega_{m0}\dot{\bar{\nu}}_1 + f_1\omega_{m0}(1+\bar{\nu}_1) + m_1 r_0^2\omega_{m0}\bar{T}_{\rm L1} &= \bar{T}_{\rm e1},\\
J_{02}\omega_{m0}\dot{\bar{\nu}}_1 + f_2\omega_{m0}(1+\bar{\nu}_2) + m_1 r_0^2\omega_{m0}\bar{T}_{\rm L2} &= \bar{T}_{\rm e2}
\end{aligned}
\tag{5}
$$

with

$$
\begin{aligned}
\bar{T}_{\rm L1} &= \chi'_{11}\dot{\bar{\nu}}_1 + \chi'_{12}\dot{\bar{\nu}}_2 + \omega_{m0}(\chi_{11}\bar{\nu}_1 + \chi_{12}\bar{\nu}_2 + \chi_a + \chi_{f1}),\\
\bar{T}_{\rm L2} &= \chi'_{21}\dot{\bar{\nu}}_1 + \chi'_{21}\dot{\bar{\nu}}_2 + \omega_{m0}(\chi_{21}\bar{\nu}_1 + \chi_{22}\bar{\nu}_2 - \chi_a + \chi_{f2}),
\end{aligned}
\tag{6}
$$

where $\chi'_{11} = -W_{c1}/2$, $\chi'_{12} = \chi'_{21} = W_c\cos(2\bar{\alpha}+\theta_c)/2$, $\chi'_{22} = -W_{c2}/2$, $\chi_{11} = W_{s1}$, $\chi_{12} = -\chi_{21} = W_c\sin(2\bar{\alpha}+\theta_c)$, $\chi_{22} = W_{s2}$, $\chi_{f1} = (W_{s1} + W_s\cos(2\bar{\alpha}+\theta_s))/2$,

$$
\begin{aligned}
&\chi_{f2} = (W_{s2} + W_s\cos(2\bar{\alpha}+\theta_s))/2, \quad \chi_a = W_c\sin(2\bar{\alpha}+\theta_s)/2;\\
&W_{s1} = r_m(\mu_x\sin\gamma_x + \mu_y\sin\gamma_y + r_{l1}^2\mu_\psi\sin\gamma_\psi),\\
&W_{s2} = \eta^2 r_m(\mu_x\sin\gamma_x + \mu_y\sin\gamma_y + r_{l2}^2\mu_\psi\sin\gamma_\psi),\\
&W_{c1} = r_m(\mu_x\cos\gamma_x + \mu_y\cos\gamma_y + r_{l1}^2\mu_\psi\cos\gamma_\psi),\\
&W_{c2} = \eta^2 r_m(\mu_x\cos\gamma_x + \mu_y\cos\gamma_y + r_{l2}^2\mu_\psi\cos\gamma_\psi);\\
&a_s = -\mu_x\sin\gamma_x + \mu_y\sin\gamma_y + r_{l1}r_{l2}\mu_\psi\sin\gamma_\psi\cos(\beta_1+\beta_2),\\
&b_s = r_{l1}r_{l2}\mu_\psi\sin\gamma_\psi\sin(\beta_1+\beta_2), \quad W_s = \eta r_m\sqrt{a_s^2+b_s^2},\\
&a_c = \mu_x\cos\gamma_x - \mu_y\cos\gamma_y - r_{l1}r_{l2}\mu_\psi\cos\gamma_\psi\cos(\beta_1+\beta_2),\\
&b_c = r_{l1}r_{l2}\mu_\psi\cos\gamma_\psi\sin(\beta_1+\beta_2), \quad W_c = \eta r_m\sqrt{a_c^2+b_c^2},\\
&\theta_s = \begin{cases} \arctan(-b_s/a_s), & a_s \geq 0,\\ \pi+\arctan(-b_s/a_s), & a_s < 0, \end{cases} \quad \theta_c = \begin{cases} \arctan(b_c/a_c), & a_c \geq 0,\\ \pi+\arctan(b_c/a_c), & a_c < 0. \end{cases}
\end{aligned}
$$

It should be noticed that ε_1 and ε_2 are replaced by ν_1 and ν_2 in Eqs. (5) and (6). Compared with the change of φ with time t, that of α, ν_1, ν_2, $\dot{\nu}_1$, and $\dot{\nu}_2$ are very small, α, ν_1, and ν_2 are considered to be slow-changing parameters in this study. During the afore-mentioned integration, α, ν_1, ν_2, $\dot{\nu}_1$, and $\dot{\nu}_2$ are assumed to be the middle values of their integration $\bar{\alpha}$, $\bar{\nu}_1$, $\bar{\nu}_2$, $\dot{\bar{\nu}}_1$, and $\dot{\bar{\nu}}_2$, respectively.[22] In addition, compared with W_c, W_s is very small in a vibrating system with small damping.[14–16] Hence, W_s can be neglected in the expressions of χ'_{12}, χ'_{21}, χ_{12}, and χ_{21}.

In Eq. (5), $\bar{T}_{\rm e1}$ and $\bar{T}_{\rm e2}$ can be expressed as[22]

$$
\bar{T}_{\rm e1} = T_{\rm e01} - k_{\rm e01}\bar{\nu}_1, \quad \bar{T}_{\rm e2} = T_{\rm e02} - k_{\rm e02}\bar{\nu}_2,
\tag{7}
$$

where $T_{\rm e01}$ and $T_{\rm e02}$ are the electromagnetic torques of the two motors operating at the angular velocity of ω_{m0}; $k_{\rm e01}$ and $k_{\rm e02}$ are the angular velocity stiffness of the two motors.

Substituting Eqs. (6) and (7) into Eq. (5), choosing the exciter 1 as the standard exciter, introducing the following non-dimensional parameters $\rho_1 = 1 - W_{c1}/2$, $\rho_2 = \eta - W_{c1}/2$, $\kappa_1 = k_{\rm e01}/m_1 r_0^2\omega_{m0}^2 + W_{s1}$, $\kappa_2 = k_{\rm e01}/m_1 r_0^2\omega_{m0}^2 + W_{s2}$

into them, and writing them into the matrix form, we obtain

$$\boldsymbol{A}\dot{\bar{\boldsymbol{\nu}}} = \boldsymbol{B}\bar{\boldsymbol{\nu}} + \boldsymbol{u}, \tag{8}$$

where $\bar{\boldsymbol{\nu}} = \{\ \bar{\nu}_1\ \ \bar{\nu}_2\ \}^{\mathrm{T}}$, $u = \{\ u_1\ \ u_2\ \}^{\mathrm{T}}$,

$$\boldsymbol{A} = \begin{bmatrix} \rho_1 & W_c\cos(2\bar{\alpha}+\theta_c) \\ W_c\cos(2\bar{\alpha}+\theta_c) & \rho_2 \end{bmatrix},$$

$$\boldsymbol{B} = -\omega_{m0}\begin{bmatrix} \kappa_1 & W_c\sin(2\bar{\alpha}+\theta_c) \\ -W_c\sin(2\bar{\alpha}+\theta_c) & \kappa_2 \end{bmatrix},$$

$$u_1 = \frac{T_{\mathrm{e}01}-f_1\omega_{m0}}{m_1r_0^2\omega_{m0}} - \frac{1}{2}\omega_{m0}(W_{s1}+W_s\cos(2\bar{\alpha}_0+\theta_s) + W_c\sin(2\bar{\alpha}_0+\theta_c)),$$

$$u_2 = \frac{T_{\mathrm{e}02}-f_2\omega_{m0}}{m_1r_0^2\omega_{m0}} - \frac{1}{2}\omega_{m0}(W_{s2}+W_s\cos(2\bar{\alpha}_0+\theta_s) - W_c\sin(2\bar{\alpha}_0+\theta_c)).$$

In Eq. (8), $\boldsymbol{A}$ is related to the moments of inertia of the two exciters and defined as the inertia coupling matrix; $\boldsymbol{B}$ is related to the stiffness of angular velocity of the two motors and defined as the stiffness coupling matrix; and $\boldsymbol{u}$ is related to electromagnetic torque and the load torques of the two motors. So Eq. (8) can describe completely dynamic coupling characteristic of the two exciters and is called the non-dimensional coupling equation (NDCE) of the two exciters.

Substituting $\nu_1 = 0$ and $\nu_2 = 0$ into Eq. (8) and rearranging them, we have

$$T_{\mathrm{e}01} - f_1\omega_{m0} = \frac{1}{2}m_1r_0^2\omega_{m0}^2(W_{s1}+W_s\cos(2\bar{\alpha}+\theta_s) + W_c\sin(2\bar{\alpha}+\theta_c)), \tag{9}$$

$$T_{\mathrm{e}02} - f_2\omega_{m0} = \frac{1}{2}m_1r_0^2\omega_{m0}^2(W_{s2}+W_s\cos(2\bar{\alpha}+\theta_s) - W_c\sin(2\bar{\alpha}+\theta_c)). \tag{10}$$

Subtracting Eq. (10) from Eq. (9) and rearranging it, we obtain

$$2\bar{\alpha}+\theta_c = \arcsin T_{\mathrm{D}}/T_{\mathrm{S}}, \tag{11}$$

where $T_{\mathrm{S}} = m_1r_0^2\omega_{m0}^2W_c$, $T_{\mathrm{D}} = T_{\mathrm{R}1} - T_{\mathrm{R}2}$, $T_{\mathrm{R}1} = T_{\mathrm{e}01} - f_1\omega_{m0} - m_1r_0^2\omega_{m0}^2W_{s1}/2$, $T_{\mathrm{R}2} = T_{\mathrm{e}02} - f_2\omega_{m0} - m_1r_0^2\omega_{m0}^2W_{s2}/2$.

From $|\sin(2\bar{\alpha}+\theta_c)| \le 1$, it can be found that when $T_{\mathrm{S}} \ge |T_{\mathrm{D}}|$, there is $\bar{\alpha}$ satisfying Eq. (11). Hence, $T_{\mathrm{S}} \ge |T_{\mathrm{D}}|$ is the condition of implementing the synchronisation of the two coupled exciters. In this case, solving Eqs. (9) and (10) by a numeric method can determine the synchronous angular velocity ω_{m0} and the phase difference $2\bar{\alpha}_0$.

When $u_1 = 0$ and $u_2 = 0$, Eq. (8) is a general system, which satisfies the general Lyapunov equations[23]

$$\boldsymbol{I}^{\mathrm{T}}\boldsymbol{B} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{I} = -2\omega_{m0}\mathrm{diag}\{\kappa_1,\kappa_2\},$$
$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{A} > 0, \tag{12}$$

when the matrix $\boldsymbol{A}$ is definite-positive.

Linearising Eq. (8) around $2\bar{\alpha}_0$, neglecting W_s and considering Eqs. (9) and (10), we obtain

$$\boldsymbol{A}_0\bar{\boldsymbol{\nu}} = \boldsymbol{B}_0\bar{\boldsymbol{v}} + \boldsymbol{u}_0, \tag{13}$$

where $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ denote the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively when $\bar{\alpha} = \bar{\alpha}_0$; $\boldsymbol{u}_0 = \omega_{m0}W_c\cos(2\bar{\alpha}_0 + \theta_c)\Delta\alpha\{-1\ \ 1\}^{\mathrm{T}}$, $\Delta\alpha = \bar{\alpha} - \bar{\alpha}_0$.

Constructing the general Lyapunov function as[23]

$$V(\boldsymbol{A}_0\bar{\boldsymbol{v}}) = \bar{\nu}^T\boldsymbol{A}_0\bar{\boldsymbol{\nu}}, \tag{14}$$

then we have

$$\dot{V}(\boldsymbol{A}_0\bar{\boldsymbol{\nu}}) = \dot{\bar{\boldsymbol{\nu}}}^{\mathrm{T}}\boldsymbol{A}_0\bar{\boldsymbol{\nu}} + \bar{\boldsymbol{\nu}}^{\mathrm{T}}\boldsymbol{A}_0\dot{\bar{\boldsymbol{\nu}}} = -2\omega_{m0}(\kappa_1\bar{\nu}_1^2 + \kappa_2\bar{\nu}_2^2) - 2W_c\cos(2\bar{\alpha}_0+\theta_c)\Delta\alpha(\bar{\nu}_1 - \bar{\nu}_2). \tag{15}$$

It should be noticed that $\Delta\alpha$ is the average phase difference over time that is the integration of $\nu_1 - \nu_2$ over time. Hence, $\Delta\alpha$ and $\bar{\nu}_1 - \bar{\nu}_2$ have the same sign, i.e. $(\bar{\nu}_1 - \bar{\nu}_2)\Delta\alpha \ge 0$. So, we have $\dot{V}(\boldsymbol{A}_0\bar{\boldsymbol{\nu}}) < 0$ when $\cos(2\alpha_0+\theta_c) > 0$. Therefore, the condition of synchronisation stability is that the inertia coupling matrix is definite positive and all its elements are greater than zero, i.e.,

$$\rho_1 > 0, \quad \rho_2 > 0, \quad 4\rho_1\rho_2 - W_c^2\cos(2\bar{\alpha}_0+\theta_c) > 0, \quad \cos(2\alpha_0+\theta_c) > 0. \tag{16}$$

3. General dynamic symmetry of the two coupled exciters

As shown in Eq. (8), the coupling characteristics of the two exciters depend on the non-dimensional parameters, W_{s1}, W_{s2}, W_{c1}, W_{c2}, W_s and W_c. According to their expressions, W_{s1} and W_{s2} are defined as the coefficients of sine effect (CSE) for the exciters 1 and 2, respectively; W_{c1} and W_{c2} are referred to as the coefficients of cosine effect (CCE) for the exciters 1 and 2, respectively; W_s and W_c are referred to as the coefficients of coupled sine and cosine effect of the two exciters (CCSE and CCCE), respectively.

The three terms on the right-hand sides of Eqs. (9) and (10) are the load torques by which the vibrating system acts on each motor, respectively. The

first term is the product of the kinetic energy of the standard exciter and its CSE, which represents the load torque stemming from the motions excited by one exciter on its own motor and is called the torque of sine effect (TSE). The last two terms represent the coupled load torque of the two exciters, i.e., they are the effect of the motions excited by one exciter on the other motor. In this paper, the second term is called the torque of coupled sine effect (TCSE) and the third is the torque of coupled cosine effect (TCCE). But the interesting is that the TCSE for the two motors has the same sign, and that the TCCE for the two motors has opposite signs. When the vibrating system operates synchronously in the steady-state, no work is done by the TCCE.

When the vibrating system is completely symmetrical, i.e., $\beta_1 + \beta_2 = \pi$, $l_1 = l_2 = l_0$, $\eta = 1$, and the parameters of the two motors are the same, we have $\alpha = 0$, and $\theta_s = \pi$ and $\theta_c = 0$ for $a_s > 0$ and $a_c > 0$. In this case, the TCCE is zero, the vibrations of the system in x- and ψ- directions excited by the two exciters are mutually eliminated, while the vibration of the system in y-direction is the sum of the vibrations excited by the two exciters. Hence, $W_s\cos(2\alpha + \theta_s)$ represents the principle of motion superposition in the vibrating system when the vibrating system is asymmetrical in the structure and the parameters of the two motors are not the same.

The TCCE acts on the motor with the phase leading exciter as load torque to limit the increase of its angular velocity and on the other motor to limit the decrease of its angular velocity. Furthermore, the increase of the TCCE is proportional to the increase of $\sin(2\alpha+\theta_c)$ and its maximum value is $T_{\rm S}$. This demonstrates that $T_{\rm S}$ plays a role of forcing the phase difference between the two exciters close to $-\theta_c$. When the vibrating system is completely symmetrical in the structure, we have $\theta_c = 0$ when $a_c > 0$ and $\theta_c = \pi$ when $a_c < 0$. When $\theta_c = 0$, $T_{\rm S}$ forces the phase difference to approach zero. As shown in Eq. (11), the greater the value of $T_{\rm S}/T_{\rm D}$ is, the less the phase difference is. In this case, the running state of zero phase difference between the two exciters means that the vibrating system is completely symmetrical. Therefore, $T_{\rm S}$ represents the ability of sustaining the symmetry of the system during its running process, while the occurrence of the phase difference between the two exciters describes the asymmetry of the load and electromagnetic torques of the two motors. When the structure of the vibrating system is asymmetrical, $T_{\rm S}$ forces the phase difference to approach $-\theta_c$. Hence, the feature that a vibrating system with two exciters can sustain the symmetric running of the two exciters is called the general dynamic symmetry (GDS), which is the intrinsical dynamic characteristic of the considered vibrating system. $T_{\rm S}$ is called the torque of general dynamic symmetry (TGDS) and $T_{\rm D}$ is the asymmetric torque (AST) of the two motors. The symbol $-\theta_c$ is called the angle of general dynamic symmetry (AGDS) of the vibrating system. The synchronisation of the two exciters rises from the characteristic of the GDS, which is independent of the parameters of the two motors. According to the torque by which the vibrating system acts on the two motors and the expression of the TGDS, the coefficient of general dynamic symmetry (CGDS) is defined as follows:

$$\zeta = \frac{2W_c}{W_{s1} + W_{s2} + 2W_s\cos(\theta_s - \theta_c)}. \qquad (17)$$

The greater the CGDS is, the stronger the ability of the GDS is. As shown in Eq. (17), the CGDS is a function of r_m, η, r_{l1} and r_{l2} in a vibrating system with small damping. Figure 2 shows the variation of the CGDS with the parameter r_{l1} when $r_l = r_{l1}/r_{l2} = 0.5$. As illustrated in Fig. 2(a), the CGDS increases gradually with the increase of r_{l1} when r_{l1} is smaller, but when r_{l1} is above a given value, denoted by r_{l10}, the CGDS increases rapidly, and r_{l10} values increase with the decrease of r_m values. When r_{l1} is greater than r_{l10}, the moment of inertia of the vibrating system increases rapidly with the increase of r_{l1}. This causes the phase angle, γ_ψ, rapidly approach zero, i.e., the load torque of the two motors stemming from the swing of the rigid frame rapidly approach zero, as shown in Fig. 2(b). On the other hand, the CCCE increases almost proportionately with the increase of r_{l1}^2. Hence, the CGDS increases rapidly when r_{l1} increases after it is over r_{l0}. Figure 2(c) is the plot of the CGDS for the different values of $\beta_1 + \beta_2$. As illustrated in Fig. 2(c), the CGDS is independent of the installation orientation angle of the two exciters. But the installation orientation angles of the two exciters has an effect on the AGDS, $-\theta_c$.

There are two values of $2\bar{\alpha}_0$ that satisfies Eq. (11), but $\cos(2\alpha_0+\theta_c) > 0$ indicates that one of the two $2\bar{\alpha}_0$-values is the stable state of synchronisation. Therefore, the stability of synchronisation is dependent on the parameters, ρ_1, ρ_2, and $W_c\cos(2\bar{\alpha}_0 + \theta_c)$, which are called the relative non-dimensional moments of inertia (RNDMI) of the two exciters, and their non-dimensional coupling moment of inertia, as shown in

Eq. (16). In the expressions of RNDMI, it can be found that the RNDMI of each exciter decreases by half its CCE. ρ_1 and ρ_2 decrease with the increase of r_{l1} and r_{l2}, and W_c increases with the increase of $r_{l1}r_{l2}$. But there are the limits of r_{l1} and r_{l2}, which can expressed as:

$$r_{l1\max}^2 = \frac{1+r_m(1+\eta)}{r_m(1+\eta r_l^2)},$$
$$r_{l2\max}^2 = r_l^2 r_{l1\,\max}^2, \tag{18}$$

where $r_l = l_2/l_1$.

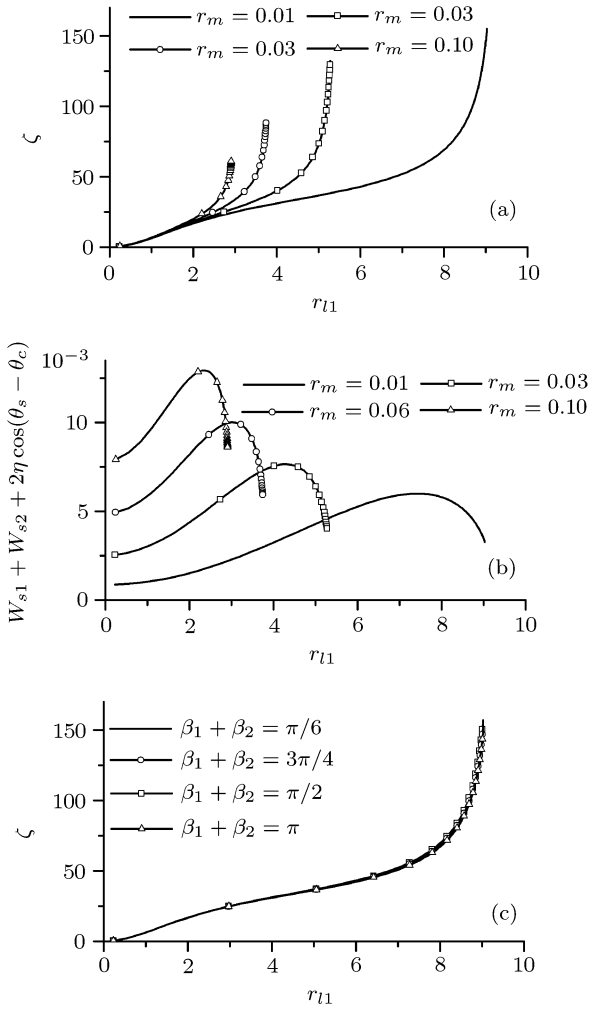


Fig. 2. Coefficients of general dynamic symmetry for $\mu_x = \mu_y = \mu_\psi = 1.05$: (a) $\beta_1 + \beta_2 = \pi/3$, $\eta = 0.5$, $r_l = 0.5$; (b) $\beta_1 + \beta_2 = \pi/3$, $\eta = 0.5$, $r_l = 0.5$; (c) $r_m = 0.01$, $\eta = 0.5$, $r_l = 0.5$.

It is assumed that $H_0 = 4\rho_1\rho_2 - W_c^2\cos(2\bar{\alpha}_0 + \theta_c)$. If the limits of r_{l1} and r_{l2} are neglected, the plots of $H_0 = 0$ and $r_{l1\,\max}$ in $r_m r_{l1}$-plane are shown in Fig. 3 for $\eta = 0.5$, $r_l = 0.5$, $\beta_1 + \beta_2 = \pi/3$, and $\mu_x = \mu_y = \mu_\psi = 1.07$. As illustrated in Fig. 3, the special structure of the vibrating system always ensures the stability of synchronisation of the two exciters.

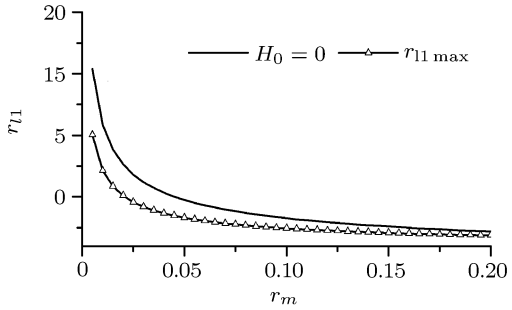


Fig. 3. The plots of $H_0 = 0$ and $r_{l1\,\max}$ in $r_m r_{l1}$-plane.

4. Computer simulation

Further analysis has been performed by means of computer simulations. Here, the parameters of the vibrating system are: $m = 2400$ kg, $m_1 = 60$ kg ($r_m = 0.025$), $m_2 = 30$ kg ($\eta = 0.5$), $r = 0.2$ m, $J_1 = 2.4$ kg·m^2, $J_2 = 1.2$ kg·m^2, $\beta_1 = 0$, $\beta_2 = \pi/3$, $r_{l2}/r_{l1} = 0.5$ and $\mu_x = \mu_y = \mu_\psi = 1.07$. The calculated value of θ_c is $-\pi/3(-60°)$. The parameters of the two motors are: rated rotational velocity 980 r/min, rated power 3.73 kW for motor 1 and 0.75 kW for motor 2. The results of the computer simulation are shown in Fig. 4, in which Figs. 4(a) and 4(b) are the rotational velocities of the two motors and the phase difference between the two exciters for $r_{l1} = 1.15$, respectively; Figures 4(c) and 4(d) show $2\bar{\alpha}_0$-values of the steady-state and the AST and the sum load torque of the two motors when r_{l1} ranges from 0.5 to 2.2. As illustrated in Figs. 4(a) and 4(b), during the starting process of the system, the rotational velocity of the motor 2 is greater than that of the motor 1 because the moment of inertia of the exciter 2 less than that of the exciter 1. But with the increase of the rotational velocity of the two exciters, the vibrations are excited and the TGDS plays the role of adjusting the load torque to make the phase difference stabilize at 64.7 ° and the AST is 23.6 N·m. As illustrated in Figs. 4(c) and 4(d), when r_{l1} changes from 0.57 to 2.13, $2\bar{\alpha}_0$ varies from 81.83° to 60.36°, $T_{\rm D}$ does from 21.25 N·m to 29.6 N·m; the sum load torque of the two motors ranges from 29.03 N·m to 40.46 N·m. This result is consistent with that in Fig. 2 and demonstrates that the greater r_{l1} and r_{l2} are, the stronger the ability of general dynamic symmetry is.

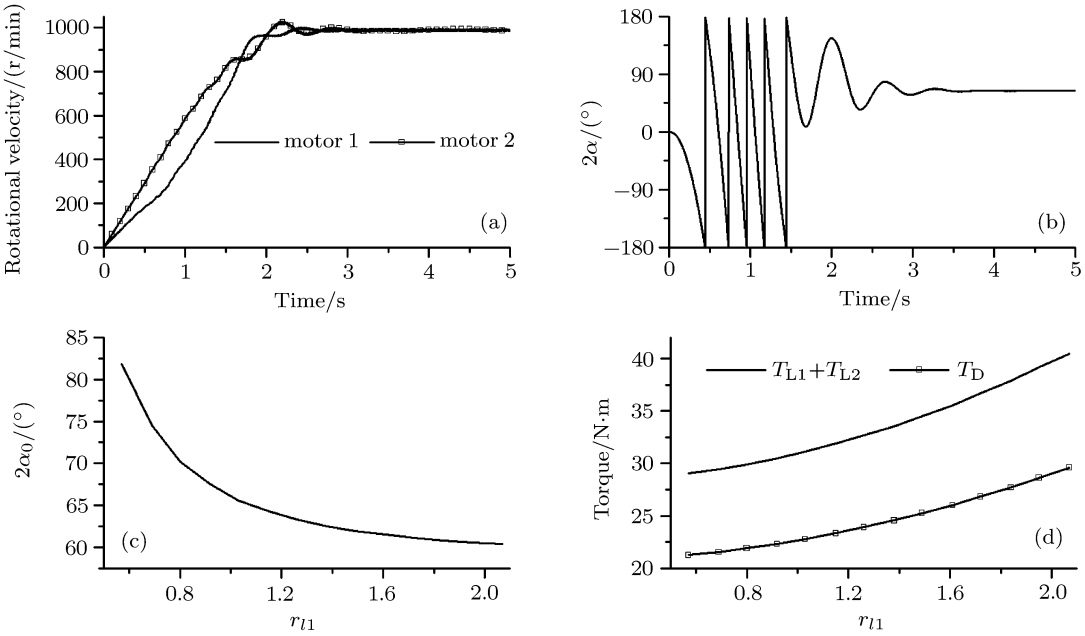


Fig. 4. Results of computer simulation.

5. Conclusions

This paper proposes an analytical approach to investigate the synchronisation of the two exciters in a vibrating system. We deduce the NDCE of the two exciters, which includes the inertia coupling, stiffness coupling of angular velocity and the load torque coupling of the two motors. The concept of GDS is proposed to physically explain the peculiarity of self-synchronisation of the two exciters. The load torque coupling between the two motors produces the TGDS to force the phase difference close to the AGDS, which is a function of the installation positions of the two exciters. Numeric results show that the greater the distances between the installation positions of the two exciters and the mass centre of the vibrating system are, the stronger the ability of GDS is. The condition of implementing synchronisation of the two exciters is that the TGDS is greater than the AST of the two motors. The non-dimensional inertia coupling matrix is symmetric and the stiffness coupling matrix is antisymmetric. The relative non-dimensional moment of inertia of each exciter decreases by half its CCE. A general Lyapunov function is constructed to derive the stability condition of synchronisation that the non-dimensional inertia coupling matrix is positive definite and all its elements is greater than zero. The numeric results show that the structure of the vibrating system can guarantee the stability of synchronisation of the two exciters.

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