Homotopic mapping solving method of the reduces equation for Kelvin waves*

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A reduces equation of the Kelvin wave is considered. By using the homotopic mapping solving method, the approximate solution is obtained. The homptopic mapping method is an analytic method, the obtained solution can analyse operations sequentially.

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1. Introduction

Emaniel[1] has shown that in small-scale short-lived two dimensional disturbances the Coriolis parameter is important in equatorial regions only when the aspect is near unity. Sun[2] reached a comparable conclusion, finding that all Coriolis terms are important for small-scale unsymmetrical symmetric instability. Tropical circulations are greatly influenced, an approximation for the Coriolis parameter describing the meridional variation over a distance in north–south near the equator.

Studying atmospheric physics is a very attractive subject academic circles. During the past decade many approximate methods have been developed and refined, including the method of averaging, boundary layer methods, methods of matched asymptotic expansion and multiple scales. Recently, many scholars such as Guarguaglini,[3] Hovhannisyan and Vulanovic,[4] Abid, Jieli and Trabelsi,[5] Graef and Kong,[6] and Barbu and Cosma[7] have done a great deal of work. Using the method of differential inequalities, Mo *et al.* also considered a class of singularly perturbed nonlinear boundary value problems for the reaction diffusion equations,[8] the activator inhibitor systems,[9] the ecological environment,[10] the shock wave,[11] the soliton,[12−15] the laser pulse,[16] the ocean science[17,18] and atmospheric physics.[19−23] In this paper, using a special homotopic mapping method, we study the Kelvin waves in atmospheric physics.

2. Kelvin wave model

The goal of the current study is to identity Kelvin wave modes associated with a stably stratified or neutral atmosphere that are independent of moist convective and nonlinear interactions.

Now we consider the following reduced equation for the Kelvin wave in $(x, z, t) \in (-\infty, +\infty)^2 \times [0, +\infty)$:[24]

$$\left(\frac{\alpha_2}{C^2}\right)\frac{\partial^4\pi}{\partial\, t^4} + \left[\left(\frac{\alpha_2}{C^2}\right)(N^2 + f'^2) + \Gamma^2\right]\frac{\partial^2\pi}{\partial\, t^2} - \alpha_1\frac{\partial^4\pi}{\partial\, t^2\partial x^2} - N^2\frac{\partial^2\pi}{\partial x^2} + 2f'\Gamma\frac{\partial^2\pi}{\partial\, t\partial x} - \frac{\partial^4\pi}{\partial\, t^2\partial z^2} = 0, \quad (1)$$

where independent variables t, x, z are measures of time and distance in the east–west and vertical coordinates, respectively, the scaled variable π represents the pressure, N^2 is the Brunt Vaisalla frequency, C is the Laplacian velocity of sound, while α_1, α_2, f' and Γ are corresponding constants.[24]

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We first write the following equation

$$\left(\frac{\alpha_2}{C^2}\right)\frac{\partial^4\pi}{\partial\, t^4}+\left[\left(\frac{\alpha_2}{C^2}\right)(N^2+f'^2)+\Gamma^2\right]\frac{\partial^2\pi}{\partial\, t^2}-\alpha_1\frac{\partial^4\pi}{\partial\, t^2\partial x^2}-\frac{\partial^4\pi}{\partial\, t^2\partial z^2}=0. \tag{2}$$

Let $w=\dfrac{\partial^2\pi}{\partial\, t^2}$, thus from Eq. (2), we have

$$\left(\frac{\alpha_2}{C^2}\right)\frac{\partial^2 w}{\partial\, t^2}+\left[\left(\frac{\alpha_2}{C^2}\right)(N^2+f'^2)+\Gamma^2\right]w-\alpha_1\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial z^2}=0. \tag{3}$$

From (3), we have

$$\frac{\partial^2 w}{\partial\, t^2}+a^2w-\frac{\partial^2 w}{\partial x_1^2}-\frac{\partial^2 w}{\partial z_1^2}=0, \tag{4}$$

where

$$a^2=(N^2+f'^2)+\frac{C^2}{\alpha_2}\Gamma^2,\quad x_1=\frac{1}{C}\sqrt{\frac{\alpha_2}{\alpha_1}}x,\quad z_1=\frac{\sqrt{\alpha_2}}{C}z.$$

Let $u(x_1,y_1,z_1,t)=(\cos\,(ay_1))\bar{w}(x_1,z_1,t)$, we obtain

$$\frac{\partial^2 u}{\partial\, t^2}-\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial y_1^2}+\frac{\partial^2 u}{\partial z_1^2}\right)=(\cos\,(ay_1))\left(\frac{\partial^2\bar{w}}{\partial\, t^2}+a^2\bar{w}-\frac{\partial^2\bar{w}}{\partial x_1^2}-\frac{\partial^2\bar{w}}{\partial z_1^2}\right).$$

Therefore, from Eq. (4), we consider only the following three dimensions wave equation:

$$\frac{\partial^2 u}{\partial\, t^2}-\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial y_1^2}+\frac{\partial^2 u}{\partial z_1^2}\right)=0. \tag{5}$$

It is easy to see that the solution of Eq. (5) is

$$\begin{aligned}u(x_1,y_1,z_1,t)=&\frac{\partial}{\partial\, t}\left[\frac{1}{4\pi\, a}\iint_{S_t^M}\frac{\phi_1(\xi,\zeta)}{\sqrt{(\xi-x_1)^2+(\eta-y_1)^2-(\zeta-z_1)^2}}\mathrm{d}S\right]\\&+\frac{1}{4\pi\, a}\iint_{S_t^M}\frac{\phi_2(\xi,\zeta)}{\sqrt{(\xi-x_1)^2+(\eta-y_1)^2+(\zeta-z_1)^2}}\mathrm{d}S,\end{aligned}$$

where S_t^M is a sphere whose radius is t and centre is $M(x_1,y_1,z_1)$, and $\phi_i(x_1,z_1),\ i=1,2$ are initial conditions of wave equation (5). Thus we have the solution of Eq. (3) as follows:

$$\begin{aligned}w(x,z,t)=&\frac{\partial}{\partial\, t}\left[\frac{1}{4\pi}\iint_{S_t^{M'}}\frac{(\cos\,(a\eta))\phi_1(\xi,\zeta)}{(\cos\,(ay))\sqrt{(\xi-\sqrt{\alpha_2/\alpha_1}x/C)^2+(\eta-y)^2-(\zeta-\sqrt{\alpha_2}z/C)^2}}\mathrm{d}S\right]\\&+\frac{1}{4\pi}\iint_{S_t^{M'}}\frac{(\cos(a\eta))\phi_2(\xi,\zeta)}{(\cos\,(ay))\sqrt{(\xi-\sqrt{\alpha_2\alpha_1}x/C)^2+(\eta-y)^2+(\zeta-\sqrt{\alpha_2}z/C)^2}}\mathrm{d}S.\end{aligned} \tag{6}$$

Therefore the solution $\tilde{\pi}(x,z,t)$ of Eq. (2) with the initial conditions $\phi_i(x,z),\ \ i=1,2,3,4$ is

$$\begin{aligned}\tilde{\pi}(x,z,t)=&\frac{1}{4\pi\, a}\int_0^t\left[\iint_{S_t^{M'}}\frac{(\cos\,(a\eta))\phi_1(\xi,\zeta)}{(\cos\,(ay))\sqrt{(\xi-\sqrt{\alpha_2/\alpha_1}x/C)^2+(\eta-y)^2-(\zeta-\sqrt{\alpha_2}z/C)^2}}\mathrm{d}S\right]\mathrm{d}t_1\\&+\frac{1}{4\pi\, a}\int_0^t\int_0^{t_1}\left[\iint_{S_t^{M'}}\frac{(\cos(a\eta))\phi_2(\xi,\zeta)}{(\cos\,(ay))\sqrt{(\xi-\sqrt{\alpha_2/\alpha_1}x/C)^2+(\eta-y)^2+(\zeta-\sqrt{\alpha_2}z/C)^2}}\mathrm{d}S\right]\mathrm{d}t_2\mathrm{d}t_1\\&+t\phi_3(\sqrt{\alpha_2/\alpha_1}x/C,\sqrt{\alpha_2}z/C)+\phi_4(\sqrt{\alpha_2/\alpha_1}x/C,\sqrt{\alpha_2}z/C^2),\end{aligned} \tag{7}$$

where $S_t^{M'}$ is a sphere with radius t and centred at $M'(x,y,z)$.

3. Homotopic mapping solving method

Now we consider the Eq. (1). Introducing a homotopic mapping[25−27] $H(\pi, p),\ R \times I \to R$:

$$H(\pi, p) = L_1\pi - L_1\bar{\pi} + p(L_1\bar{\pi} + L_2\pi), \tag{8}$$

where $R = (-\infty, +\infty),\ I = [0, 1],\ \bar{\pi}$ is an initial approximations of Eq. (1), and linear operators $L_i\ (i = 1, 2)$, are

$$L_1 = \left(\frac{\alpha_2}{C^2}\right)\frac{\partial^4}{\partial\, t^4} + \left[\left(\frac{\alpha_2}{C^2}\right)(N^2 + f'^2) + \Gamma^2\right]\frac{\partial^2}{\partial\, t^2} - \alpha_1\frac{\partial^4}{\partial\, t^2\partial x^2} - \frac{\partial^4}{\partial\, t^2\partial x^2},$$
$$L_2 = -N^2\frac{\partial^2}{\partial x^2} + 2f'\Gamma\frac{\partial^2}{\partial\, t\partial x}.$$

Obviously, from mapping (8), $H_i\ (\pi, 1) = 0$, it is the same as the Eq. (1). Then the solution of the equation (1) is the same as the solution of $H(\pi, p) = 0$ as $p \to 1$.

In order to obtain the approximate solution of Eq. (1), let

$$\pi = \sum_{i=0}^{\infty}\bar{\pi}_i p^i, \quad p \in I. \tag{9}$$

From Eq. (6) we first assume that the zero-th order approximate solution $\bar{\pi}_0$ is a solution $\tilde{\pi}(x, z, t)$ for Eq. (2).

From homotopic mapping (8), the first order approximate solution of Eq. (1) is

$$L_1\bar{\pi}_1 = N^2\frac{\partial^2\bar{\pi}_0}{\partial x^2} - 2f'\Gamma\frac{\partial^2\bar{\pi}_0}{\partial\, t\partial x} \equiv F_1(x, y, z, t). \tag{10}$$

It is easy to see that the solution π_1 of Eq. (10) with zero initial conditions is

$$\bar{\pi}_1 = \frac{1}{4\pi}\int_0^t\int_0^{t_1}\iiint_{r'\le t_2}\frac{(\cos(a\eta))F_1(\xi, \eta, \zeta, t_2 - r')}{(\cos(ay_1))r'}\mathrm{d}V\mathrm{d}t_2\mathrm{d}t_1, \tag{11}$$

where $r' = \sqrt{(\xi - \sqrt{\alpha_2/\alpha_1}x/C)^2 + (\eta - y)^2 + (\zeta - \sqrt{\alpha_2}z/C)^2}$.

Analogously, using the same method, we can obtain the higher order approximate solutions as follow

$$\bar{\pi}_i = \frac{1}{4\pi}\int_0^t\int_0^{t_1}\iiint_{r'\le t_2}\frac{(\cos(a\eta))F_i(\xi, \eta, \zeta, t_2 - r')}{(\cos(ay_1))r'}\mathrm{d}V\mathrm{d}t_2\mathrm{d}t_1, \quad i = 2, 3, \ldots, \tag{12}$$

where $F_i(x, y, z, t) = N^2\dfrac{\partial^2\bar{\pi}_{i-1}}{\partial x^2} - 2f'\Gamma\dfrac{\partial^2\bar{\pi}_{i-1}}{\partial\, t\partial x}$ which are well determined functions successively.

Thus from homotopic theory and Eqs. (6), (11), (12) and (9), we can obtain the solution $\pi(x, z, t)$

$$\begin{aligned}\pi(x, z, t) = {} & \frac{\partial}{\partial\, t}\left[\frac{1}{4\pi}\iint_{S_t^{M'}}\frac{(\cos(a\eta))\phi_1(\xi, \zeta)}{(\cos(ay))\sqrt{(\xi - \sqrt{\alpha_2/\alpha_1}x/C)^2 + (\eta - y)^2 - (\zeta - \sqrt{\alpha_2}z/C)^2}}\mathrm{d}S\right] \\ & + \frac{1}{4\pi}\iint_{S_t^{M'}}\frac{(\cos(a\eta))\phi_2(\xi, \zeta)}{(\cos(ay))\sqrt{(\xi - \sqrt{\alpha_2/\alpha_1}x/C)^2 + (\eta - y)^2 + (\zeta - \sqrt{\alpha_2}z/C)^2}}\mathrm{d}S \\ & + t\phi_3(\sqrt{\alpha_2/\alpha_1}x/C, \sqrt{\alpha_2}z/C) + \phi_4(\sqrt{\alpha_2/\alpha_1}x/C, \sqrt{\alpha_2}z/C) \\ & + \frac{1}{4\pi}\int_0^t\int_0^{t_1}\iiint_{r'\le t_2}\frac{\sum\limits_{i=1}^{\infty}(\cos(a\,\eta))F_i(\xi, \eta, \zeta, t_2 - r')}{(\cos(ay_1))r'}\mathrm{d}V\mathrm{d}t_2\mathrm{d}t_1.\end{aligned}$$

4. Exemple

In order to obtain an approximate expression of the solution of Eq. (1) using the homotopic mapping method, we now select some peculiar parameters of the model (1) as follows:[24] $\alpha_i = 1,\ \ i = 1, 2$, $C = 325\ \mathrm{m{\cdot}s^{-1}}$, $N^2 = 10^{-4}\ \mathrm{s^{-2}}$, $f' = 1.4 \times 10^{-4}\ \mathrm{s^{-1}}$, $\Gamma = 4.5 \times 10^{-5}\ \mathrm{m^{-1}}$, and the initial conditions: $\phi_1(x, z) = x + z$, $\phi_i(x, z) = 0,\ i = 2, 3, 4$. Then from Eq. (12), we have the zero-th and the first order approximate solutions $\pi_0(x, z, t)$ and $\pi_i(x, z, t)$ of Eq. (1):

$$\pi_0(x,z,t) = -(2.10\times10^{-1})(x+z)\cos[(2.18)t],$$
$$\pi_1(x,z,t) = -(2.10\times10^{-1})(x+z)\cos[(2.18)t] + (8.68\times10^{-11})[-t\cos[(2.18)t] + (9.17\times10^{-1})\sin(2.18)t - t].$$

5. Summary and discussion

Kelvin wave is a complicated natural phenomenon in atmospheric physics. Hence we need to be reduced to basic models. We solve them using the approximate method. The homotopic mapping method is a simple and valid method. Using the homotopic mapping method, the precision solution is decided by the initial approximation of the original system (1)–(3). In the homotopic mapping, we first assume the zero-th order approximate solution π_0 of the reduced equation for the Eq. (2). In this way, we can approach faster the solution of the original equation (1). The homotopic mapping method is an approximate analytic method, which differs from the general numerical method. The expansions of the solution using the homotopic mapping method can be continuously analytically operated. Thus, from approximate solutions, we can study the qualitative and quantitative behaviours of the other physical quantities in future.

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