The periodic solution to the model for the El Niño-Southern oscillation*

Li Xiao-Jing(李晓静)†

College of Mathematics and Physics, Jiangsu Teachers University of Technology, Changzhou 213001, *China*

(Received 14 June 2009; revised manuscript received 28 August 2009)

A class of oscillator of the El Niño-Southern oscillation model is considered. Using Mawhin's continuation theorem, a result on the existence of periodic solutions for ENSO model is obtained.

Keywords: nonlinear, coincidence degree, periodic solution, ENSO model
PACC: 0230

1. Introduction

The El Niño-Southern oscillation (ENSO) is a natural phenomenon involved in the tropical Pacific ocean–atmosphere interactions. The phenomenon of the ENSO is a very attractive subject of study in international academic circles.[1−6] Mo *et al.*[7−9] also considered a class of nonlinear problems in atmospheric physics, ocean weather, dynamic system and so on. The ENSO is an interannual climate variability. Recently, Li have studied the periodic behaviour of the corresponding problems.[10−13] In this study, we investigate the periodic solutions of a sea–air oscillator model for the ENSO by using Mawhin's continuation theorem.

We now consider the following sea–air oscillator of the ENSO model:[4]

$$\frac{\mathrm{d}T}{\mathrm{d}t} = CT + Dh - \epsilon T^3, \tag{1}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -ET - R_h h, \tag{2}$$

where T is the SST anomaly in the equatorial eastern Pacific, h is the thermocline depth anomaly, C, D, E, R_h and ϵ are positive model parameters, defined in Refs. [1]–[3]. Using the variational iteration method, the approximation of the solution for the ENSO model is obtained in Ref. [8].

In this paper, we continue to study the ENSO model. By using Mawhin's continuation theorem, a result on the existence of periodic solutions for the sea–air oscillator model is obtained. Throughout this paper we assume that C, D, E, R_h and ϵ are positive parameters.

2. The periodic solutions for model (1) and (2)

Theorem 2.1 If $CR_h < DE + \dfrac{\epsilon}{E^2}R_h^3$ and $3R_h^2\omega^2 + 3R_h\omega < 1$, where ω is a given constant, then system (1) and (2) has at least one ω-periodic solution.

Proof In order to use Mawhin's continuation theorem to study the existence of ω-periodic solutions for system (1) and (2), we should consider the following equation:

$$h''(t) = (CR_h - DE)h(t) + (C - R_h)h'(t) - \frac{\epsilon}{E^2}(h'(t) + R_h h(t))^3. \tag{3}$$

Clearly, if $h(t)$ is a ω-periodic solution of Eq. (3), then $\left(-\dfrac{h'(t) + R_h h(t)}{E}, h(t)\right)^{\mathrm{T}}$ must be a ω-periodic solution of system (1) and (2).

Obviously, 0 is a trivial periodic solution of Eq. (3). Without the loss of generality, we may assume that $h(t)$ is a nontrivial solution of Eq. (3).

Now, we let

$$L : D(L) \subset X \longrightarrow Y, Lh = h''$$

*Project supported by the National Natural Science Foundation of China (Grant No. 40676016), the Natural Science Foundation of Jiangsu Province of China (Grant Nos. BK2009105 and BK2008119), the Natural Science Foundation of Jiangsu Education Committee, China (Grant Nos. 09kjd110001 and 08kjb110011), Key Natural Science Foundation by the Bureau of Education of Anhui Province of China (Grant No. KJ2008A05ZC) and Jiangsu Teachers University of Technology Foundation (Grant No. KYY08033).

†Corresponding author. E-mail: lixiaojing14@tom.com

© 2010 Chinese Physical Society and IOP Publishing Ltd

and $N : X \longrightarrow Y$,

$$[Nh](t) = (CR_h - DE)h(t) + (C - R_h)h'(t) - \frac{\epsilon}{E^2}(h'(t) + R_h h(t))^3, \quad (4)$$

where $Y = C_\omega = \{x | x \in C(\boldsymbol{R}, \boldsymbol{R}) : x(t+\omega) \equiv x(t)\}$ with the norm $|\varphi|_0 = \max\limits_{t\in[0,\omega]} |\varphi(t)|, \forall \varphi \in C_\omega$, $X = C^1_\omega = \{x | x \in C^1(\boldsymbol{R}, \boldsymbol{R}) : x(t+\omega) \equiv x(t)\}$ with the norm $\|\varphi\| = \max\{|\varphi|_0, |\varphi'|_0\}, \forall \varphi \in C^1_\omega$, and $D(L) = \{h \in C_\omega : h'' \in C(\boldsymbol{R}, \boldsymbol{R})\}$. It is easy to see that Eq. (3) can be converted into abstract equation $Lh = Nh$. Moreover, from the definition of L, we see that $\ker L = \boldsymbol{R}$, $\mathrm{Im}\, L = \{h : h \in Y, \int_0^\omega h(t)\mathrm{d}t = 0\}$. So L is a Fredholm operator with index 0. Also, let projectors $P : X \longrightarrow \ker L$ and $Q : Y \longrightarrow \mathrm{Im}\, Q$ be defined by

$$Ph = h(0)\ ;\ Qh = \frac{1}{\omega}\int_0^\omega h(t)\mathrm{d}t,$$

and let K represent the inverse of $L|_{\ker P \bigcap D(L)}$. Obviously, $\mathrm{Im}\, P = \ker L = \boldsymbol{R}, \ker Q = \mathrm{Im}\, L$ and $\forall y \in \mathrm{Im}\, L$,

$$[Ky](t) = -\frac{t}{\omega}\int_0^\omega (\omega - s)y(s)\mathrm{d}s + \int_0^t (t-s)y(s)\mathrm{d}s. \quad (5)$$

From Eqs. (4) and (5), one can easily see that N is L-compact on $\overline{\Omega}$, where Ω is an arbitrary open, bounded subset of X. Meanwhile, take $\Omega_1 = \{h : h \in D(L) \subset X, Lh = \lambda Nh, \lambda \in (0,1)\}$, $\forall h \in \Omega_1$, then h must satisfy

$$h''(t) = \lambda(CR_h - DE)h(t) + \lambda(C - R_h)h'(t) - \lambda\frac{\epsilon}{E^2}(h'(t) + R_h h(t))^3. \quad (6)$$

Set $h(t_{\max}) = \max\limits_{t\in\boldsymbol{R}} h(t)$, $h(t_{\min}) = \min\limits_{t\in\boldsymbol{R}} h(t)$, where $t_{\max}, t_{\min} \in \boldsymbol{R}$, then we have

$$h'(t_{\max}) = 0, h''(t_{\max}) \le 0, h'(t_{\min}) = 0, h''(t_{\min}) \ge 0, \quad (7)$$

which, together with Eq. (6) and intermediate value theorem yields that there exists a constant $t_1 \in \boldsymbol{R}$ such that

$$(CR_h - DE)h(t_1) - \frac{\epsilon}{E^2}R_h^3 h^3(t_1) = 0. \quad (8)$$

We now prove that: if $h(t)$ is a ω-periodic solution of Eq. (6), then there exists a constant $t_2 \in \boldsymbol{R}$ such that

$$|h(t_2)| \le 1. \quad (9)$$

Assume, by way of contradiction, that formula (9) does not hold. Then

$$|h(t)| > 1, \quad \text{for all} \quad t \in \boldsymbol{R},$$

which, together with assumption $CR_h < DE + \frac{\epsilon}{E^2}R_h^3$ yields that

$$\begin{aligned}&(CR_h - DE)h^2(t_1) - \frac{\epsilon}{E^2}R_h^3 h^4(t_1)\\ &< (CR_h - DE)h^2(t_1) - \frac{\epsilon}{E^2}R_h^3 h^2(t_1)\\ &< (CR_h - DE - \frac{\epsilon}{E^2}R_h^3)h^2(t_1) < 0,\end{aligned}$$

i.e.

$$(CR_h - DE)h(t_1) - \frac{\epsilon}{E^2}R_h^3 h^3(t_1) \ne 0,$$

which contradicts Eq. (8). This contradiction implies that Eq. (9) is true.

Let $t_2 = m\omega + t_0$, where $t_0 \in [0, \omega]$ and m is an integer, then $|h(t_0)| \le 1$. Hence we have

$$|h|_0 \le 1 + \int_0^\omega |h'(t)|\mathrm{d}t. \quad (10)$$

Multiplying the two sides of Eq. (6) by $h'(t)$ and integrating them over $[0, \omega]$, we get

$$\begin{aligned}0 = {}&(C - R_h)\int_0^\omega |h'(t)|^2\mathrm{d}t - \frac{\epsilon}{E^2}\int_0^\omega |h'(t)|^4\mathrm{d}t\\ &- \frac{3\epsilon R_h}{E^2}\int_0^\omega h(t)(h'(t))^3\mathrm{d}t\\ &- \frac{3\epsilon R_h^2}{E^2}\int_0^\omega |h'(t)|^2|h(t)|^2\mathrm{d}t,\end{aligned}$$

which, together with formula (10) and Hölder inequality yields

$$\begin{aligned}&\left(\frac{\epsilon}{E^2} - \frac{3\epsilon R_h\omega}{E^2} - \frac{3\epsilon R_h^2\omega^2}{E^2}\right)\int_0^\omega |h'(t)|^4\mathrm{d}t\\ &\le \left(C + R_h + \frac{3R_h^2\epsilon}{E^2}\right)\omega^{1/2}\left(\int_0^\omega |h'(t)|^4\mathrm{d}t\right)^{1/2}\\ &\quad + \left(\frac{6R_h^2\epsilon\omega^{5/4}}{E^2} + \frac{3R_h\epsilon\omega^{1/4}}{E^2}\right)\left(\int_0^\omega |h'(t)|^4\mathrm{d}t\right)^{3/4}.\end{aligned}$$

As $3R_h^2\omega^2 + 3R_h\omega < 1$, there is a constant $M > 0$ such that

$$\int_0^\omega |h'(t)|^4\mathrm{d}t < M. \quad (11)$$

From formulas (10) and (11), there exist two constants M_0 and M_1 independent of λ and $h(t)$ such that

$$|h|_0 < M_0\ ,\ |h'|_0 < M_1.$$

Let $\widetilde{M} = \max\{M_0 + 1, M_1 + 1\}$, $\Omega = \{h : \|h\| \le \widetilde{M}\}$ and $\Omega_2 = \{h \in \partial\Omega : h \in \ker L\}$, then $\forall h \in \Omega_2$,

$$QNh = \frac{1}{\omega}\int_0^\omega \left[(CR_h - DE)h - \frac{\epsilon}{E^2}R_h^3 h^3\right]\mathrm{d}t \ne 0.$$

On the other hand, let $J : \mathrm{Im}\, Q \longrightarrow \mathrm{ker}L$ as follows

$$H(h,\mu) = -\mu h + (1-\mu)JQNh,\ (h,\mu) \in \Omega \times [0,1].$$

Then we have $\forall (h,\mu) \in (\partial\Omega \bigcap \mathrm{ker}L) \times [0,1]$,

$$H(h,\mu) = \frac{1-\mu}{\omega}\int_0^{\omega}[(CR_h - DE)h - \frac{\epsilon}{E^2}R_h^3 h^3]\mathrm{d}t - \mu h \neq 0.$$

Hence,

$$\begin{aligned} &\deg\{JQN, \Omega \cap \mathrm{ker}L, 0\} \\ &= \deg\{H(h,0), \Omega \cap \mathrm{ker}L, 0\} \\ &= \deg\{H(h,1), \Omega \cap \mathrm{ker}L, 0\} \\ &= \deg\{-I, \Omega \cap \mathrm{ker}L, 0\} \neq 0. \end{aligned}$$

Thus we conclude that[14] equation $Lh = Nh$ has a ω-periodic solution $h(t)$, i.e. system (1) and (2) has ω-periodic solution

$$(T(t), h(t))^{\mathrm{T}} = \left(-\frac{h'(t) + R_h h(t)}{E}, h(t)\right)^{\mathrm{T}}.$$

3. Conclusions

ENSO is a complicated natural phenomenon, and it describes an interannual climate variability. Hence it is necessary to deduce basic models for the sea–air oscillator and prove the existence of periodic solutions of corresponding problems in mathematical theory. The interesting thing in this study is that a result on the existence of periodic solutions for the ENSO model is obtained. Thus, the authors who study atmospheric physics can forecast when the ENSO will appear by combining the result presented in this paper and practice.

References

[1] Feng G L, Dong W J, Jia X J and Cao H X 2002 *Acta Phys. Sin.* **51** 1181 (in Chinese)
[2] Liu S K, Fu Z T, Liu S D and Zhao Q A 2002 *Acta Phys. Sin.* **51** 10 (in Chinese)
[3] Feng G L, Dai X G and Wang A H 2001 *Acta Phys. Sin.* **50** 606 (in Chinese)
[4] Wang C Z 2001 *Advances in Atmospheric Sciences* **18** 674
[5] McPhaden M J and Zhang D 2002 *Nature* **415** 603
[6] Gu D F and Philander S G H 1997 *Science* **275** 805
[7] Mo J Q and Lin W T 2005 *Acta Phys. Sin.* **54** 993 (in Chinese)
[8] Mo J Q, Lin W T and Zhu J 2006 *Advances in Math.* **35** 232
[9] Mo J Q, Lin W T and Wang H 2007 *Chin. Phys.* **16** 578
[10] Li X J 2007 *Chin. Phys.* **16** 2837
[11] Li X J 2008 *Chin. Phys.* B **17** 1946
[12] Li X J 2008 *Acta Phys. Sin.* **57** 5366 (in Chinese)
[13] Li X J 2010 *Chin. Phys.* B **19** 020202
[14] Gaines R E and Mawhin J L 1977 *Coincidence Degree and Nonlinear Differential Equations* (Berlin: Springer)