Chaos in the perturbed Korteweg-de Vries equation with nonlinear terms of higher order*

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The dynamical behaviour of the generalized Korteweg-de Vries (KdV) equation under a periodic perturbation is investigated numerically. The bifurcation and chaos in the system are observed by applying bifurcation diagrams, phase portraits and Poincaré maps. To characterise the chaotic behaviour of this system, the spectra of the Lyapunov exponent and Lyapunov dimension of the attractor are also employed.

Keywords: generalized KdV equation, bifurcation, chaos
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1. Introduction

A considerable amount of research has been focused on the solitary waves and integrable properties of some fundamental nonlinear models over several decades.[1−7] It is well known that a completely integrable nonlinear system possesses some positive properties such as the Lax pair, N-soliton solutions, infinite conservation laws, Painlevé property and bi-Hamiltonian structure. The famous Korteweg-de Vries (KdV) equation is a prototype of completely integrable models. A number of theoretical issues concerning it have attracted considerable attention. The initial value problem of the KdV equation can be solved by means of the inverse scattering transform. Its other integrable properties have also been investigated in detail.

However, there often exist various perturbations in many real physical processes. The addition of a perturbation or forcing term to an integrable equation can lead to chaotic dynamics, while deterministic chaos is one of the most interesting nonlinear phenomena. For example, chaos was found in the perturbed sine-Gordon equation,[8] and in the cubic nonlinear Schrödinger equation.[9] A route to chaos by a period-doubling sequence was also investigated in a perturbed sine-Gordon system.[10] Furthermore, significant attention has recently been paid to the study of soliton equations under external perturbations.[11−15]

In this paper, we will consider a generalized form of the KdV (gKdV) equation with nonlinear, dispersive, damping and perturbing terms

$$u_t + au^p u_x + bu_{xxx} + \nu u_{xx} = f_x, \qquad p > 0, \tag{1}$$

where ν is a damping parameter, f is a perturbation, a and b are real parameters. Equation (1) without damping and forcing terms (i.e. gKdV equation), arising from fluid mechanics and other fields of physics, is an important model in nonlinear wave theory. It has been found to possess travelling wave solutions in the form of solitary or periodic solutions, and the stability of these solutions was extensively applied in quantum field theory and solid state physics. For Eq. (1), the initial value problem was considered by using Sobolev space and finite element method,[16] and the harmonic solutions of non-resonance and primary resonance were obtained with the perturbed method.[12] Though the chaotic behaviour under an external force has been predicted by the Melnikov theory in Ref. [12], a further study on its bifurcation structure and the dynamical behaviour for the system is still needed.

2. Unperturbed system

We assume that the forcing term f is a periodic function. Let $u(x,t) = \varphi(\xi)$ with $\xi = x - ct$ be a travelling wave solution to Eq. (1). Substituting $\varphi(\xi)$ into Eq. (1) yields

$$-c\varphi_\xi + a\varphi^p\varphi_\xi + \nu\varphi_{\xi\xi} + b\varphi_{\xi\xi\xi} = f_\xi, \tag{2}$$

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where $\varphi_\xi = \mathrm{d}\varphi/\mathrm{d}\xi$. Taking f as $g_0\cos(\omega\xi)$, and then integrating Eq. (2) once leads to

$$\varphi_{\xi\xi} + \gamma\varphi_\xi - \alpha\varphi + \frac{\beta}{p+1}\varphi^{p+1} = f_0\cos(\omega\xi), \qquad (3)$$

where $\gamma = \nu/b$, $\alpha = c/b$, $\beta = a/b$ and $f_0 = g_0/b$. Equation (3) for $p = 1$ and $p = 2$ can reduce respectively to well known Helmholtz oscillator and Duffing oscillator, and they are found to play an important role in the understanding of some physical problems.

The equivalent form of system (3) can be rewritten as ($X = \varphi$)

$$X' = Y,$$

$$Y' = f_0\cos(\omega\xi) - \gamma Y + \alpha X - \frac{\beta}{p+1}X^{p+1}, \qquad (4)$$

where the prime denotes the derivative with respect to ξ.

In the absence of forcing and damping terms, system (4) is an integrable Hamiltonian system with a Hamiltonian function

$$H(X,Y) = \frac{1}{2}Y^2 - \frac{\alpha}{2}X^2 + \frac{\beta}{(p+1)(p+2)}X^{p+2}. \quad (5)$$

According to the bifurcation theory of dynamical systems, the phase orbits defined by the vector fields of system (4) determine all travelling wave solutions of Eq. (1). Usually the homoclinic and heteroclinic orbits of nonlinear ordinary differential equations correspond to the solitary and kink wave solutions of nonlinear partial differential ones, respectively.

It is easy to find, for the unperturbed system ($f_0, \gamma = 0$ in Eq. (4)), that there exist two equilibrium points: $(0,0)$ and $([\frac{\alpha}{\beta}(p+1)]^{1/p}, 0)$ when p is odd, or three equilibrium points: $(0,0)$ and $(\pm[\frac{\alpha}{\beta}(p+1)]^{1/p}, 0)$ when p is even.

Let $\boldsymbol{J}$ be the Jacobi matrix of the unperturbed system in equilibrium. Suppose that $(X_i, 0)$ be an equilibrium point of the system, at the point

$$\boldsymbol{J}(X_i, 0) = \begin{pmatrix} 0 & 1 \\ \alpha - \beta X_i^p & 0 \end{pmatrix}.$$

By the theory of the planar dynamical system,[17] for an equilibrium point of an integrable system, the determinant of its Jacobi matrix plays an important role. If $\det(\boldsymbol{J}(X_i,0)) < 0$, the equilibrium point is a saddle point; if $\det(\boldsymbol{J}(X_i,0)) > 0$, the equilibrium point is a centre point; if $\det(\boldsymbol{J}(X_i,0)) = 0$, and the Poincaré index of the equilibrium point is 0, it is then a cusp. Hence, it is not difficult to check that the equilibrium point $(0,0)$ is a saddle point for $\alpha > 0$. If p is odd, the saddle point $(0,0)$ is connected by one homoclinic orbit in which there is a centre point given by $([\frac{\alpha}{\beta}(p+1)]^{1/p}, 0)$. If p is even, the saddle point $(0,0)$ is connected by two homoclinic orbits in which there is a centre point given by $(\pm[\frac{\alpha}{\beta}(p+1)]^{1/p}, 0)$. Both cases correspond to the existence of solitary wave solutions of the gKdV equation. Here we give a bell-profile solitary wave solution[18]

$$u(x,t) = \varphi(\xi) = A\mathrm{sech}^{2/p}[k\xi], \qquad (6)$$

where

$$A = \left[\frac{\alpha}{2\beta}(p+1)(p+2)\right]^{1/p},$$

and

$$k = \pm\frac{p}{2}\sqrt{\alpha}(\alpha > 0).$$

We will now be interested in the stability of the solitary wave solution (6) because of its importance in the theory and application of differential equations. Applying Lyapunov's direct method, we may analyse briefly the stability of the solution (6).

Let $\varphi(\xi)$ denote the exact bell-profile solitary wave solution (6), and $v(\xi)$ be a small disturbance of the travelling wave form for the solution $\varphi(\xi)$, i.e.

$$u = \varphi + v, \quad |v| \ll |\varphi|. \qquad (7)$$

Substituting u into Eq. (1) (f_x and $\nu = 0$), and neglecting small quantities of high orders $O(v^2)$, the following equation related to disturbance $v(\xi)$

$$-cv + a\varphi^p v + bv_{\xi\xi} = 0 \qquad (8)$$

is derived. Introducing the notation w, we have

$$v_\xi = w,$$

$$w_\xi = \alpha v - \beta\varphi^p v. \qquad (9)$$

In order to prove the stability of the solitary wave solution, we have to construct the Lyapunov function candidate

$$V(\xi) = vw. \qquad (10)$$

The derivative of V with respect to ξ becomes

$$\frac{\mathrm{d}V}{\mathrm{d}\xi} = w^2 + \alpha v^2\frac{1 + \mathrm{e}^{4k\xi} - 2(p^2+3p+1)\mathrm{e}^{2k\xi}}{(1+\mathrm{e}^{2k\xi})^2}. \quad (11)$$

Note that the term $\dfrac{\mathrm{e}^{2k\xi}}{(1+\mathrm{e}^{2k\xi})^2}(k > 0)$ is monotonically decreasing if $\xi \geq 0$, then we have

$$\frac{\mathrm{d}V}{\mathrm{d}\xi} \geq w^2 + v^2\frac{\alpha}{(2p^2+6p+3)^2}, \qquad (12)$$

when

$$\xi \geq \frac{\ln 2(p^2+3p+1)}{p\sqrt{\alpha}}. \tag{13}$$

Thus we can infer from the unstable theorem of Lyapunov that the solution (6) of the gKdV equation is unstable (in the sense of Lyapunov) with respect to infinitely small perturbation. The inference we obtained can modify and generalize the results in Refs.[19]–[21] for the gKdV equation in the case of $a = 1$.

3. Chaos in perturbed system

In order to explore the possible chaotic behaviour and properties of the perturbed system (4), it is necessary to locate regions of chaotic and regular behaviour within the parameter space. Since the system we study is dissipative, most of the information regarding its dynamics is obtained by studying its attractor. It is important to analyse the geometry of the system's attractor for different values of the system parameters and calculate the dimensions of the attractor. Here several different numerical tools are applied: (i) bifurcation diagram, (ii) largest Lyapunov exponent, (iii) Poincaré map and phase plane plot, and (iv) fractal dimension. They are all very useful tools for examining chaotic properties of dynamic systems and detecting chaotic attractors. For example, bifurcation diagrams show the evolution of the attractor for a dissipative system with a change of the bifurcation parameter. Such diagrams are a convenient tool for a qualitative examination of the system's behaviour, and help us to determine regions within the parameter space where the system has a chaotic attractor; i.e. where it exhibits chaos. The Lyapunov exponent is used to characterise the stability of a dynamical system with respect to a small perturbation of its phase space trajectory. The spectral analysis of the Lyapunov exponents has been proven to be the most useful dynamical diagnosis for a chaotic system because the solid proof of chaos is the existence of at least one positive Lyapunov exponent.

System (4) depends on five independent parameters. An investigation of such a system for complete range of parametric space or the influence of each parameter is complicated and difficult. To simplify the analysis, all parameters are kept as constants except one to be changed. We choose the perturbing frequency ω as control parameter. We could in fact vary any of the other parameters, but this does not give us any significantly different qualitative results.

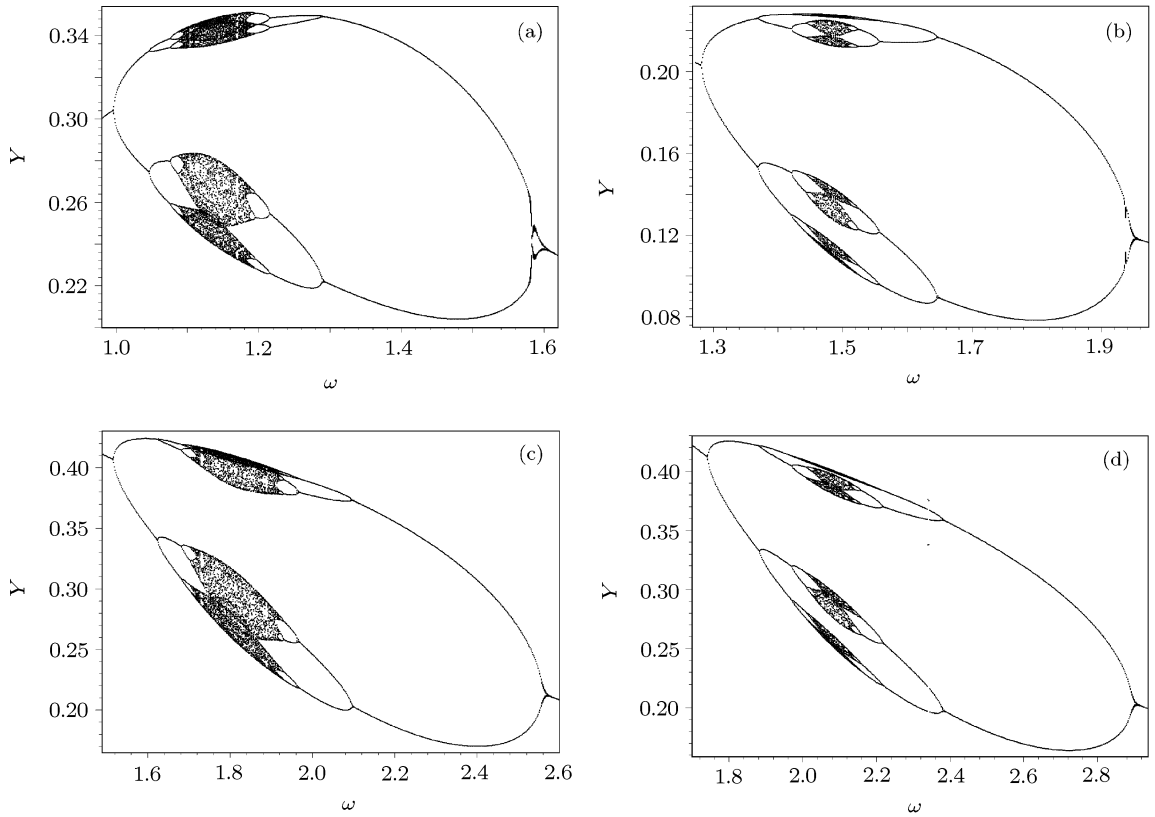


Fig. 1. Bifurcation diagrams in $(\omega - Y)$ plane of Eq. (4) for (a) $p = 3$, $f_0 = 0.8$, $\gamma = 0.7$, $\alpha = 0.3$ and $\beta = 2.2$; (b) $p = 4$, $f_0 = 1.0$, $\gamma = 0.4$, $\alpha = 0.2$ and $\beta = 3.0$; (c) $p = 5$, $f_0 = 1.8$, $\gamma = 0.7$, $\alpha = 0.3$ and $\beta = 3.0$; (d) $p = 6$, $f_0 = 2.24$, $\gamma = 0.7$, $\alpha = 0.295$ and $\beta = 4.34$.

Taking $p = 3$, 4, 5 and 6 in Eq. (4), we have investigated respectively the dynamical responses of the system X or Y varying with bifurcation parameter ω. The corresponding bifurcation diagrams in $(\omega - Y)$ plane of Eq. (4) for different parameter spaces are shown in Fig. 1. It actually describes the evolution of the system's attractor with parameter ω. The smeared region in diagrams corresponds to a chaotic behaviour of the system.

We can see from Fig. 1(a)–1(d) that the system exhibits periodic and chaotic behaviours when ω increases. The system starts from period-1, and enters period-2 by a pitchfork bifurcation. Then it goes through a cascade of period doubling, leading to chaos, and becomes chaotic. The attractor is now an infinite set of points, within the basin of attraction. Furthermore, all unstable orbits generated by period doubling are embedded in the attractor. Such an object is called a chaotic attractor. As ω further increases, the system comes back to a periodic state again via a sequence of inverse period doubling, leading from chaos to periodicity. Obviously, Fig. 1 exhibits a complete period doubling and inverse period doubling phenomena leading to chaos, while period doubling is at present the most commonly known route to chaos.

Let us analyse the dynamical behaviour of system (4) in detail. By choosing the parameters $p = 4$, $f_0 = 0.35$, $\gamma = 0.4$, $\alpha = 0.5$ and $\beta = 3.0$, the dynamical responses of system (4) with a change of ω are described in Fig. 2(a). The corresponding largest Lyapunov exponents are given in Fig.2 (b), that confirm the existence of the chaotic regions and periodic orbits in the parametric space.

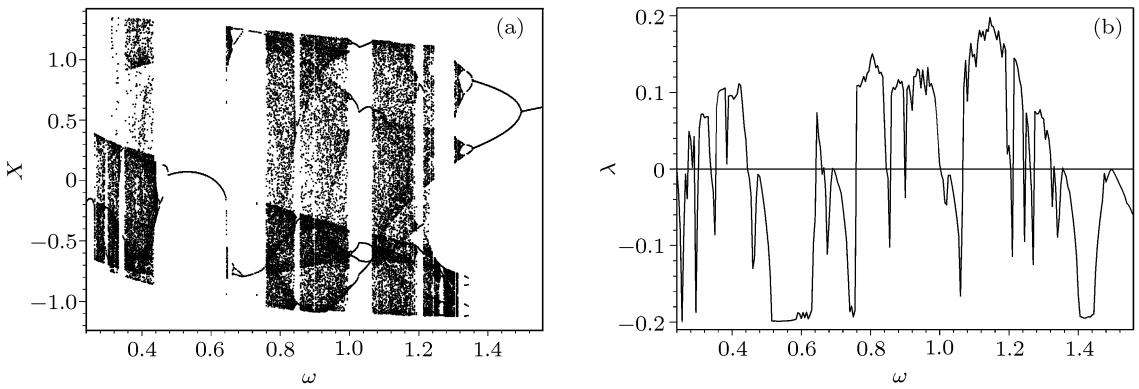


Fig. 2. (a) Bifurcation diagram in $(\omega - X)$ plane of Eq. (4) for $p = 4$, $f_0 = 0.35$, $\gamma = 0.4$, $\alpha = 0.5$ and $\beta = 3.0$; (b) the largest Lyapunov exponent corresponding to (a).

We can see from Fig.2 that the system (4) also exhibits periodic and chaotic behaviours when ω changes. After the response of the system becomes chaotic a number of windows of periodic behaviour appear. There are period-3 windows for $\omega \in$(0.847,0.856) and (1.203,1.213), and period-5 windows for $\omega \in$(0.999,1.064) and (1.263,1.272). Clearly, the system comes into regular motions from chaos usually through a sequence of inverse period-doubling bifurcations. There are also other types of routes to chaos, say, intermittency. For example, the system goes to a transient chaos from period-1 via an intermittent or explosive bifurcation at $\omega \sim 0.64$.

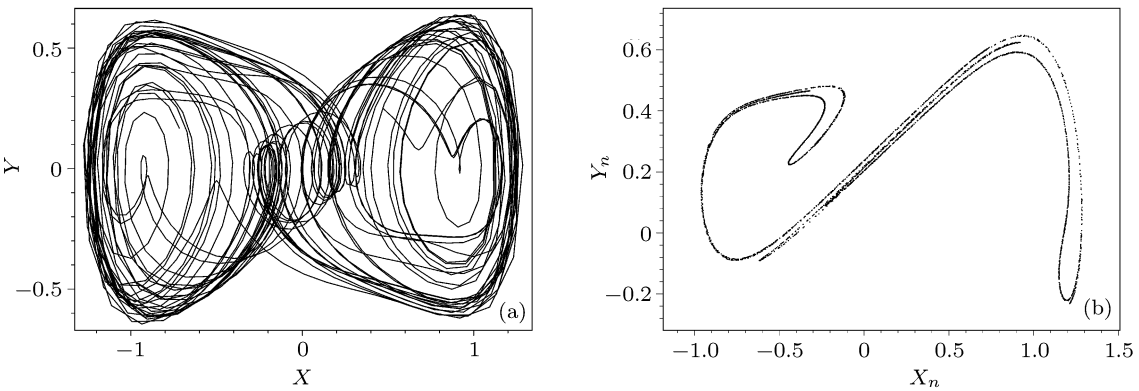


Fig. 3. (a) Phase projection of the chaotic state at $\omega = 1.145$ in Fig. 2(a); (b) the Poincaré map corresponding to (a).

As mentioned above, a very useful tool for exploring chaotic attractors is Poincaré sections. Poincaré sections provide us with images of the attractor in a $(d-1)$-dimensional hyperplane of d-dimensional phase

space. In other words, with a Poincaré section we actually choose to observe the geometric form of a smaller dimension that contains the same amount of information about the motion. Figure 3 displays the phase portrait and Poincaré map of the chaotic state at $\omega = 1.145$ in Fig. 2(a). The corresponding largest Lyapunov exponent is found to be positive: $\lambda = 0.198$, indicating chaotic dynamics.

In addition, one can also observe some crisis phenomena from Fig. 2(a). An interior crisis results in a sudden change in the size of an attractor, and it occurs when an attractor collides with an unstable orbit. The interior crises take place at $\omega \sim 0.435$, 0.840 and 1.188, respectively. Figure 4 is used to demonstrate the interior crisis, and the Poincaré maps at $\omega = 0.839$ and $\omega = 0.840$ are depicted respectively in Figs. 4(a) and 4(b).

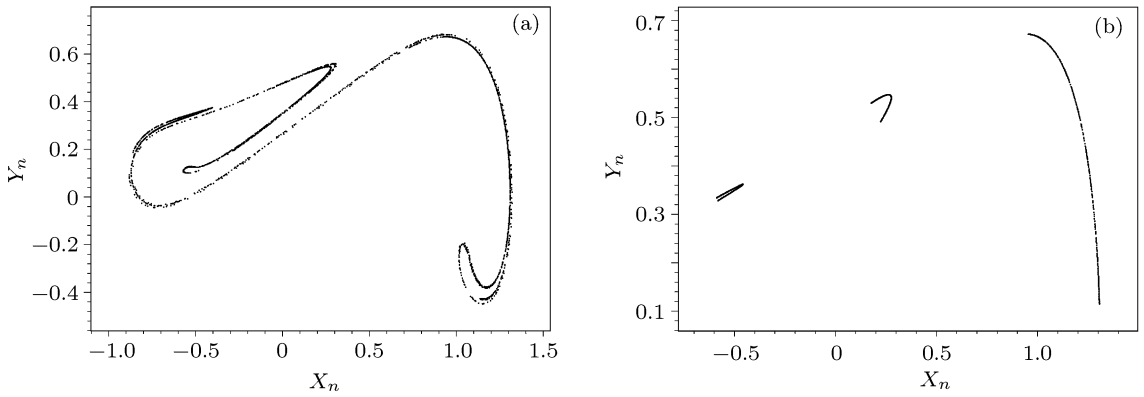


Fig. 4. Poincaré maps of the chaotic states in Fig.2(a) for showing the interior crisis: (a) $\omega = 0.839$; (b) $\omega = 0.840$.

To gain a better insight into the chaotic feature, we may calculate the fractal dimension, called the Lyapunov dimension, of the strange attractor. According to the definition of Kaplan and Yorke,[22] the Lyapunov dimension is defined as

$$D_L = k + \frac{1}{\mid \lambda_{k+1} \mid} \sum_{i=1}^{k} \lambda_i, \tag{14}$$

where k satisfies $\sum_{i=1}^{k} \lambda_i \geq 0$ and $\sum_{i=1}^{k+1} \lambda_i < 0$. By a complicated computing, the fractal dimension of the strange attractors shown in Fig. 3(b) is 2.317.

4. Conclusion

A number of numerical computations for the generalised KdV equation subject to a periodic perturbation with $p = 3$, 4, 5 and 6 are carried out. With the help of bifurcation diagrams and the largest Lyapunov exponents, abundant dynamical phenomena in the perturbed system (4) such as the complex bifurcation structure, the cascade of period-doubling leading to chaos, intermittency behaviour and interior crisis have been found. To characterise the chaotic behaviour of this system, the spectra of the Lyapunov exponent and Lyapunov dimension are also employed.

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